A generalization of the classical association schemes to a higher dimension is developed which is called an association scheme on triples, in short, an AST, and a corresponding ternary, non-associative algebra is developed generalizing the Bose-Mesner algebra. A large number of examples of ASTs are constructed using block designs, permutation groups, and two-graphs. Several identities in the ternary algebra are obtained. It is shown that, given an AST, one can obtain a 2-design and also a family of (classical) association schemes. The concept of a 2-design which is partially balanced with respect to 3-subsets, is introduced.

1. Introduction

The concepts of an association scheme and the related Bose–Mesner algebra are well known in combinatorics [1] with applications in coding theory [7; 11, Chap. 21]. Here we develop a generalization of an association scheme to a higher dimensional combinatorial structure which we call an association scheme on triples, or in short, an AST. One motivation to study ASTs is that we can formulate the concept of a block design which is partially balanced for 3-subsets; this will be considered at the end of Section 3. We also develop a related ternary, non-associative algebra as a generalization of the Bose–Mesner algebra.
DEFINITION. An association scheme on triples (AST) on a set $\Omega$ is a partition of $\Omega \times \Omega \times \Omega$ into relations $R_0, \ldots, R_m$, where $m \geq 4$, subject to certain regularity conditions which are stated (informally) as follows:

(i) Given any two distinct elements $x, y \in \Omega$, for each $i \in \{0, \ldots, m\}$, the pair $(x, y)$ is in a uniform number of triples $(x, y, z) \in R_i$.

(ii) (Principal regularity condition). For any (not necessarily distinct) $i, j, k, l \in \{0, \ldots, m\}$ and given a triple $(x, y, z) \in R_i$, there is a uniform number of elements $w \in \Omega$ such that $(w, y, z) \in R_j$, $(x, w, z) \in R_k$, and $(x, y, w) \in R_l$. This constant which depends only on $i, j, k, l$ and not on $(x, y, z)$, is denoted by $p_{ijk}$.

(iii) For any relation $R_i$, $i \in \{0, \ldots, m\}$, and any permutation $\sigma$ of $\{1, 2, 3\}$, the set

$$\{(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) : (x_1, x_2, x_3) \in R_i\}$$

equals the relation $R_j$ for some $j \in \{0, \ldots, m\}$. (If $i = j$, then we say that $R_i$ is symmetric with respect to the permutation $\sigma$.)

(iv) The relations $R_i$, $0 \leq i \leq 3$ (called the trivial relations) are chosen such that $R_0$, the identity relation, consists of all triples $(x, x, x)$, $x \in \Omega$ and $R_1, R_2, R_3$ consist of all triples of the form $(x, y, y)$, $(x, x, y)$, $(x, x, y)$, respectively, $x, y \in \Omega$, $x \neq y$.

In what follows, a three-dimensional array of real numbers will be called a three-dimensional matrix, or briefly, a matrix. The addition and scalar multiplication of these matrices are defined componentwise in a natural way. Now, if $|\Omega| = v$, then each relation $R_i$ has a $v \times v \times v$ adjacency matrix $A_i$ indexed by $\Omega$. Thus if the $(x, y, z)$ entry of $A_i$ is denoted by $(A_i)_{xyz}$ then one has

$$(A_i)_{xyz} = \begin{cases} 1 & \text{if } (x, y, z) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

Since the trivial relations must occur in every AST, it is sometimes convenient to use a special notation for their adjacency matrices. We denote $I_0, I_1, I_2, I_3$ to be the adjacency matrices of $R_0$, $R_1, R_2, R_3$, respectively. Also, let $K$ denote the adjacency matrix of the relation consisting of all triples $(x, y, z)$ in which $x, y, z$ are all distinct elements of $\Omega$.

EXAMPLE 1.1. The matrices $I_0, I_1, I_2, I_3$, and $K$ are the adjacency matrices of an AST with precisely five relations on a set $\Omega$ and this AST is the coarsest possible AST on $\Omega$ in the sense that any other AST defined on the same set has its relations such that all non-trivial relations are
refinements of the relation corresponding to the adjacency matrix $K$. In Table I we list the adjacency matrices $I_0, \ldots, I_3, K$ of this coarsest association scheme for $v = 5$. In Table I the first five columns represent the adjacency matrix $I_0$, where the five vertical planes of the three-dimensional matrix correspond to sets of five rows, each set is followed by a blank line, the first set represents the first plane, etc. Further, the sets of remaining five columns, each punctuated by a blank column, represent in a similar manner the matrices $I_1, \ldots, I_3, K$, respectively. (This resembles a standard notation in APL, the programming language used in some of our computations.)

