LINEAR ALGEBRA AND ITS APPLICATIONS 8, 477–482 (1974)

Sharpness of Generalized Gerschgorin Disks*

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ABSTRACT

The generalized Gerschgorin disks of a partitioned matrix A, as introduced by D. G. Feingold and R. S. Varga (*Pacif. J. Math.* **12** 1241–1250 (1962)), are shown to give the smallest region covering all eigenvalues of all matrices from a certain class related to A.

1. INTRODUCTION

Given an arbitrary square matrix A of order n with complex entries, we will call a region of the complex plane containing all of the eigenvalues of A an inclusion region. Many generalizations of the inclusion region of Gerschgorin [3] are known. We shall investigate an inclusion region given in [2] for A partitioned into blocks as follows:

$$A=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix}. \tag{1}$$

Here A_{ij} is an $n_i \times n_j$ matrix ($n_i \geqslant 1$ for $1 \leqslant i \leqslant N$, $n_1 + n_2 + \cdots + n_N = n$), regarded also as a linear mapping of $V_j \to V_i$, where each V_i is the n_i-dimensional space of

*This research was supported in part by the National Research Council of Canada, grants A8201 and A8214.

complex column vectors. Thus A may be viewed as a linear transformation of $V_1 \oplus V_2 \oplus \cdots \oplus V_N$.

We suppose throughout this paper that a vector norm $\|\cdot\|_i$ is defined on each V_i. With respect to these norms, a linear mapping T: $V_j \to V_i$ has the induced operator norm

$$\|T\| \equiv \sup_{\|\mathbf{x}\|_j = 1} \|T\mathbf{x}\|_i,$$

and the greatest lower bound

$$m(T) \equiv \inf_{\|\mathbf{x}\|_j = 1} \|T\mathbf{x}\|_i.$$

We note that the supremum and infimum are actually attained because of compactness of the unit ball in V_j.

If $\sigma(A)$ denotes the set of all eigenvalues of A, it is known [2] that

$$\sigma(A) \subseteq G \equiv \bigcup_{i=1}^{N} \left\{ z : m(zI - A_{ii}) \leqslant \sum_{1 \leqslant j \leqslant N, j \neq i} \|A_{ij}\| \right\}. \tag{2}$$

In Sec. 3 we exhibit a class $\mathfrak{B}$ of matrices, related in a certain way to A, for which G is the minimal inclusion region, i.e.,

$$G = \bigcup_{B \in \mathfrak{B}} \sigma(B).$$

When $N = n$, the region G is simply the classical inclusion region of Gerschgorin, and consequently, we also obtain a precise characterization of the Gerschgorin circles.

2. PRELIMINARY RESULTS

For any fixed integer $n \geqslant 1$, let $\mathfrak{S}$ denote the set of all singular square matrices of order n. Let $\|\cdot\|$ denote an arbitrary vector norm. In this section, $\|\cdot\|$ will also denote the associated induced operator norm and the associated dual vector norm. It is well known that if B is nonsingular, then

$$m(B) = \|B^{-1}\|^{-1}.$$

Another representation of the greatest lower bound is given in the following lemma.

LEMMA *For an arbitrary square matrix B of order n,*

$$m(B) = \operatorname{dist}(B; \mathfrak{S}),$$

where

$$\operatorname{dist}(B;\mathbb{S}) \equiv \inf_{S\in\mathbb{S}} \|B-S\| \equiv \inf_{(B+H)\in\mathbb{S}} \|H\|.$$

(*Note*: *By a simple compactness argument,*

$$\operatorname{dist}(B;\mathbb{S}) = \min_{S\in\mathbb{S}} \|B-S\|.)$$

Proof. Since the lemma is clearly true if $B\in\mathbb{S}$, we consider the case $B\notin\mathbb{S}$. If

$$\|H\|<\|B^{-1}\|^{-1},$$

then

$$\|B^{-1}H\|\leqslant\|B^{-1}\|\,\|H\|<1,$$

and consequently

$$B+H=B(I+B^{-1}H)\notin\mathbb{S}.$$

Therefore, we have

$$\|B^{-1}\|^{-1}\leqslant\operatorname{dist}(B;\mathbb{S}).$$

To prove the reverse inequality, let $\mathbf{x}$ be such that $\|\mathbf{x}\|=1$ and $m(B)=\|B\mathbf{x}\|$, and let $\mathbf{v}'$ be such that $\|\mathbf{v}'\|=1=\mathbf{v}'\mathbf{x}$. Defining

$$H=-B\mathbf{x}\mathbf{v}',$$

we have

$$(B+H)\mathbf{x}=B\mathbf{x}-B\mathbf{x}\mathbf{v}'\mathbf{x}=0,$$

and thus $(B+H)\in\mathbb{S}$. Moreover,

$$\|H\|=\|B\mathbf{x}\|\,\|\mathbf{v}'\|=m(B),$$

so that

$$\|B^{-1}\|^{-1}\geqslant\operatorname{dist}(B;\mathbb{S}).$$

Q.E.D.

