Exact analytical solutions for axial flow of a fractional second grade fluid between two coaxial cylinders

M. Imran, a) M. Kamran, and M. Athar
Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan

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Abstract The velocity field and the adequate shear stress corresponding to the longitudinal flow of a fractional second grade fluid, between two infinite coaxial circular cylinders, are determined by applying the Laplace and finite Hankel transforms. Initially the fluid is at rest, and at time \( t = 0^+ \), the inner cylinder suddenly begins to translate along the common axis with constant acceleration. The solutions that have been obtained are presented in terms of generalized \( G \) functions. Moreover, these solutions satisfy both the governing differential equations and all imposed initial and boundary conditions. The corresponding solutions for ordinary second grade and Newtonian fluids are obtained as limiting cases of the general solutions. Finally, some characteristics of the motion, as well as the influences of the material and fractional parameters on the fluid motion and a comparison between models, are underlined by graphical illustrations. © 2011 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1102202]

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Navier-Stokes equations are nonlinear partial differential equations. For this reason, there exists only a limited number of exact solutions in which the nonlinear inertial terms do not disappear automatically. Exact solutions are very important not only because they are solutions of some fundamental flows but also because they serve as accuracy checks for experimental, numerical, and asymptotic methods. The inadequacy of the classical Navier-Stokes theory to describe rheologically complex fluids such as polymer solutions, blood, paints, certain oils and greases, has led to the development of several theories of non-Newtonian fluids. Amongst the many models which have been used to describe the non-Newtonian behavior exhibited by certain fluids, the fluids of different types have received special attention. The fluids of second grade, which form a subclass of the fluids of different types, have been studied successfully in various types of flow situations. Here, we mention some of the studies. Sometimes since the equations governing the flow of second-grade fluids are one order higher than the Navier-Stokes equations, one would require boundary conditions in addition to the "nonslip" condition to have a well-posed problem.

In recent years, fractional calculus has achieved much success in the description of complex dynamics. Fractional derivative models are used quite often to describe viscoelastic behavior of polymers in the glass transition and the glassy state. The starting point is usually classical differential equation which is modified by replacing the classical, time derivatives of an integer order by the so-called Riemann-Liouville operator. This generalization allows one to define precisely non integer order integrals or derivatives. Fractional calculus has also gained much fame in the description of viscoelasticity. Tan and Xu discussed the flow of a generalized second grade fluid due to the impulsive motion of a flat plate. Relevant studies involving generalized second grade fluid in bounded domains are presented.

In this work, we consider the viscoelastic fluid to be modeled as fractional second grade fluid (FSGF) and study the flow starting from rest due to the sliding of the inner cylinder along its axis with a constant acceleration. The velocity and adequate shear stress, obtained by means of the finite Hankel and Laplace transforms, are presented under series form in terms of the generalized \( G \) functions. The similar solutions for the ordinary second grade or Newtonian fluids, performing the same motion, are respectively obtained as special cases when \( \beta \to 1 \), or \( \beta \to 1 \) and \( \alpha_1 \to 0 \).

The flows to be considered here have the form of

\[
v = v(r, t) = v(r, t) e_z ,
\]

where \( e_z \) is the unit vector in the \( z \)-direction of the cylindrical coordinates system \( r, \theta \) and \( z \). For such flows, the constraint of incompressibility is automatically satisfied.

Furthermore, if initially the fluid is at rest, then

\[
v(r, 0) = 0 .
\]

The governing equations, corresponding to such motions for second grade fluid are

\[
\tau(r, t) = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \frac{\partial v(r, t)}{\partial r} ,
\]

\[
\frac{\partial v(r, t)}{\partial t} = \left( \nu + \alpha_1 \frac{\partial}{\partial t} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) v(r, t) ,
\]

\[a)\text{Corresponding author. Email: drmimranchaudhry@gmail.com.}\]
where $\mu$ is the dynamic viscosity, $\nu = \mu/\rho$ is the kinematic viscosity, $\rho$ being the constant density of the fluid, $\alpha_1$ is a material constant and $\alpha = \alpha_1/\rho$.

