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Discretization of Semilinear Differential Equations with an Exponential Dichotomy

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Abstract—A structural stability result for one-step discretizations of semilinear differential equations with an exponential dichotomy in its linear part is presented and interpreted as a shadowing result. Estimates are given in terms of the stepsize and of the order of the discretization method chosen.

1. INTRODUCTION

The main result of this paper (Theorem 3.1 and Corollary 3.1) is rather technical in nature, and therefore, we give a somewhat informal description in this introductory section. Throughout this paper, we consider differential equations of the form

$$\dot{y} = A(t)\,y + a(t,y) \tag{1}$$

with $t \in \mathbb{R}$ and y belonging to a Banach space $\mathcal{Y}$. We assume that $A \in C^{p+1}$, $a \in C^{p+1}$, and $a(t,0) = 0$, $a'_y(t,0) = 0$ for all $t \in \mathbb{R}$. The main assumption we make is that the linear equation $\dot{y} = A(t)y$ admits an exponential dichotomy on $\mathbb{R}$. We show in this paper that locally, in a small neighborhood of $0 \in \mathcal{Y}$, the differential equation (1) is correctly reproduced by one-step numerical methods of order $p \geq 2$. For sufficiently small stepsize h and initial time $t_0 = 0$ (this is just for simplicity), two "time variant discrete dynamical systems" are compared:

(a) the solution operator of the difference equation obtained by discretizing the differential equation (1) with stepsize h,
(b) the time-h-solution operator of the differential equation (1).

The "time variant dynamical systems" described in (a) and (b) are in fact no dynamical systems in the strict mathematical sense, they are rather cocycles associated with nonautonomous difference equations. Equations of this type have not attracted much attention until only recently. Things have changed, however. In recent years, the theory of difference equations has experienced a tremendous renaissance which is demonstrated by the list [1–4] of new monographs in this field. In [5,6], the time variance of the dynamics is emphasized. There we speak of "dynamical processes" instead of "time variant dynamical systems," and e.g., the "time variant family of stable manifolds" is called "stable fiber bundle." According notions are used for the other types

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of invariant manifolds. Also, in the present paper, we use these new notions in order to distinguish our nonautonomous point of view from the autonomous one.

The main result of this paper (see Theorem 3.1) provides a coordinate transformation that carries the above (a) and (b) onto each other. To be more precise, we present a sequence of self-homeomorphisms $\{\mathcal{H}_{kh}\}_{k\in\mathbb{N}}$ of $\mathcal{Y}$, $O(h^p)$-near to the identity, indexed by the time sequence $\{kh\}_{k\in\mathbb{N}}$ and carrying (a) onto (b). The corresponding stable fiber bundles are mapped onto each other, and the same holds true for the unstable fiber bundles. Geometrically this means that the "rotating saddle structure" inherent in the given differential equation persists under discretization. It also means that exact and numerical solutions are mutually shadowing each other.

The present paper is a natural extension of methods and results from [7–10] elaborated for the autonomous case. In the present paper, we start with a rather formal generalization of the Grobman-Hartman lemma and apply it to discretizations about equilibria of nonautonomous ordinary differential equations. Naturally enough, it turns out later on that the assumptions we have to impose for purely technical reasons are just the conditions defining an exponential dichotomy for the linear part.

2. A NONAUTONOMOUS GROBMAN-HARTMAN LEMMA

It is an outstanding theorem in differentiable dynamics that locally, in a neighbourhood of a hyperbolic equilibrium point, diffeomorphisms are conjugate to their linear part. A global version of this result can be formulated as follows.

THEOREM 2.1. (GROBMAN-HARTMAN LEMMA, [11, THEOREM 5.14]) *Let $\mathcal{U}$, $\mathcal{S}$, $\mathcal{X}$ be Banach spaces, $U \in L(\mathcal{U},\mathcal{U})$, $S \in L(\mathcal{S},\mathcal{S})$, $X \in L(\mathcal{X},\mathcal{X})$, $\mathcal{X} = \mathcal{U} \times \mathcal{S}$, $X = \operatorname{diag}(U,S)$. Let $\xi, \eta \in C^0(\mathcal{X},\mathcal{X})$ be Lipschitzian and $\operatorname{Lip}(\xi)$, $\operatorname{Lip}(\eta) \leq \kappa$. Assume that X is invertible and $\kappa < \min\{1-a, 1/|S^{-1}|\}$, where $a = \max\{|U^{-1}|, |S|\}$ and $a < 1$. Then there exists a unique $h \in C^0(\mathcal{X},\mathcal{X})$ such that, with $\mathcal{H} = \mathrm{id}_{\mathcal{X}} + h$, the relation*

$$(X+\eta)\,\mathcal{H} = \mathcal{H}\,(X+\xi) \tag{2}$$

holds true. Moreover, $\mathcal{H}$ is a homeomorphism of $\mathcal{X}$ onto $\mathcal{X}$, and therefore, it is a conjugacy from $X+\xi$ to $X+\eta$.

In this theorem and throughout this paper, we use standard notation and terminology. In the following, K_1, K_2, etc., will denote positive constants. The positive constants $K_1(\varepsilon)$, $K_2(\varepsilon)$, etc., will depend on some parameter ε. The constants K_1, $K_2(\varepsilon)$, etc., will not necessarily be the same at different appearances. Given Banach spaces $\mathcal{X}, \mathcal{Y}$, for each $j = 0, 1, 2, \dots$, $C^j(\mathcal{X},\mathcal{Y})$ denotes, with the norm

$$\max\Big\{\sup\Big\{\big|f^{(m)}(x)\big| \mid x \in \mathcal{X}\Big\} \mid m = 0, 1, \dots, j\Big\},$$

the Banach space of all j times continuously differentiable functions from $\mathcal{X}$ to $\mathcal{Y}$ with bounded derivatives. For brevity, we write $C^0(\mathcal{X},\mathcal{Y}) = C(\mathcal{X},\mathcal{Y})$. Partial derivates are denoted by f'_u, f'_s, φ'_τ, φ''_τ, $\varphi^{(m)}_\tau$, etc. The Banach space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $L(\mathcal{X},\mathcal{Y})$. The norm of $X \in L(\mathcal{X},\mathcal{Y})$ is defined by $\inf\Big\{c \in \mathbb{R} \mid |Xx| \leq c|x|, \text{ for all } x \in \mathcal{X}\Big\}$. The spectrum of $X \in L(\mathcal{X},\mathcal{X})$ is denoted by $\sigma(X)$. In product spaces, the norm is defined by $\max\{|u|, |s|\}$. (Though single bars denote norms in different spaces, no confusion should arise.) Lipschitz constants are denoted by $\operatorname{Lip}(\cdot)$.

The proof [11, Exercise 5.19] of Theorem 2.1 begins with the observation that, in virtue of the Lipschitz inverse function theorem [11, Exercise C.11], the mapping $X+\xi$ is a homeomorphism of $\mathcal{X}$ onto $\mathcal{X}$. The second step is to split the conjugacy equation (2) and to rewrite it as system

$$\begin{aligned} h^{\mathcal{U}} &= U^{-1}\Big\{h^{\mathcal{U}}(X+\xi) - \eta^{\mathcal{U}}(\mathrm{id}_{\mathcal{X}}+h) + \xi^{\mathcal{U}}\Big\} \\ h^{\mathcal{S}} &= \Big\{Sh^{\mathcal{S}} + \eta^{\mathcal{S}}(\mathrm{id}_{\mathcal{X}}+h) - \xi^{\mathcal{S}}\Big\}(X+\xi)^{-1}, \end{aligned} \tag{3}$$

where, of course, superscripts $\mathcal{U}$ and $\mathcal{S}$ stay for the $\mathcal{U}$- and $\mathcal{S}$-coordinate, respectively. A straightforward application of the contraction mapping principle yields that system (3) has a unique solution in $C(\mathcal{X},\mathcal{X}) = C(\mathcal{X},\mathcal{U}) \times C(\mathcal{X},\mathcal{S})$. The last step in the proof of Theorem 2.1 is to point out that the resulting map $\mathcal{H} = (\mathrm{id}_{\mathcal{U}} + h^{\mathcal{U}},\ \mathrm{id}_{\mathcal{S}} + h^{\mathcal{S}})$ is a homeomorphism of $\mathcal{X}$ onto $\mathcal{X}$.