**Example 1.2.** Let $Ω$ denote the set of points of the 7-point projective plane. We define a partition of $Ω \times Ω \times Ω$ into six relations as follows. Let $R_0, \ldots, R_3$ denote the trivial relations according to the notation set up in the definition of an AST. Now, define

$$R_4 = \{(x, y, z) : x, y, z \text{ are distinct collinear points}\}$$

$$R_5 = \{(x, y, z) : x, y, z \text{ are distinct non-collinear points}\}.$$
Then, we claim that the relations \( R_i, 0 \leq i \leq 5 \) form an AST. To prove this, let \( a, b \) be any two distinct points of \( \Omega \) and let \( c \) be the remaining point on the line \( ab \). To verify condition (i) of the definition of an AST, take \((x, y) = (a, b)\) and observe that

\[
(a, b, a) \in R_2, \quad (a, b, b) \in R_1, \quad (a, b, c) \in R_4,
\]

and \((a, b, z) \in R_5\), otherwise. Thus, for \( 0 \leq i \leq 5 \), the uniform number in condition (i) for an AST is respectively \(0, 1, 1, 0, 1, 4\). To verify the principal regularity condition (ii), for \( l = 1 \) and given \( i, j, k \), take \((x, y, z) = (a, b, b) \in R_1\) and consider the set

\[
S = \{ w : (w, b, b) \in R_i, (a, w, b) \in R_j, (a, b, w) \in R_k \}.
\]

We see that \( S = \{a\} \) for \( i = 1, j = 3, k = 2 \), verifying \( p_{132}^1 = |\{a\}| = 1 \). Similarly,

\[
p_{011}^1 = |\{b\}| = 1, \quad p_{144}^1 = |\{c\}| = 1, \quad p_{155}^1 = |\Omega \setminus \{a, b, c\}| = 4.
\]

For all other \( i, j, k \), one has that \( p_{ijk}^1 = 0 \). Other choices of \((x, y, z)\) are used to complete the verification of (ii) and determine the remaining \( p_{ijk}^1 \). The remaining results are summarized below:

\[
p_{000}^0 = 1, \\
p_{202}^2 = p_{321}^3 = p_{424}^4 = 1, \\
p_{330}^3 = p_{213}^3 = p_{443}^4 = 1, \\
p_{213}^4 = p_{341}^4 = p_{432}^4 = 1, \\
p_{215}^5 = p_{351}^5 = p_{455}^5 = p_{532}^5 = p_{545}^5 = p_{554}^5 = p_{555}^5 = 1, \\
p_{155}^1 = p_{525}^2 = p_{553}^3 = p_{555}^4 = 4, \quad p_{123}^0 = 6, \\
p_{ijk}^1 = 0 \quad \text{otherwise.}
\]

We remark that all of these values of \( p_{ijk}^1 \) are special cases of some theorems to be proved later in Section 3. However, we have computed them here directly. The adjacency matrices \( A_4, A_5 \) of this AST are given in Table II, using the notation explained in Example 1.1. The reader can easily visualize the adjacency matrices of the trivial relations from the previous example and so they are omitted.

We now give below a preview of many more examples of ASTs constructed by using results obtained later on.
### TABLE II