The governing equations corresponding to an incompressible fractional second grade fluid, performing the same motion, are

$$\frac{\partial v(r,t)}{\partial t} = (\nu + \alpha D_t^\beta) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) v(r,t),$$

(5)

$$\tau(r,t) = (\mu + \alpha D_t^\beta) \frac{\partial v(r,t)}{\partial r},$$

(6)

where the fractional differential operator $D_t^\beta$ is defined by Ref. 26 as

$$D_t^\beta f(t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^\beta};\ 0 \leq \beta < 1,$$

(7)

and $\Gamma(\cdot)$ is the Gamma function. Of course, the new material constant $\alpha_1$, although for simplicity we keep the same notation, tends to the original $\alpha_1$ as $\beta \to 1$. In the following the system of fractional partial differential Eqs. (5) and (6), with appropriate initial and boundary conditions, will be solved by means of Laplace and finite Hankel transforms. In order to avoid lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform method will be used.

Let us consider an incompressible FSGF at rest in the annular region between two straight circular cylinders of radii $R_1$ and $R_2 (> R_1)$. At time $t = 0^+$, the inner cylinder suddenly slides to begin along the common axis with constant accelerations $A$. Owing to the shear, the fluid is gradually moved, its velocity being of the form of Eq. (1). The governing equations are Eqs. (5) and (6), while the appropriate initial and boundary conditions are

$$v(r,0) = 0, \quad r \in [R_1, R_2],$$

(8)

$$v(R_1,t) = At, \quad v(R_2,t) = 0 \quad \text{for} \quad t \geq 0,$$

(9)

where $A$ is a real constant.

Applying the Laplace transform to Eqs. (5) and (9), we get

$$q \varpi(r,q) = (\nu + \alpha q^\beta) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \varpi(r,q),$$

(10)

$$\varpi(R_1,q) = \frac{A}{q^2}, \quad \varpi(R_2,q) = 0,$$

(11)

where $r \in (R_1, R_2)$, $\varpi(r,q)$ is the Laplace transform of function $v(r,t)$. In the following, let us denote$^{27}$

$$\varpi_H(r_n,q) = \int_{R_1}^{R_2} r \varpi(r,q) B(r,r_n) dr, \quad n = 1, 2, 3, \ldots,$$

(12)

by the finite Hankel transform of the function $\varpi(r,q)$

$$B(r,r_n) = J_0(r r_n) Y_0(R_2 r_n) - J_0(R_2 r_n) Y_0(r r_n),$$

(13)

where $r_n$ are the positive roots of the transcendental equation $B(R_1, r) = 0$, and $J_p(\cdot)$ and $Y_p(\cdot)$ are Bessel functions of the first and second kind of order $p$.

The inverse Hankel transform of $\varpi_H(r_n,q)$ is given by

$$\varpi(r,q) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^2 J_2^2 (R_1 r_n) B(r,r_n)}{J_0^2 (R_1 r_n) - J_0^2 (R_2 r_n)} \varpi_H(r_n,q).$$

(14)

Multiplying now both sides of Eq. (10) by $r B(r,r_n)$, then integrating it with respect to $r$ from $R_1$ to $R_2$, and taking into account Eq. (11) along with the following relations

$$\frac{d}{dr} B(r,r_n) = - r_n [J_1 (r r_n) Y_0 (R_2 r_n) - J_0 (R_2 r_n) Y_1 (r r_n)],$$

and

$$J_0(z) Y_1(z) - J_1(z) Y_0(z) = - \frac{2}{\pi z},$$

and the result which we can easily prove

$$\int_{R_1}^{R_2} r \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \varpi(r,q) B(r,r_n) dr = - \frac{2 J_0 (R_2 r_n) \varpi(R_1,q)}{\pi J_0 (R_1 r_n)} - r_n^2 \varpi_H(r_n,q),$$

(15)

we find that

$$\varpi_H(r_n,q) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^2 J_2^2 (R_1 r_n) B(r,r_n)}{J_0^2 (R_1 r_n) - J_0^2 (R_2 r_n)} \frac{1}{q^2 (q + \nu r_n^2 + \alpha r_n^2 q^2)}.$$