In order to extend the proof of Theorem 2.1 we just sketched to a nonautonomous setting, we replace the one-cell conjugacy diagram (cf. equation (2))

$$\begin{tikzcd}
\mathcal{U}\times\mathcal{S} \arrow[r, "X+\xi"] \arrow[d, "\mathcal{H}"'] & \mathcal{U}\times\mathcal{S} \arrow[d, "\mathcal{H}"] \\
\mathcal{U}\times\mathcal{S} \arrow[r, "X+\eta"] & \mathcal{U}\times\mathcal{S}
\end{tikzcd}$$

by the doubly-infinite ($n \in \mathbb{Z}$) conjugacy diagram

$$\begin{tikzcd}
{} \arrow[r] & \mathcal{U}_n\times\mathcal{S}_n \arrow[r, "X_n+\xi_n"] \arrow[d, "\mathcal{H}_n"'] & \mathcal{U}_{n+1}\times\mathcal{S}_{n+1} \arrow[d, "\mathcal{H}_{n+1}"] \arrow[r] & {} \\
{} \arrow[r] & \mathcal{U}_n\times\mathcal{S}_n \arrow[r, "X_n+\eta_n"] & \mathcal{U}_{n+1}\times\mathcal{S}_{n+1} \arrow[r] & {}
\end{tikzcd}$$

Given a sequence $\left\{(\mathcal{X}_n, |\cdot|_n)\right\}_{-\infty}^{\infty}$ of Banach spaces, let $\mathcal{C}$ denote the Banach space of all doubly-infinite bounded sequences $h = (\dots, h_n, h_{n+1}, \dots)$ with coordinates in $C(\mathcal{X}_n, \mathcal{X}_n)$. The norm on $\mathcal{C}$ is defined as $\|h\| = \sup\left\{|h_n| \mid n \in \mathbb{Z}\right\}$, where $|h_n| = \sup\left\{|h_n(x_n)|_n \mid x_n \in \mathcal{X}_n\right\}$, $n \in \mathbb{Z}$. The following theorem is the discrete analogue of the main result in [12]. For other discrete-time nonautonomous variants of the Grobman-Hartman Lemma, see [13,14] as well as the forthcoming monograph [6]. (Essentially, all continuous- and discrete-time invariant manifold and linearization results treated in [15] are valid (see [6]) in the nonautonomous case as well.)

THEOREM 2.2. *Let $\mathcal{U}_n$, $\mathcal{S}_n$, $\mathcal{X}_n$ be Banach spaces, $U_n \in L(\mathcal{U}_n,\mathcal{U}_{n+1})$, $S_n \in L(\mathcal{S}_n,\mathcal{S}_{n+1})$, $X_n \in L(\mathcal{X}_n,\mathcal{X}_{n+1})$, $\mathcal{X}_n = \mathcal{U}_n \times \mathcal{S}_n$, $X_n = \operatorname{diag}(U_n, S_n)$, $n \in \mathbb{Z}$. Assume that*

$$X_n \text{ is invertible and } a = \sup\left\{\max\left\{|U_n^{-1}|, |S_n|\right\} \mid n \in \mathbb{Z}\right\} < 1. \tag{4}$$

Furthermore, let $\xi_n, \eta_n \in C(\mathcal{X}_n, \mathcal{X}_{n+1})$ be Lipschitzian and satisfy

$$|\xi_n|, |\eta_n| \le K \quad \text{and} \quad \operatorname{Lip}(\xi_n), \operatorname{Lip}(\eta_n) \le \kappa, \qquad \text{for all } n \in \mathbb{Z}, \tag{5}$$

where K, κ are positive constants and

$$\kappa < \min\left\{1 - a, \frac{1}{|S_n^{-1}|}\right\}, \qquad \text{for all } n \in \mathbb{Z}. \tag{6}$$

Then there exists a unique $h = (\dots, h_n, h_{n+1}, \dots) \in \mathcal{C}$ such that, with $\mathcal{H}_n = \mathrm{id}_n + h_n$, we obtain the relations (cf. equation (2))

$$(X_n + \eta_n)\,\mathcal{H}_n = \mathcal{H}_{n+1}\,(X_n + \xi_n), \qquad \text{for all } n \in \mathbb{Z}. \tag{7}$$

Moreover, for all $n \in \mathbb{Z}$, $\mathcal{H}_n$ is a homeomorphism of $\mathcal{X}_n$ onto $\mathcal{X}_n$ and $\mathcal{H} = (\dots, \mathcal{H}_n, \mathcal{H}_{n+1}, \dots)$ is, therefore, a conjugacy from $\left\{X_n + \xi_n\right\}_{-\infty}^{\infty}$ to $\left\{X_n + \eta_n\right\}_{-\infty}^{\infty}$.

PROOF. Using (4)–(6), it is immediately seen that $\mathrm{Lip}\,(\xi_n) \le \kappa$ and $|X_n^{-1}| = \max\{|U_n^{-1}|, |S_n^{-1}|\} = |S_n^{-1}| < 1/\kappa$. Thus, the conditions of the Lipschitz inverse function theorem [11, Exercise C.11] are satisfied and so $X_n + \xi_n$ is a homeomorphism of $\mathcal{X}_n$ onto $\mathcal{X}_{n+1}$, $n \in \mathbb{Z}$. It follows that system (7) can be reformulated as a fixed-point-equation $\mathcal{F}(h) = h$, where (cf. (3))

$$\begin{aligned}
(\mathcal{F}(h))_n^{\mathcal{U}} &= U_n^{-1}\Big\{h_{n+1}^{\mathcal{U}}(X_n + \xi_n) - \eta_n^{\mathcal{U}}(\mathrm{id}_n + h_n) + \xi_n^{\mathcal{U}}\Big\}\\
(\mathcal{F}(h))_n^{\mathcal{S}} &= \Big\{S_{n-1}h_{n-1}^{\mathcal{S}} + \eta_{n-1}^{\mathcal{S}}(\mathrm{id}_{n-1} + h_{n-1}) - \xi_{n-1}^{\mathcal{S}}\Big\}(X_{n-1} + \xi_{n-1})^{-1},
\end{aligned}$$

for all $n \in \mathbb{Z}$. We claim that $\mathcal{F}$ is a contraction on $\mathcal{C}$. In fact, given $h, j \in \mathcal{C}$ arbitrarily, it is routine to check that

$$\begin{aligned}
\big|(\mathcal{F}(h))_n\big| &\le \max\Big\{a\left(|h_{n+1}| + 2K\right), a|h_{n-1}| + 2K\Big\} \le a\|h\| + 2K,\\
\Big|(\mathcal{F}(h))_n^{\mathcal{U}} - (\mathcal{F}(j))_n^{\mathcal{U}}\Big| &\le a\left(|h_{n+1}^{\mathcal{U}} - j_{n+1}^{\mathcal{U}}| + \kappa|h_n - j_n|\right) \le a(1+\kappa)\|h - j\|,\\
\Big|(\mathcal{F}(h))_n^{\mathcal{S}} - (\mathcal{F}(j))_n^{\mathcal{S}}\Big| &\le a|h_{n-1}^{\mathcal{S}} - j_{n-1}^{\mathcal{S}}| + \kappa|h_{n-1} - j_{n-1}| \le (a+\kappa)\|h - j\|,
\end{aligned}$$

for all $n \in \mathbb{Z}$ and consequently, $\mathcal{F}(h) \in \mathcal{C}$ and $\|\mathcal{F}(h) - \mathcal{F}(j))\| \le (a+\kappa)\|h - j\|$. This proves the claim. In what follows, let h denote the unique solution of $\mathcal{F}(h) = h$ in $\mathcal{C}$. (Applying (4)–(6) again, we obtain that

$$\begin{aligned}
|h_n^{\mathcal{U}}| &\le a\left(|h_{n+1}^{\mathcal{U}}| + \kappa|h_n| + |-\eta_n^{\mathcal{U}} + \xi_n^{\mathcal{U}}|\right) \le a(1+\kappa)\|h\| + |\eta_n - \xi_n|\\
|h_n^{\mathcal{S}}| &\le a|h_{n-1}^{\mathcal{S}}| + \kappa|h_{n-1}| + |\eta_{n-1}^{\mathcal{S}} - \xi_{n-1}^{\mathcal{S}}| \le (a+\kappa)\|h\| + |\eta_{n-1} - \xi_{n-1}|,
\end{aligned}$$

for all $n \in \mathbb{Z}$, and consequently,

$$\|h\| \le (1 - a - \kappa)^{-1}\sup\Big\{|\eta_n - \xi_n| \mid n \in \mathbb{Z}\Big\}. \tag{8}$$

In several cases, inequality (8) is sharper than $\|h\| \le (1-a)^{-1}2K$, a consequence of $|(\mathcal{F}(h))_n| \le a\|h\| + 2K$.)