We now prove

THEOREM 1. *Let B be an arbitrary square matrix of order n, let* $\kappa\geqslant 0$, *and let* $\mathcal{C}$ *denote the set of all square matrices of order n such that* $C\in\mathcal{C}$ *if and only if*

$$\|C-B\|\leqslant\kappa,$$

relative to a fixed, but arbitrary, operator norm. Then, denoting

$$\bigcup_{C\in\mathcal{C}}\sigma(C)\quad\text{by}\quad\sigma(\mathcal{C}),$$

we have

$$\sigma(\mathcal{C}) = \{z : m(zI - B) \leqslant \kappa\}.$$

Proof. Let $C \in \mathcal{C}$ and $\lambda \in \sigma(C)$, so that $m(\lambda I - C) = 0$. Using the property [4] that

$$m(X + Y) \leqslant m(X) + \|Y\|,$$

we have

$$m(\lambda I - B) \leqslant m(\lambda I - C) + \|C - B\|$$
$$\leqslant \kappa,$$

and thus

$$\sigma(\mathcal{C}) \subseteq \{z : m(zI - B) \leqslant \kappa\}.$$

To prove the reverse inclusion, let ξ be such that $m(\xi I - B) \leqslant \kappa$. Using the above lemma, there exists a matrix $S \in \mathcal{S}$ such that $\|\xi I - B - S\| \leqslant \kappa$. Then, $(\xi I - S) \in \mathcal{C}$ and $\xi \in \sigma(\xi I - S)$. Thus

$$\{z : m(zI - B) \leqslant \kappa\} \subseteq \sigma(\mathcal{C}) \qquad \text{Q.E.D.}$$

We note that the statement

$$\sigma(\mathcal{C}) = \{z : m(zI - B) \leqslant \kappa\}$$

can be rewritten as

$$\sigma(\mathcal{D}) = \sigma(B) \cup \{z : z \not\subseteq \sigma(B) \text{ and } \|(zI - B)^{-1}\|\kappa \geqslant 1\},$$

where $D \in \mathcal{D}$ if and only if $D = B + H$ and $\|H\| \leqslant \kappa$. Thus, Theorem 1 includes a result of Bauer and Fike [1, Theorem II], viz.

$$\sigma(C) \subseteq \sigma(B) \cup \{z : z \not\subseteq \sigma(B) \text{ and } \|(zI - B)^{-1}\|^{-1} \leqslant \|C - B\|\}.$$

3. MAIN RESULT

We state our main result as the following theorem.

THEOREM 2. *Let A be partitioned as in* (1) *and let $\mathcal{B}$ denote the set of all square matrices of order n, partitioned conformally with A, such that $B \in \mathcal{B}$ if and only if*

$$\|B_{ii} - A_{ii}\| + \sum_{1 \leqslant j \leqslant N, j \neq i} \|B_{ij}\| \leqslant \sum_{1 \leqslant j \leqslant N, j \neq i} \|A_{ij}\|, \quad 1 \leqslant i \leqslant N.$$

Then

$$\sigma(\mathfrak{B}) = G.$$

Proof. Let $B \in \mathfrak{B}$ and $\lambda \in \sigma(B)$. Then, from (2), there exists i, $1 \leqslant i \leqslant N$, such that

$$m(\lambda I - B_{ii}) \leqslant \sum_{1 \leqslant j \leqslant N, j \neq i} \|B_{ij}\|.$$

Therefore,

$$m(\lambda I - A_{ii}) \leqslant m(\lambda I - B_{ii}) + \|B_{ii} - A_{ii}\|$$

$$\leqslant \sum_{1 \leqslant j \leqslant N, j \neq i} \|A_{ij}\|,$$

and thus

$$\sigma(\mathfrak{B}) \subseteq G.$$

To prove the reverse inclusion, suppose $m(\xi I - A_{ii}) \leqslant \sum_{1 \leqslant j \leqslant N, j \neq i} \|A_{ij}\|$ for some i, $1 \leqslant i \leqslant N$. By Theorem 1, there exists a matrix C_i of order n_i such that

$$\|C_i - A_{ii}\| \leqslant \sum_{1 \leqslant j \leqslant N, j \neq i} \|A_{ij}\|$$

and $\xi \in \sigma(C_i)$. Form a block-diagonal matrix B of order n, partitioned conformally with A, as follows:

$$B = \begin{bmatrix} A_{11} & & & & & & \\ & \ddots & & & & \bigcirc & \\ & & A_{i-1,i-1} & & & & \\ & & & C_i & & & \\ & & & & A_{i+1,i+1} & & \\ & \bigcirc & & & & \ddots & \\ & & & & & & A_{NN} \end{bmatrix}.$$

Then $B \in \mathfrak{B}$ and $\xi \in \sigma(B)$. Thus $G \subseteq \sigma(\mathfrak{B})$.

Q.E.D.

As a special case of Theorem 2, we obtain the following characterization of Gerschgorin's inclusion region.

COROLLARY *If $N = n$, then $B \in \mathfrak{B}$ if and only if*

$$|b_{ii} - a_{ii}| + \sum_{1 \leq j \leq n, j \neq i} |b_{ij}| \leq \sum_{1 \leq j \leq n, j \neq i} |a_{ij}|, \qquad 1 \leq i \leq n,$$

and

$$\sigma(\mathfrak{B}) = \bigcup_{i=1}^{n} \left\{ z : |z - a_{ii}| \leq \sum_{1 \leq j \leq n, j \neq i} |a_{ij}| \right\}.$$

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Received August 7, 1972