(16)

It can also be written in the equivalent form of

$$\varpi_H(r_n,q) = \frac{-2 A J_0 (R_2 r_n)}{\pi r_n^2 J_0 (R_1 r_n)} \frac{1}{q^2 + 2 A J_0 (R_2 r_n)} \frac{1}{q (q + \nu r_n^2 + \alpha r_n^2 q^2)}.$$

(17)

Applying the inverse Hankel transform to Eq. (16), and using identity

$$\ln \left( \frac{R_2}{R_1} \right) = - \pi \sum_{n=1}^{\infty} \frac{J_0 (R_1 r_n) J_0 (R_2 r_n) B(r,r_n)}{J_0^2 (R_1 r_n) - J_0^2 (R_2 r_n)},$$

we get

$$\varpi(r,q) = \frac{\ln \left( \frac{R_2}{R_1} \right)}{\ln \left( R_2/R_1 \right)} \frac{A}{q^2} + \pi A \sum_{n=1}^{\infty} \left( \frac{J_0 (R_1 r_n) J_0 (R_2 r_n) B(r,r_n)}{J_0^2 (R_1 r_n) - J_0^2 (R_2 r_n)} \times \frac{1}{q (q + \nu r_n^2 + \alpha r_n^2 q^2)} \right).$$
Using identity
\[ \frac{1}{q(q + \nu r_n^2 + \alpha r_n^2 \beta)} = \sum_{k=0}^{\infty} \frac{(-\nu r_n^2)^k q^{-\beta k - \beta - 1}}{(q^{1-\beta} + \alpha r_n^2)^{k+1}}, \]

Equation (17) can be further simplified to
\[ \tau(r, q) = \ln(R_2/r) \frac{A}{\ln(R_2/R_1)} q^2 \]
\[ \pi A \sum_{n=1}^{\infty} J_0(R_1 r_n) J_0(R_2 r_n) B(r, r_n) \times \]
\[ \sum_{k=0}^{\infty} (-\nu r_n^2)^k q^{-\beta k - \beta - 1} \frac{J_0^2(R_1 r_n) - J_0^2(R_2 r_n)}{(q^{1-\beta} + \alpha r_n^2)^{k+1}}. \]...

Eq. (21), we get the shear stress of the form of
\[ \tau(r, t) = \frac{-A}{r \ln \left( \frac{R_2}{R_1} \right)} \left( \frac{r t + \frac{\alpha_1 t^{1-\beta}}{\Gamma(2-\beta)}}{r^2} \right) - \]
\[ \pi A \sum_{n=1}^{\infty} r_n J_0(R_1 r_n) J_0(R_2 r_n) B^*(r, r_n) \times \]
\[ \sum_{k=0}^{\infty} (-\nu r_n^2)^k \left[ \mu G_{1-\beta, -\beta k - \beta - 1, k+1} \left( -\alpha r_n^2, t \right) \right] + \]
\[ \alpha_1 G_{1-\beta, -\beta - 1, k+1} \left( -\alpha r_n^2, t \right) \].

By introducing Eq. (19) into Eq. (6) and using the results of
\[ D_{\alpha, b, c}^\beta g_a, b, c(d, t) = G_{a, b, c}(d, t), \]...

and
\[ D_{\alpha, b, c}^\beta g_a, b, c(d, t) = \Gamma(\gamma + 1) \Gamma(\gamma - \beta + 1), \]

we can also find shear stress as
\[ \tau(r, t) = \]
\[ \frac{-A}{r \ln \left( \frac{R_2}{R_1} \right)} \left( \frac{r t + \frac{\alpha_1 t^{1-\beta}}{\Gamma(2-\beta)}}{r^2} \right) - \]
\[ \pi A \sum_{n=1}^{\infty} r_n J_0(R_1 r_n) J_0(R_2 r_n) B^*(r, r_n) \times \]
\[ \sum_{k=0}^{\infty} (-\nu r_n^2)^k \left[ \mu G_{1-\beta, -\beta k - \beta - 1, k+1} \left( -\alpha r_n^2, t \right) \right] + \]
\[ \alpha_1 G_{1-\beta, -\beta - 1, k+1} \left( -\alpha r_n^2, t \right) \].