It is left to prove that, for all $n \in \mathbb{Z}$, $\mathcal{H}_n$ is a homeomorphism of $\mathcal{X}_n$ onto $\mathcal{X}_n$. Interchanging the role of $\{\xi_n\}_{-\infty}^{\infty}$ and $\{\eta_n\}_{-\infty}^{\infty}$, the previous argument yields the existence of a unique $j = (\ldots, j_n, j_{n+1}, \ldots) \in \mathcal{C}$, such that, with $\mathcal{J}_n = \mathrm{id}_n + j_n$, we get $(X_n + \xi_n)\mathcal{J}_n = \mathcal{J}_{n+1}(X_n + \eta_n)$. Since $(X_n + \xi_n)\mathcal{J}_n\mathcal{H}_n = \mathcal{J}_{n+1}(X_n + \eta_n)\mathcal{H}_n = \mathcal{J}_{n+1}\mathcal{H}_{n+1}(X_n + \xi_n)$ and $(\ldots, \mathcal{J}_n\mathcal{H}_n - \mathrm{id}_n, \mathcal{J}_{n+1}\mathcal{H}_{n+1} - \mathrm{id}_{n+1}, \ldots) \in \mathcal{C}$, the uniqueness property in (7) shows that $\mathcal{J}_n\mathcal{H}_n = \mathrm{id}_n$, for all $n \in \mathbb{Z}$. A similar reasoning yields that $\mathcal{H}_n\mathcal{J}_n = \mathrm{id}_n$, for all $n \in \mathbb{Z}$. ∎

REMARK 2.1. Assume, in addition to the assumptions of Theorem 2.2, that

$$\text{there exists a constant } L \text{ such that } |U_n| \le L, \qquad \text{for all } n \in \mathbb{Z}. \tag{9}$$

Let $\mathcal{X}$ denote the Banach space of all doubly-infinite bounded sequences $x = (\ldots, x_n, x_{n+1}, \ldots)$ with coordinates in $\mathcal{X}_n$. The norm on $\mathcal{X}$ is defined as $|||x||| = \sup\{|x_n|_n \mid n \in \mathbb{Z}\}$. By letting $Xx = (\ldots, X_{n-1}x_{n-1}, X_nx_n, \ldots)$, a bounded linear operator $X \in L(\mathcal{X}, \mathcal{X})$ is defined. Observe that $X = \mathrm{diag}(U, S)$, where $U \in L(\mathcal{U}, \mathcal{U})$, $S = L(\mathcal{S}, \mathcal{S})$ and $\mathcal{U} = \{x \in \mathcal{X} \mid x_n \in \mathcal{U}_n$, for all $n \in \mathbb{Z}\}$, $\mathcal{S} = \{x \in \mathcal{X} \mid x_n \in \mathcal{S}_n$ for all $n \in \mathbb{Z}\}$, $U = (\ldots, U_{n-1}u_{n-1}, U_nu_n, \ldots)$ for $u \in \mathcal{U}$, $S = (\ldots, S_{n-1}s_{n-1}, S_ns_n, \ldots)$ for $s \in \mathcal{S}$. Since X is invertible and $|||U^{-1}|||$, $|||S||| \le a < 1$, X is a hyperbolic linear operator on $\mathcal{X}$ and, in case condition (9) is satisfied, the statement of Theorem 2.2 about a *nonautonomous* Grobman-Hartman lemma can be interpreted as the classical *autonomous* Grobman-Hartman result in a suitably chosen sequence space. Though Theorem 2.2 is not in the most general form (nonlinear perturbations of the form $\xi, \eta \in C(\mathcal{X}, \mathcal{X})$ can also be allowed), it is general enough for our purposes.

REMARK 2.2. (continuation): By elementary Taylor-Laurent expansions, it is easily seen that $\sigma(X)$ is contained in the closed annular hull of the set $\cup\{\sigma(X_n) \mid n \in \mathbb{Z}\}$. The notion $(\sigma(X_n))$ makes sense here because the Banach spaces $\mathcal{X}_n$ and $\mathcal{X}_{n+1}$ can be identified via the linear homeomorphism $X_n \in L(\mathcal{X}_n, \mathcal{X}_{n+1})$. Thus, we get $\sigma(X) \subset \mathrm{cl}(\{\mu \in \mathbb{C} \mid$ there exist $n \in \mathbb{Z}$ and $\lambda \in \sigma(X_n)$, such that $|\mu| = |\lambda|\})$. On the other hand, it follows directly from the definition and from the closedness of the spectrum that

$$\sigma(X) \supset \mathrm{cl}\Big(\cup\{\sigma(X_n) \mid n \in \mathbb{Z}\}\Big).$$

In what follows, we present an example showing that, in general, $\sigma(X)$ is different from $\mathrm{cl}\big(\cup\{\sigma(X_n) \mid n \in \mathbb{Z}\}\big)$. In fact, consider the separable Hilbert space $(l_2, |\cdot|)$. For $n \geq 3$, let Q_n be a weighted Toeplitz permutation matrix defined by

$$q_{ij}^n = \begin{cases} 1, & \text{if } (i,j) = (1,n) \text{ or } (i,j) = (k,k),\ k \geq n+1 \\ n-1, & \text{if } (i,j) = (2,1) \\ (n-k+2)^{-1}(n-k+1), & \text{if } (i,j) = (k,k-1),\ k = 3,4,\ldots,n \\ 0, & \text{otherwise}. \end{cases}$$

It is immediate that Q_n is invertible and $R_n = Q_n^{-1}$ is given by $r_{ij}^n = 1/q_{ji}^n$, $i,j = 1,2,\ldots$; $n \geq 3$. A direct computation shows that

$$Q_n, R_n \in L(l_2, l_2), \quad |Q_n| = n-1, \quad |R_n| = 2, \qquad n \geq 3, \tag{10}$$

$$Q_n^n = R_n^n = \mathrm{id}_{l_2}, \qquad n \geq 3. \tag{11}$$

For $n \leq 2$, set $Q_n = Q_3$, $R_n = R_3$. Finally, for $n \in \mathbb{Z}$, set $\mathcal{X}_n = l_2$, $X_n = 10 \cdot \mathrm{id}_{l_2} - R_n$. Using the R-part of (10), it is easily seen that $X_n \in L(l_2, l_2)$, $(\mathcal{U}_n = \mathcal{X}_n$, $\mathcal{S}_n = \{O\})$, $|X_n| \leq 12$, X_n is invertible and $|X_n^{-1}| \leq 1/8$. In virtue of (11), the spectral theorem yields that $\sigma(-R_n) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, $\sigma(X_n) \subset \{\lambda \in \mathbb{C} \mid |\lambda - 10| = 1\}$. Thus, $10 \notin \sigma(X_n)$ and, by the construction, $\left(10 \cdot \mathrm{id}_{l_2} - X_n\right)^{-1} = Q_n$, for all $n \in \mathbb{Z}$. With the notation adopted in Remark 2.1, consider now the operator $10 \cdot \mathrm{id}_{\mathcal{X}} - X \in L(\mathcal{X}, X)$. If it were invertible, we had $\sup\{|Q_n| \mid n \in \mathbb{Z}\} = \sup\left\{\left|\left(10 \cdot \mathrm{id}_{l_2} - X_n\right)^{-1}\right| \mid n \in \mathbb{Z}\right\} < \infty$, contradicting the Q-part of (10). Summing up the previous considerations, we get $\mathrm{cl}(\cup\{\sigma(X_n) \mid n \in \mathbb{Z}\}) \subset \{\lambda \in \mathbb{C} \mid |\lambda - 10| = 1\}$ but $10 \in \sigma(X)$. (A somewhat more detailed analysis shows that $\mathrm{cl}(\cup\{\sigma(X_n) \mid n \in \mathbb{Z}\}) = \{\lambda \in \mathbb{C} \mid |\lambda - 10| = 1\}$ and $\sigma(X) = \{\lambda \in \mathbb{C} \mid |\lambda - 10| \leq 1\}$. We do not know if there is any theoretical relationship between $\sigma(X)$ and the convex hull of the set $\mathrm{cl}(\cup\{\sigma(X_n) \mid n \in \mathbb{Z}\})$.)

REMARK 2.3. The argument which led from Theorem 2.1 to Theorem 2.2 works equally well when generalizing basic invariant manifold theory to the nonautonomous case. For example, consider the classical stable manifold theorem [11, Theorem 6.5] for mappings of the form

$$\mathcal{U} \times \mathcal{S} \to \mathcal{U} \times \mathcal{S}, \qquad (u,s) \to (Uu + \xi^{\mathcal{U}}(u,s), Ss + \xi^{\mathcal{S}}(u,s)).$$

As it is well known, the stable manifold $\mathcal{M}$ is obtained as

$$\mathrm{graph}(\mu) = \Big\{(\mu(s), s) \in \mathcal{U} \times \mathcal{S} \mid s \in \mathcal{S}\Big\},$$

where μ is the unique bounded Lipschitzian solution of the invariance equation

$$\mu(s) = U^{-1}\mu\Big(Ss + \xi^{\mathcal{S}}(\mu(s), s)\Big) - U^{-1}\xi^{\mathcal{U}}(\mu(s), s), \qquad s \in \mathcal{S}.$$

In order to extend the stable manifold theorem to the nonautonomous case, i.e., to state and prove a stable fiber bundle theorem for nonautonomous difference equations, the only thing to do is the introduction and the solution—as in the proof of Theorem 2.2, via the contraction mapping principle—of the doubly-infinite system

$$\mu_n(s) = U_n^{-1}\mu_{n+1}\Big(S_n s + \xi_n^{\mathcal{S}}(\mu_n(s), s)\Big) - U_n^{-1}\xi_n^{\mathcal{U}}(\mu_n(s), s), \qquad s \in \mathcal{S}_n,\ n \in \mathbb{Z}$$

of invariance equations. For a precise statement see the proof of Theorem 4.1.

3. AN APPLICATION TO NUMERICAL ANALYSIS

Let $\mathcal{Y}$ be a Banach space, $f \in C^{p+1}(\mathbb{R} \times \mathcal{Y}, \mathcal{Y})$, and consider the ordinary differential equation $y' = f(t, y)$. The solution through $(t_0, z) \in \mathbb{R} \times \mathcal{Y}$ is denoted by $\Phi(\cdot, t_0, z)$. The τ-discretization of order $p = 1, 2, \ldots$ is defined as a mapping $\varphi \in C^{p+1}(\mathbb{R}^+ \times \mathbb{R} \times \mathcal{Y}, \mathcal{Y})$ satisfying, with some positive constants K and τ_0 (independent of τ, t_0 and z) the inequality

$$|\Phi(t_0 + \tau, t_0, z) - \varphi(\tau, t_0, z)| \leq K\tau^{p+1}, \qquad \text{for all } t_0 \in \mathbb{R},\ \tau \in (0, \tau_0],\ z \in \mathcal{Y}. \tag{12}$$

(Actually, inequality (12) is needed only on a $\mathcal{Y}$-ball of sufficiently large radius. Also the boundedness of derivatives of f and φ is required only in this sense.) It is well known (see [16]) that, for each $T > 0$, there exists a constant $K = K(T)$ such that $K(T_2) < K(T_1)$ whenever $T_2 < T_1$ and, with $\varphi(k, \tau, t_0, z)$ defined by the recursion $\varphi(0, \tau, t_0, z) = z$, $\varphi(k + 1, \tau, t_0, z) = \varphi(\tau, t_0 + k\tau, \varphi(k, \tau, t_0, z))$, $k \in \mathbb{N}$, we get

$$\left|\Phi(t_0 + T, t_0, z) - \varphi\left(N, \frac{T}{N}, t_0, z\right)\right| \leq \frac{K(T)}{N^p}, \tag{13}$$

whenever $t_0 \in \mathbb{R}$, $N \in \mathbb{N}$, $T/N < \tau_0$ and $z \in \mathcal{Y}$.

In other words, on any interval of finite length T, the exact solution is $O(N^{-p})$-approximated by the numerical solution obtained by solving the nonautonomous difference equation (with $z_0 = z$)

$$z_{k+1} = \varphi\left(\frac{T}{N}, t_0 + \frac{kT}{N}, z_k\right), \qquad k = 0, 1, \ldots, N - 1.$$

LEMMA 3.1. *Under the previous conditions, the estimate*

$$\left|\Phi'_z(t_0 + T, t_0, z) - \varphi'_z\left(N, \frac{T}{N}, t_0, z\right)\right| \leq \frac{K(T)}{N^{p-1}} \tag{14}$$

holds true whenever $t_0 \in \mathbb{R}$, $N \in \mathbb{N}$, $(T/N) < \tau_0$ *and* $z \in \mathcal{Y}$.

PROOF. In virtue of (12), $\Phi_\tau^{(m)}(t_0, t_0, z) = \varphi_\tau^{(m)}(0, t_0, z)$, $m = 0, 1, \ldots, p$, and hence, differentiating Taylor's expansion formula

$$\Phi(t_0 + \tau, t_0, z) - \varphi(\tau, t_0, z) = \big((p-1)!\big)^{-1}\int_0^1 (1-s)^{p-1}\Big(\Phi_\tau^{(p)}(t_0 + s\tau, t_0, z) - \varphi_\tau^{(p)}(s\tau, t_0, z)\Big)\, ds\, \tau^p,$$

with the remainder in integral form, we obtain that

$$|\Phi'_z(t_0 + \tau, t_0, z) - \varphi'_z(\tau, t_0, z)| \leq K\tau^p, \qquad \text{for all } t_0 \in \mathbb{R},\ \tau \in (0, \tau_0],\ z \in \mathcal{Y}. \tag{15}$$