(1) Making $\beta \to 1$ in Eqs. (19) and (23) we find the velocity field and the shear stress
\[ v_{SGC}(r, t) = \]
\[ \ln(R_2/r) \frac{A}{\ln(R_2/R_1)} q^2 \]
\[ \pi A \sum_{n=1}^{\infty} J_0(R_1 r_n) J_0(R_2 r_n) B^*(r, r_n) \times \]
\[ \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{0, -k - 2, k+1} \left( -\alpha r_n^2, t \right), \]...

\[ \tau_{SGC}(r, t) = \]
\[ \frac{-\mu A}{r \ln \left( \frac{R_2}{R_1} \right)} \left( t + \frac{\alpha_1}{\mu} \right) - \]
\[ \pi A \sum_{n=1}^{\infty} r_n J_0(R_1 r_n) J_0(R_2 r_n) B^*(r, r_n) \times \]
\[ \sum_{k=0}^{\infty} (-\nu r_n^2)^k \left[ \mu G_{0, -k - 2, k+1} \left( -\alpha r_n^2, t \right) \right] + \]
\[ \alpha_1 G_{0, -k - 1, k+1} \left( -\alpha r_n^2, t \right) \].
corresponding to an ordinary second grade fluid, performing the same motion. These solutions can also be simplified to (see also Eqs. (A2) and (A3) from Appendix)

\[ v_{SC}(r, t) = \frac{\ln(R_2/r)}{\ln(R_2/R_1)} At + \frac{\pi A}{\nu} \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) J_0(R_2 r_n) B(r, r_n)}{r_n^2 [J_0^2(R_1 r_n) - J_0^2(R_2 r_n)]} \times \left[ 1 - \exp \left( -\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right) \right], \] (29)

and

\[ \tau_{SC}(r, t) = \frac{-\mu A}{r \ln(R_2/R_1)} \left( t + \frac{\alpha_1}{\mu} \right) - \frac{\pi \rho A}{r \ln(R_2/R_1)} \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) J_0(R_2 r_n) B^*(r, r_n)}{r_n^2 [J_0^2(R_1 r_n) - J_0^2(R_2 r_n)]} \times \left[ 1 - \frac{1}{1 + \alpha r_n^2} \exp \left( -\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right) \right]. \] (30)

(2) Now making \( \alpha_1 \rightarrow 0 \) in Eqs. (29) and (30), we find the velocity field and the shear stress

\[ v_N(r, t) = \frac{\ln(R_2/r)}{\ln(R_2/R_1)} At + \frac{\pi A}{\nu} \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) J_0(R_2 r_n) B(r, r_n)}{r_n^2 [J_0^2(R_1 r_n) - J_0^2(R_2 r_n)]} \times \left( 1 - e^{-\nu r_n^2 t} \right), \] (31)

and

\[ \tau_N(r, t) = \frac{-\mu A}{r \ln(R_2/R_1)} \left( t + \frac{\alpha_1}{\mu} \right) - \frac{\pi \rho A}{r \ln(R_2/R_1)} \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) J_0(R_2 r_n) B^*(r, r_n)}{r_n^2 [J_0^2(R_1 r_n) - J_0^2(R_2 r_n)]} \times \left( 1 - e^{-\nu r_n^2 t} \right), \] (32)

which are the same for both types of fluids.