For brevity, we write $F_k(z) = \Phi(t_0 + kT/N, t_0 + (k-1)T/N, z)$, $G_k(z) = \varphi(T/N, t_0 + (k-1)T/N, z)$ and observe that $\Phi(t_0 + kT/N, t_0, \cdot) = F_k F_{k-1} \ldots F_1$ and $\varphi(k, T/N, t_0, \cdot) = G_k G_{k-1} \ldots G_1$, $k = 1, 2, \ldots, N$. Applying Gronwall's lemma to the variational equation of $\dot{y} = f(t, y)$, it follows that $|F'_k(z)| \leq 1 + a/N$ and (using (15),(13)) $|G'_k(z)| \leq 1 + b/N$ and

$$\begin{aligned} |F'_k(F_{k-1} \ldots F_1) - G'_k(G_{k-1} \ldots G_1)| &\leq |F'_k(F_{k-1} \ldots F_1) - G'_k(F_{k-1} \ldots F_1)| \\ &\quad + |G'_k(F_{k-1} \ldots F_1) - G'_k(G_{k-1} \ldots G_1)| \\ &\leq K\left(\frac{T}{N}\right)^p + c\,|F_{k-1} \ldots F_1 - G_{k-1} \ldots G_1| \\ &\leq \frac{d}{N^p}, \end{aligned}$$

for all $k = 1, 2, \ldots, N$, $t_0 \in \mathbb{R}$, $T/N < \tau_0$, $z \in \mathcal{Y}$ (where, of course, a, b, c, d (and e) are constants independent of N). The desired inequality follows now via

$$
\begin{aligned}
&\left|\Phi_z'(t_0+T,t_0,z)-\varphi_z'\left(N,\frac{T}{N},t_0,z\right)\right| \\
&\qquad = \left|(F_N\dots F_1)_z'-(G_N\dots G_1)_z'\right| \\
&\qquad = \Big|F_N'(F_{N-1}\dots F_1)\cdot F_{N-1}'(F_{N-2}\dots F_1)\cdot\dots\cdot F_1' \\
&\qquad\qquad -G_N'(G_{N-1}\dots G_1)\cdot G_{N-1}'(G_{N-2}\dots G_1)\cdot\dots\cdot G_1'\Big| \\
&\qquad \le \sum_{k=1}^{N}\Big|F_N'(F_{N-1}\dots F_1)\cdot\dots\cdot F_{k+1}'(F_k\dots F_1)\cdot\Big[F_k'(F_{k-1}\dots F_1) \\
&\qquad\qquad -G_k'(G_{k-1}\dots G_1)\Big]\cdot G_{k-1}'(G_{k-2}\dots G_1)\cdot\dots\cdot G_1'\Big| \\
&\qquad \le N\left(1+\frac{\max\{a,b\}}{N}\right)^{N-1}\frac{d}{N^p} \\
&\qquad \le \frac{e}{N^{p-1}}.
\end{aligned}
$$

This completes the proof of Lemma 3.1. ∎

REMARK 3.1. Similar computations yield also that

$$\left|\Phi_z^{(m)}(t_0+T,t_0,z)-\varphi_z^{(m)}\left(N,\frac{T}{N},t_0,z\right)\right|\le\frac{K(T)}{N^{p-m}},$$

whenever $m=0,1,\dots,p$, $t_0\in\mathbb{R}$, $N\in\mathbb{N}$, $T/N<\tau_0$ and $z\in\mathcal{Y}$.

From now on, assume that $f(t,y)=A(t)y+a(t,y)$, where $A\in C^{p+1}(\mathbb{R},L(\mathcal{Y},\mathcal{Y}))$, $a\in C^{p+1}(\mathbb{R}\times\mathcal{Y},\mathcal{Y})$ and, for all $t\in\mathbb{R}$, $a(t,0)=0$, $a_y'(t,0)=0$. The solution of $\dot y=A(t)\,y$ through $(t_0,z)\in\mathbb{R}\times\mathcal{Y}$ is denoted by $\Psi(\cdot,t_0)z$.

Further, assume there exists a function $\mu\in C^{p+1}(\mathcal{Y},\mathbb{R}^+)$, with the properties that $\mu(y)=0$ whenever $|y|\ge 1$, and $\mu(y)=1$ whenever $|y|\le\Delta$, for some $\Delta>0$. (This requirement concerns the finer structure of the Banach space and is met, e.g., if $\mathcal{Y}$ is finite-dimensional or a Hilbert space.) With $a(t,y;\varepsilon)=a(t,y)\mu(y/\varepsilon)$, consider now the ordinary differential equation

$$\dot y=A(t)y+a(t,y;\varepsilon),\qquad t\in\mathbb{R},\ y\in\mathcal{Y},\ \varepsilon>0. \tag{16}$$

The solution of (16) through $(t_0,z)\in\mathbb{R}\times\mathcal{Y}$ is denoted by $\Phi(\cdot,t_0,z;\varepsilon)$. For t, $t_0\in\mathbb{R}$, $z\in\mathcal{Y}$, $\varepsilon>0$, set $r(t,t_0,z;\varepsilon)=\Phi(t,t_0,z;\varepsilon)-\Psi(t,t_0)z$.

LEMMA 3.2. *There exists a continuous function $\Omega:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, increasing in both variables, with $\Omega(0,T)=0$, for all $T\ge 0$, such that*

$$|r(t,t_0,\cdot;\varepsilon)|\le\Omega(\varepsilon,T)\varepsilon(t-t_0),\qquad \textit{whenever } t_0\in\mathbb{R},\ t_0\le t\le t_0+T,\ T\ge 0,\ \varepsilon>0, \tag{17}$$

$$|r_z'(t,t_0,\cdot;\varepsilon)|\le\Omega(\varepsilon,T)(t-t_0),\qquad \textit{whenever } t_0\in\mathbb{R},\ t_0\le t\le t_0+T,\ T\ge 0,\ \varepsilon>0. \tag{18}$$

PROOF. The differentiability assumptions on a imply that, for a suitable bounded continuous function $\omega:\mathbb{R}^+\to\mathbb{R}^+$ with $\omega(0)=0$, one has

$$\big|a(t,z;\varepsilon)-a(t,w;\varepsilon)\big|\le\omega(\varepsilon)|z-w|,\qquad \text{whenever } t\in\mathbb{R},\ z,w\in\mathcal{Y},\ \varepsilon>0.$$

In particular, $|a(t,z;\varepsilon)|\le\omega(\varepsilon)\varepsilon$ for all $t\in\mathbb{R}$, $z\in\mathcal{Y}$, $\varepsilon>0$. Since

$$\dot r(t,t_0,z;\varepsilon)=A(t)r(t,t_0,z;\varepsilon)+a(t,\Phi(t,t_0,z;\varepsilon);\varepsilon) \tag{19}$$

and $r(t_0, t_0, z; \varepsilon) = 0$, an elementary application of Gronwall's lemma yields that, with $\alpha = \sup\{|A(t)| \mid t \in \mathbb{R}\}$,

$$|r(t, t_0, z; \varepsilon)| \leq \omega(\varepsilon)\varepsilon\frac{1}{\alpha}\Big(\exp(\alpha(t - t_0)) - 1\Big), \qquad \text{for all } t_0 \in \mathbb{R},\ t \geq t_0,\ z \in \mathcal{Y},\ \varepsilon > 0.$$

Thus, (17) follows. Furthermore, since r is obviously differentiable in z, (18) is equivalent to $\operatorname{Lip}(r(t, t_0, \cdot; \varepsilon)) \leq \Omega(\varepsilon, T)(t - t_0)$, and this latter inequality is a direct consequence of Gronwall's lemma when applied to the difference of equations (19) and $\dot{r}(t, t_0, w; \varepsilon) = A(t)r(t, t_0, w; \varepsilon) + a(t, \Phi(t, t_0, w; \varepsilon); \varepsilon)$. ∎

In what follows, we put further restrictions on $\dot{y} = A(t)y$ and assume there are positive constants μ, T_0, M satisfying $\mu < 1$ and, for each $n \in \mathbb{Z}$, we assume there exists a splitting $\mathcal{V}_n \times \mathcal{W}_n = \mathcal{Y}$ with projections $P_n \in L(\mathcal{Y}, \mathcal{V}_n)$ and $Q_n \in L(\mathcal{Y}, \mathcal{W}_n)$, $P_n + Q_n = \mathrm{id}_{\mathcal{Y}}$ such that

$$\Psi\big(nT_0, (n+1)T_0\big)\mathcal{V}_{n+1} = \mathcal{V}_n \quad \text{and} \quad \big|\Psi(nT_0, (n+1)T_0)v\big| \leq \mu|v|, \quad \text{whenever } v \in \mathcal{V}_{n+1} \tag{20}$$

$$\Psi\big((n+1)T_0, nT_0\big)\mathcal{W}_n = \mathcal{W}_{n+1} \quad \text{and} \quad \big|\Psi((n+1)T_0, nT_0)w\big| \leq \mu|w|, \quad \text{whenever } w \in \mathcal{W}_n \tag{21}$$

$$|P_n|, |Q_n| \leq M, \qquad \text{for all } n \in \mathbb{Z}. \tag{22}$$

From now on, we consider equation (16) with the particular initial condition $y(0) = z$. Also conditions (20)–(22) were already formulated in anticipation of this technical simplification. There is another technical simplification: the stepsize is chosen to be T_0/N, ($N \in \mathbb{N}$, large). The first paragraph of Section 4 helps the reader to reformulate the results for arbitrary initial time t_0. (With ε, N fixed, the sequence of homeomorphisms is indexed then by $\{t_0/T_0 + m/N\}_{m\in\mathbb{Z}}$. It is not hard to show that $\mathcal{H}_{t_0/T_0+m/N}$ depends continuously on t_0. (The basic reason for this is that in Theorem 2.1 the homeomorphism $\mathcal{H}$ depends continuously on ξ and η.))

Now we are in a position to state and prove the main results of this paper. Our aim is to compare exact and numerical solutions of equation (16). More precisely, the time-T_0/N-sequence $\{\Phi(kT_0/N, 0, z; \varepsilon)\}_0^\infty$ of the exact solution through $(0, z) \in \mathbb{R} \times \mathcal{Y}$ is compared with a numerical solution suitably chosen, i.e., for some $w \in \mathcal{Y}$ suitably chosen, with the solution sequence of the nonautonomous difference equation $w_{k+1} = \varphi(T_0/N, kT_0/N, w_k; \varepsilon), k = 0, 1, 2, \ldots$ (and $w_0 = w$). With the notations of Theorem 3.1, below, $w = \mathcal{H}_0(z)$ and $|w_k - \Phi(kT_0/N, 0, z; \varepsilon)| \leq L(\varepsilon)/N^p$ for all $k \in \mathbb{N}$. Similarly, since $\mathcal{H}_0$ is a homeomorphism, given $w \in \mathcal{Y}$ arbitrarily, there exists a $z \in \mathcal{Y}$ (as a matter of fact, $z = \mathcal{H}_0^{-1}(w)$), such that $|w_k - \Phi(kT_0/N, 0, z; \varepsilon)| \leq L(\varepsilon)/N^p$, for all $k \in N$. Thus, using a terminology dating back (see [17]) to Anosov and Bowen, exact and numerical solutions are mutually shadowing each other. In particular, the $O(N^{-(p+1)})$-pseudo-orbit $\{w_k\}_0^\infty$ is $O(N^{-p})$-shadowed by the time-T_0/N-sequence of the exact solution $\Phi(t, 0, \mathcal{H}_0^{-1}(w); \varepsilon)$. For the reader's convenience, we recall the notions appearing in this shadowing statement: Suppose we are given a mapping $F : \mathcal{Y} \to \mathcal{Y}$. The sequence $\{x_k\}_0^\infty \subset \mathcal{Y}$ is then said to be a δ-pseudo-orbit if $|x_{k+1} - F(x_k)| < \delta$, $k \in \mathbb{N}$. This pseudo-orbit is ε-shadowed by the true orbit $\{y_k\}_0^\infty \subset \mathcal{Y}$ if $(y_{k+1} = F(y_k)$ and $|x_k - y_k| < \varepsilon$, $k \in \mathbb{N}$.

In describing the relationship between true orbits and orbits found by computers (round-off-errors!), the concept of shadowing has found some nice applications (see, e.g., [18,19] and the references therein). While most of these papers are devoted to explaining numerical results, especially numerically computed chaotic orbits, our approach is analytical and more traditional.

Heading for the main results of this paper, we first present a conjugacy result for N^{th} power subsequences. By considering the collection of initial conditions $\{y(nT_0) = z\}_{n\in\mathbb{Z}}$, we will see that Theorem 2.2 is directly applicable for N^{th} power (exact resp., numerical) solution subsequences. The extension to the full solution sequences is accomplished in Corollary 3.1.

THEOREM 3.1. *Assume that all conditions preceding Lemmas 3.1 and 3.2 are satisfied. Furthermore, assume that the conditions (20)–(22) are met and that $p \geq 2$. Consider equation (16). Its solution operator and τ-discretization are denoted by $\Phi(t, t_0, \cdot; \varepsilon)$ and $\varphi(\tau, t_0, \cdot; \varepsilon)$,* *respectively. Let $\varphi(0,\tau,t_0,z;\varepsilon) = z$ and recursively, for $k \in \mathbb{N}$, define $\varphi(k+1,\tau,t_0,z;\varepsilon) = \varphi(\tau, t_0 + k\tau, \varphi(k,\tau,t_0,z;\varepsilon);\varepsilon)$. Then there is a positive constant ε_0 with the following property. Given $\varepsilon \in (0,\varepsilon_0]$ arbitrarily, there exist an integer $N_0 = N_0(\varepsilon)$ and a constant $L = L(\varepsilon)$ and, for each integer $N \geq N_0$, there exists a unique sequence $\{\mathcal{H}_n\}_{-\infty}^{\infty} = \{\mathcal{H}_n(\varepsilon,N)\}_{-\infty}^{\infty}$ of homeomorphisms of $\mathcal{Y}$ onto $\mathcal{Y}$ such that (provided $\varepsilon \in (0,\varepsilon_0]$ and $N \geq N_0$), for all $n \in \mathbb{Z}, z \in \mathcal{Y}$, we get*

$$\varphi\left(kN, \frac{T_0}{N}, nT_0, \mathcal{H}_n(z);\varepsilon\right) = \mathcal{H}_{k+n}\Big(\Phi\left((n+k)T_0, nT_0, z;\varepsilon\right)\Big), \qquad k = 0,1,2,\dots \tag{23}$$

and

$$|\mathcal{H}_n(z) - z| \leq \frac{L(\varepsilon)}{N^p}. \tag{24}$$

PROOF. It is enough to show that (23) is true for $k = 1$. In fact, (23) is trivially satisfied for $k = 0$ and ("extrapolating" in the long conjugacy diagram prior to Theorem 2.2, i.e., by induction, starting from the $k = 1$ case) the $k \geq 2$ case follows from the cocycle formulas

$$\Phi(t,t_0,z;\varepsilon) = \Phi\big(t,s,\Phi(s,t_0,z;\varepsilon);\varepsilon\big),$$
$$\varphi\big(k+\ell,\tau,t_0,z;\varepsilon\big) = \varphi\big(k,\tau,t_0+\ell\tau,\varphi(\ell,\tau,t_0,z;\varepsilon);\varepsilon\big).$$

Set $\mathcal{X}_n = \mathcal{Y}$, $|y|_n = \max\{|P_n y|, |Q_n y|\}$, for all $y \in \mathcal{X}_n$, $\mathcal{U}_n = \mathcal{V}_n$, $\mathcal{S}_n = \mathcal{W}_n$, $X_n = \Psi((n+1)T_0, nT_0)$, $U_n = X_n \mid \mathcal{U}_n$, $S_n = X_n \mid \mathcal{S}_n$, $\xi_n = r((n+1)T_0,\ nT_0, \cdot;\varepsilon)$, $\eta_n = \xi_n + \varphi(N, T_0/N, nT_0, \cdot;\varepsilon) - \Phi((n+1)T_0,\ nT_0, \cdot;\varepsilon)$, $n \in \mathbb{Z}$.

We point out that the conditions of Theorem 2.2 are satisfied. Observe that $|u|_n = |u|$, whenever $u \in \mathcal{U}_n$ and $|s|_n = |s|$, whenever $s \in \mathcal{S}_n, n \in \mathbb{Z}$. Hence, by virtue of (20) and (21), the constant a can be chosen for μ and condition (4) is satisfied.