(3) For large values of \( t \), these solutions as well as those corresponding to second grade fluids approach to large time solutions

\[ v_L(r, t) = \frac{\ln(R_2/r)}{\ln(R_2/R_1)} At + \frac{\pi A}{\nu} \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) J_0(R_2 r_n) B(r, r_n)}{r_n^2 [J_0^2(R_1 r_n) - J_0^2(R_2 r_n)]}, \] (33)

Fig. 1. The time required to reach large time solutions for Newtonian, second grade and fractional second grade fluids, for \( R_1 = 0.3, R_2 = 0.5, A = 2, \alpha = 0.003, \nu = 0.001 \) and \( \beta = 0.5 \).
In conclusion, after some time, the behavior of the second grade fluid can be well enough approximated by that of a Newtonian fluid. Our interest here is to show that such a property also holds for fractional second grade fluid whose exact solutions, as well as those of the second grade fluid (see Eqs. (27) and (28)), are written in terms of generalized $G_{a,b,c}(\cdot,t)$ functions. More exactly, we must show that large time solutions corresponding to fractional second grade fluid is also given by Eqs. (33) and (34). Indeed from Fig. 1 we also have the time required to reach the large-time solutions for these fluids. This time (20s) is the lowest for Newtonian fluid and the highest (30s) for fractional second grade fluid.

In this paper, we find the velocity field and the adequate tangential shear stress, corresponding to the flow of a fractional second grade fluid between two infinite circular cylinders in which the inner cylinder slides along the axis, are determined. At time $t = 0^+$, the inner cylinder begins to move along the common axis with constant acceleration $A$. The solutions, determined by employing the Laplace and finite Hankel transforms, and presented in terms of Bessel functions $J_0(\cdot)$, $Y_0(\cdot)$, $J_1(\cdot)$ and $Y_1(\cdot)$ and generalized $G$ functions, satisfy the corresponding governing equations as well as all imposed initial and boundary conditions. In the special cases, when $\beta \to 1$ or $\beta \to 1$ and $\alpha_1 \to 0$, the corresponding solutions for a second grade fluid or for the Newtonian fluid, performing the same motion, are respectively obtained from the general solutions.

Now, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity $v(r,t)$ are depicted against $r$ for different values of time $t$, and for different material and fractional parameters. Figure 2 clearly shows that the velocity is an increasing function of $t$. The influence of the kinematic viscosity $\nu$ on the fluid motion is shown in Fig. 3. The velocity $v(r,t)$ is a decreasing function of $\nu$.

The influences of material parameter $\alpha$ on the velocity is shown in Fig. 4. As expected, the velocity is a decreasing function with respect to $\alpha$. Figure 5 shows the influence of the fractional parameter $\beta$ on the fluid motion. Its effect on the fluid motion is qualitatively the same as that of $\alpha$. More exactly, the velocity $v(r,t)$ is a decreasing function with regards to $\beta$.

Finally, for comparison, the profiles of $v(r,t)$ corresponding to the motion of the Newtonian, ordinary second grade and fractional second grade are depicted together in Fig. 6 for the same $t$ and the same common
material and fractional parameters. The fractional second grade fluid, as resulting from these figures, is the slowest and the Newtonian fluid is the swiftest on the whole flow domain. In practice, it is necessary to know the approximate time after which the fluid is moving according to the large time solutions. This time, as resulting from Fig. 1, is the smallest for the Newtonian fluid and the highest for fractional second grade fluid. The units of the material constants are SI units in all fluids, for $R_1 = 0.3$, $R_2 = 0.5$, $t = 3$ $s$, $A = 2$, $\alpha = 0.2$, $\nu = 0.003$ and $\beta = 0.2$

Appendix

\[ L^{-1}\left\{ \frac{t^b}{(q^a-d)^c} \right\} = G_{a,b,c}(d,t) \]
\[ = \sum_{j=0}^{\infty} \frac{d^j \Gamma(c+j) t^{(c+j)(a-b)-1}}{\Gamma(c)\Gamma(j+1)\Gamma[(c+j)a-b]} \]

where $Re(ac-b) > 0$, $|d/q^a| < 1$,

\[ \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{0,-k-1,k+1}(-\alpha r_n^2,t) \]
\[ = \frac{1}{1+\alpha^2 r_n^2} \exp \left( -\frac{\nu r_n^2 t}{1+\alpha^2 r_n^2} \right), \]
\[ \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{0,-k-2,k+1}(-\alpha r_n^2,t) \]
\[ = \frac{1}{\nu r_n^2} \left[ 1 - \exp \left( -\frac{\nu r_n^2 t}{1+\alpha^2 r_n^2} \right) \right]. \]