To check (5) and (6), a little more care is needed. Since

$$|y| \leq |P_n y| + |Q_n y| \leq 2\max\Big\{|P_n y|,\ |Q_n(y)|\Big\} = 2|y|_n$$

and, using (22),

$$|y|_n \leq M|y|, \qquad \text{for all } y \in Y,$$

the norms $|\cdot|$ and $|\cdot|_n$, $n \in \mathbb{Z}$, are uniformly equivalent. Therefore, though being somewhat altered by the renorming, the lemmas are still directly applicable. Fix ε_0 so that, with the notation adopted in Lemma 3.2, $2M\Omega(\varepsilon_0,T_0)T_0 < (1-\mu)/3$. Further, given $\varepsilon \in (0,\varepsilon_0]$ arbitrarily, choose $N_0 = N_0(\varepsilon)$ so that, with the notation adopted in Lemma 3.1 (τ_0 and $K(T_0)$ may depend on ε now), $T_0/N_0 < \tau_0$ and $2MK(T_0)/N_0^{p-1} < (1-\mu)/3$. By (13), (17) and the renorming inequality $|y|_n \leq M|y|$, the constant K in (5) can be chosen for $M\Omega(\varepsilon_0,T_0)\varepsilon_0 T_0 + MK(T_0)/N_0^p$. Similarly, (14), (18), and the renorming inequality $|y| \leq 2|y|_n \leq 2M|y|$ imply that κ can be chosen for $2M\Omega(\varepsilon_0,T_0)T_0 + 2MK(T_0)/N_0^{p-1}$ and so condition (5) is satisfied. There is no loss of generality in assuming that $2M\kappa < \exp(-\alpha T_0)$, where $\alpha = \sup\{|A(t)| \mid t \in \mathbb{R}\}$. Since $|S_n^{-1}| \leq |X_n^{-1}| \leq 2M\exp(\alpha T_0)$, it follows that also condition (6) and therefore, all conditions of Theorem 2.2 are satisfied.

Observe that $X_n + \xi_n = \Phi((n+1)T_0, nT_0, \cdot;\varepsilon)$ and $X_n + \eta_n = \varphi(N, T_0/N, nT_0, \cdot;\varepsilon)$. Applying Theorem 2.2, we see that (23) is identical to (7), and (24) is implied by (8). It is straightforward to check that $L(\varepsilon)$ can be chosen, for $(1-\mu)^{-1}6MK(T_0)$ (where $K(T_0)$ may depend on ε, too). ∎

REMARK 3.2. Further restrictions on the numerical method guarantee that the constant $K(T)$ in (13) and (14) does not depend on ε if the differential equation $\dot{y} = f(t,y) \equiv A(t)y + a(t,y)$ is replaced by the one-parameter family of equations (16). As it is shown—*mutatis mutandis*—by the arguments in [20, pp. 385–391] elaborated for autonomous equations, this is the case for explicit Runge-Kutta methods. As an easy consequence of this, it follows then that also the constants N_0 and L in (24) are independent of ε varying in $(0, \varepsilon_0]$. Another advantage of Runge-Kutta methods is that then $\varphi(\tau, t_0, z; \varepsilon_0) = \varphi(\tau, t_0, z)$ for τ and $|z|$ sufficiently small. Thus (for Runge-Kutta methods), locally, in a small neighborhood of $0 \in \mathcal{Y}$, Theorem 3.1 can be interpreted as a result on the original ordinary differential equation $\dot{y} = A(t)y + a(t, y)$.

COROLLARY 3.1. *The conjugacy obtained in Theorem 3.1 can be extended to the full solution set* $\{\varphi(k, T_0/N, mT_0/N, z; \varepsilon) \mid k \in \mathbb{N}, m \in \mathbb{Z}\}$ *of the nonautonomous difference equations obtained via* T_0/N*-discretization.*

PROOF. We "interpolate" in the long conjugacy diagram prior to Theorem 2.2. For $n \in \mathbb{Z}, \ell \in \mathbb{N}$, $z \in \mathcal{Y}$, we set

$$\mathcal{H}_{n+\ell/N}(z) = \varphi\left(\ell, \frac{T_0}{N}, nT_0, \mathcal{H}_n\left(\Phi\left(nT_0, \left(n + \frac{\ell}{N}\right)T_0, z; \varepsilon\right)\right); \varepsilon\right).$$

It is easy to see that $\mathcal{H}_{n+\ell/N}$ is well defined and that this mapping is a homeomorphism of $\mathcal{Y}$ onto $\mathcal{Y}$,

$$\varphi\left(k, \frac{T_0}{N}, \left(n+\frac{\ell}{N}\right)T_0, \mathcal{H}_{n+\ell/N}(z); \varepsilon\right) = \mathcal{H}_{n+(k+\ell)/N}\left(\Phi\left(\left(n+\frac{(k+\ell)}{N}\right)T_0, \left(n + \frac{\ell}{N}\right)T_0, z; \varepsilon\right)\right).$$

Furthermore, we get

$$\left|\mathcal{H}_{n+\ell/N}(z) - z\right| \le \frac{L(\varepsilon)}{N^p},$$

for all $n \in \mathbb{Z}, k \in \mathbb{N}, \ell \in \mathbb{N}$, $z \in \mathcal{Y}$ (providing $N \ge N_0(\varepsilon), \varepsilon \in (0, \varepsilon_0]$). ∎

COROLLARY 3.2. *The sequence of homeomorphisms obtained in Corollary 3.1 shares possible periodicity properties of (16).*

PROOF.

(a) Assume that both $\Phi(t_0 + \tau, t_0, z; \varepsilon)$ and $\varphi(\tau, t_0, z; \varepsilon)$ are periodic in t_0 with minimal period $p > 0$. Analyzing (20)–(22), we see that T_0 can be chosen for an integer multiple of p, say $T_0 = j_0 p$ for some $j_0 \in \mathbb{N}$. Assume that $N = i\, j_0$ for some positive integer i. Then the uniqueness property in Theorem 3.1 yields that $\mathcal{H}_{n+\ell/N} = \mathcal{H}_{m+(\ell+ki)/N}$, for all $n, m, k, l \in \mathbb{Z}$.

(b) For N fixed, the uniqueness property in Theorem 3.1 implies also that $\mathcal{H}_{n+\ell/N}$ is independent of n and ℓ provided that $\Phi(t_0 + \tau, t_0, z; \varepsilon)$ and $\varphi(\tau, t_0, z; \varepsilon)$ are independent of t_0. ∎

4. REMARKS ON EXPONENTIAL DICHOTOMIES

We now return to the linear equation $\dot{y} = A(t)y$ and examine conditions (20)–(22). It is an elementary observation that (20)–(22) are consequences of an exponential dichotomy of this equation. Moreover, in case the estimate

$$\sup\{|A(t)| \mid t \in \mathbb{R}\} = \alpha < \infty \tag{25}$$

is valid, (20)–(22) are equivalent to an exponential dichotomy (and (22) is a consequence of (20),(21) [21, Section IV.3]). For convenience of the reader, we recall one of the various equivalent definitions [21, Section IV.3] of exponential dichotomy: The linear equation $\dot{y} = A(t)y$ is said to have an exponential dichotomy if there exist positive constants M, γ and, for each $t \in \mathbb{R}$, there exists a splitting $\mathcal{U}_t \times \mathcal{S}_t = \mathcal{Y}$ with projections $P_t \in L(\mathcal{Y}, \mathcal{U}_t)$ and $Q_t \in L(\mathcal{Y}, \mathcal{S}_t)$, $P_t + Q_t = \mathrm{id}_{\mathcal{Y}}$ such that

(a) $\Psi(t_0,t)\mathcal{U}_t = \mathcal{U}_{t_0}$ and $|\Psi(t_0,t)u| \le Me^{-\gamma(t-t_0)}|u|$ whenever $u \in \mathcal{U}_t$, $t \ge t_0$,

(b) $\Psi(t,t_0)\mathcal{S}_{t_0} = \mathcal{S}_t$ and $|\Psi(t,t_0)s| \le Me^{-\gamma(t-t_0)}|s|$ whenever $s \in \mathcal{S}_{t_0}$, $t \ge t_0$,

(c) $|P_t|$, $|Q_t| \le M$ for all $t \in \mathbb{R}$.

By kinematic similarities, i.e., by linear time-dependent coordinate transformations, systems with an exponential dichotomy can be carried over (see [21, Sections IV.2 and IV.6]) to the simpler form where $\mathcal{U}_t = \mathcal{U}_0$, $\mathcal{S}_t = \mathcal{S}_0$, for all $t \in \mathbb{R}$.

The standard reference books on ordinary differential equations with exponential dichotomies are [21,22]. Though most results in those books were proved to remain valid (see, e.g., [16,23] and the references therein) in the discrete case, we do not know of any published expository treatment of exponential dichotomies for difference equations (see [24], however). Neither do we know of any research papers devoted to the combinations of discretizations and exponential dichotomies. The predecessors (see [7,9,10]) of our Theorem 3.1 concern the time-independent case. The analogue of Theorem 3.1 for autonomous ordinary differential equations was proved in [10]. On the link of structural stability and exponential dichotomies, see [14,23,25].

Concluding this paper, we present a perturbation result on how the stable fiber bundle, i.e., the time variant family of stable manifolds is affected by discretizations. The corresponding statement for autonomous equations was proved, with increasing generality, in [7,9,10]. We use the notation adopted in the previous section and assume, of course, that all the conditions imposed in Section 3 are satisfied.

THEOREM 4.1. *The stable fiber bundle $\{\mathcal{M}_n^\varepsilon\}_{n\in\mathbb{Z}}$ of the time-T_0/N-sequence of the solution operator of (16) is $O(N^{-p})$-approximated by the stable fiber bundle $\{\mathcal{M}_n^{\varepsilon,N}\}_{n\in\mathbb{Z}}$ of the nonautonomous difference equation obtained via T_0/N-discretization of order p.*

PROOF. The argument we used in proving Corollary 3.1 shows that it is enough to prove the corresponding statement for N^{th} power subsequences.

Modifying the proof of the classical stable manifold theorem in the same way (cf. Remark 2.3) we modified the proof of Theorem 2.1, a simple computation yields the existence of a family of bounded Lipschitzian functions

$$\mu_n^\varepsilon : \mathcal{W}_n \to \mathcal{V}_n, \quad \varepsilon \in (0,\varepsilon_0], \qquad n \in \mathbb{Z},$$

such that, with

$$\mathcal{M}_n^\varepsilon = \left\{ (\mu_n^\varepsilon(w), w) \in \mathcal{V}_n \times \mathcal{W}_n = \mathcal{Y} \mid w \in \mathcal{W}_n \right\},$$

we get

$$\mathcal{M}_n^\varepsilon = \left\{ z \in \mathcal{Y} \mid \Phi\left((k+n)\,T_0, nT_0, z; \varepsilon\right) \to 0, \qquad \text{as } k \to \infty \right\} \tag{26}$$

and

$$\Phi\left((n+1)T_0, nT_0, \mathcal{M}_n^\varepsilon\right) = \mathcal{M}_{n+1}^\varepsilon, \qquad \varepsilon \in (0,\varepsilon_0],\ n \in \mathbb{Z}.$$

Similarly, for each $N \ge N_0(\varepsilon)$, there exists a family of bounded Lipschitzian functions

$$\mu_n^{\varepsilon,N} : \mathcal{W}_n \to \mathcal{V}_n, \qquad \varepsilon \in (0,\varepsilon_0],\ n \in \mathbb{Z},$$

such that, with

$$\mathcal{M}_n^{\varepsilon,N} = \left\{ (\mu_n^{\varepsilon,N}(w), w) \in \mathcal{V}_n \times \mathcal{W}_n = \mathcal{Y} \mid w \in \mathcal{W}_n \right\},$$

one obtains

$$\mathcal{M}_n^{\varepsilon,N} = \left\{ z \in \mathcal{Y} \mid \varphi\left(kN, \frac{T_0}{N}, nT_0, z; \varepsilon\right) \to 0, \qquad \text{as } k \to \infty \right\} \tag{27}$$

and

$$\varphi\left(N, \frac{T_0}{N}, nT_0, \mathcal{M}_n^{\varepsilon,N}; \varepsilon\right) = \mathcal{M}_{n+1}^{\varepsilon,N}, \qquad \varepsilon \in (0,\varepsilon_0],\ n \in \mathbb{Z}.$$

Further, for some positive constant $l(\varepsilon)$ independent of n and N, $N \geq N_0(\varepsilon)$, we have

$$\operatorname{Lip}(\mu_n^\varepsilon), \operatorname{Lip}(\mu_n^{\varepsilon,N}) \leq l(\varepsilon), \qquad \varepsilon \in (0, \varepsilon_0],\ n \in \mathbb{Z}$$

(and $l(\varepsilon) \to 0$ as $\varepsilon \to 0$).

It is well known that the homeomorphism $\mathcal{H}$ in Theorem 2.1 is Hölder continuous and has a Hölder continuous inverse [11, Exercise 5.47]. By a quick analysis of the parameters in solving [11, Exercise 5.47], $\mathcal{H}_n(0) = 0$ and there are positive constants a, b, $b < 1$, such that

$$|\mathcal{H}_n(z) - \mathcal{H}_n(\tilde{z})|, \left|\mathcal{H}_n^{-1}(z) - \mathcal{H}_n^{-1}(\tilde{z})\right| \leq a\,|z - \tilde{z}|^b$$

for all $n \in \mathbb{N}$, $z, \tilde{z} \in \mathcal{Y}$, $|z|, |\tilde{z}| \leq 1$. In particular, the collection of homeomorphisms $\{\mathcal{H}_n, \mathcal{H}_n^{-1}\}_{-\infty}^{\infty}$ is equicontinuous in a neigborhood of $0 \in \mathcal{Y}$.

It follows now from (23) and from the stable fiber bundle limiting characterizations (26),(27) that

$$\mathcal{H}_n(\mathcal{M}_n^\varepsilon) = \mathcal{M}_n^{\varepsilon,N}, \qquad n \in \mathbb{Z}.$$

Pick an arbitrary $(\mu_n^\varepsilon(w), w) = z \in \mathcal{M}_n^\varepsilon$ and consider also $(\mu_n^{\varepsilon,N}(\tilde{w}), \tilde{w}) = \mathcal{H}_n(z) \in \mathcal{M}_n^{\varepsilon,N}$. By virtue of (24) and (22),

$$\left|\mu_n^\varepsilon(w) - \mu_n^{\varepsilon,N}(\tilde{w})\right|, |w - \tilde{w}| \leq M\,|z - \mathcal{H}_n(z)| \leq \frac{ML(\varepsilon)}{N^p},$$

and consequently, for all $w \in \mathcal{W}_n$,

$$\begin{aligned}\left|\mu_n^\varepsilon(w) - \mu_n^{\varepsilon,N}(w)\right| &\leq \left|\mu_n^\varepsilon(w) - \mu_n^{\varepsilon,N}(\tilde{w})\right| + \left|\mu_n^{\varepsilon,N}(\tilde{w}) - \mu_n^{\varepsilon,N}(w)\right| \\ &\leq \frac{ML(\varepsilon)}{N^p} + l(\varepsilon)\,|\tilde{w} - w| \leq \big(1 + l(\varepsilon)\big)\frac{ML(\varepsilon)}{N^p}. \qquad \blacksquare\end{aligned}$$

Obviously, the same result is true for the unstable fiber bundle as well. Thus, the persistence of the "rotating saddle structure" is established. Together with Theorem 3.1, the previous theorem shows that equation (16) is correctly reproduced by numerical methods.

REMARK 4.1. Starting from the inequality stated in Remark 3.1, we may repeat all considerations from [10, Section 3]. After some lengthy computations, we obtain that μ_n^ε, $\mu_n^{\varepsilon,N} \in C^{p+1}(\mathcal{W}_n, \mathcal{V}_n)$ and that, with some positive constant $k = k(\varepsilon)$ (independent of n and m),

$$\left|(\mu_n^\varepsilon)^{(m)} - (\mu_n^{\varepsilon,N})^{(m)}\right| \leq \frac{k(\varepsilon)}{N^{p-m}}, \qquad \varepsilon \in (0, \varepsilon_0],\ N \geq N_0(\varepsilon),\ n \in \mathbb{Z},\ m = 0, 1, \dots, p.$$

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