<table>
<thead>
<tr>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000000</td>
<td>0000000</td>
</tr>
<tr>
<td>0001000</td>
<td>0011011</td>
</tr>
<tr>
<td>0000001</td>
<td>0101110</td>
</tr>
<tr>
<td>0100000</td>
<td>0010111</td>
</tr>
<tr>
<td>0000010</td>
<td>0111001</td>
</tr>
<tr>
<td>0000100</td>
<td>0111001</td>
</tr>
<tr>
<td>0010000</td>
<td>0101110</td>
</tr>
<tr>
<td>0001000</td>
<td>0101110</td>
</tr>
<tr>
<td>0000000</td>
<td>0000000</td>
</tr>
<tr>
<td>0000100</td>
<td>1001011</td>
</tr>
<tr>
<td>0100000</td>
<td>1001011</td>
</tr>
<tr>
<td>0001000</td>
<td>1100101</td>
</tr>
<tr>
<td>0010000</td>
<td>1001011</td>
</tr>
<tr>
<td>1000000</td>
<td>0101110</td>
</tr>
<tr>
<td>0100000</td>
<td>0010111</td>
</tr>
<tr>
<td>1000000</td>
<td>1100101</td>
</tr>
<tr>
<td>0000010</td>
<td>1111001</td>
</tr>
<tr>
<td>0000100</td>
<td>1111001</td>
</tr>
<tr>
<td>0001000</td>
<td>1100101</td>
</tr>
<tr>
<td>0010000</td>
<td>1111001</td>
</tr>
<tr>
<td>0000100</td>
<td>0111001</td>
</tr>
<tr>
<td>0010000</td>
<td>1001011</td>
</tr>
<tr>
<td>0100000</td>
<td>1001011</td>
</tr>
<tr>
<td>0000001</td>
<td>1111010</td>
</tr>
<tr>
<td>0000000</td>
<td>0000000</td>
</tr>
<tr>
<td>1000000</td>
<td>0111001</td>
</tr>
<tr>
<td>0001000</td>
<td>1111001</td>
</tr>
<tr>
<td>0010000</td>
<td>1100101</td>
</tr>
<tr>
<td>0100000</td>
<td>1011000</td>
</tr>
<tr>
<td>0000100</td>
<td>0111001</td>
</tr>
<tr>
<td>0000001</td>
<td>1011000</td>
</tr>
<tr>
<td>0001000</td>
<td>1100101</td>
</tr>
<tr>
<td>0100000</td>
<td>1111001</td>
</tr>
<tr>
<td>0001000</td>
<td>1011110</td>
</tr>
<tr>
<td>0000010</td>
<td>1011110</td>
</tr>
<tr>
<td>1000000</td>
<td>0101110</td>
</tr>
<tr>
<td>0000100</td>
<td>1111001</td>
</tr>
<tr>
<td>0000000</td>
<td>0000000</td>
</tr>
<tr>
<td>0100000</td>
<td>1011110</td>
</tr>
<tr>
<td>0000000</td>
<td>0000000</td>
</tr>
</tbody>
</table>
Example 1.3. (i) The AST described in Example 1.2 is a particular case of a more general construction given in Theorem 3.1 where one constructs an AST from any 2-design with \( \lambda = 1 \). There are many familiar examples of 2-designs with \( \lambda = 1 \) which include projective planes, Steiner systems etc. Thus we get a large number of examples of ASTs which are constructed in this manner.

(ii) We shall prove in Theorem 4.1 that any 2-transitive permutation group gives rise to an AST in a natural way. Thus a large number of ASTs can be constructed using this method. In Section 4 we shall also examine some particular classes of ASTs constructed from the groups PSL(2, \( q \)), the affine group AGL(1, \( q \)), the Suzuki groups, and the Higman–Sims group.

Now, we develop an analog of the Bose–Mesner algebra corresponding to an AST.

**Definition.** For any three matrices \( A, B, C \) of size \( v \times v \times v \) indexed by the set \( \Omega \), the ternary product \( D = ABC \) is the \( v \times v \times v \) matrix given by

\[
(D)_{xyz} = \sum_{w \in \Omega} (A)_{wyz} (B)_{wyv} (C)_{vwy},
\]

where \( (A)_{wyz} \) denotes the \((w, y, z)\) entry of \( A \) etc.

It is straightforward to verify that this ternary product is multi-linear. Thus the set of all \( v \times v \times v \) matrices indexed by \( \Omega \) forms clearly an algebra under the ternary product (1). However, this ternary algebra is not associative. For example, for some fixed \( \alpha, \beta, \gamma \in \Omega \), let \( E^{\alpha\beta\gamma} \) be the \( v \times v \times v \) matrix whose \((x, y, z)\) entry is 1 if \( x = \alpha, y = \beta, \) and \( z = \gamma \) and 0 elsewhere. Then one can verify that if \( \Omega = \{1, 2\} \),

\[
E^{122}E^{112}E^{112}E^{121} = 0,
\]

where 0 denotes the \( v \times v \times v \) matrix whose all the entries are zero; on the other hand, one has that

\[
(E^{122}E^{112}E^{121})E^{112}E^{121} = E^{122}.
\]

Thus, associativity of the ternary product does not hold even for \( 2 \times 2 \times 2 \) matrices. More generally, for an arbitrary set \( \Omega \) one can verify that the multiplication rule for the \( E^{\alpha\beta\gamma} \) matrices is as follows. For any \( \alpha, \beta, \gamma, \delta \) (distinct or not) in \( \Omega \), one has that

\[
E^{\delta\beta\gamma}E^{\alpha\beta\gamma}E^{\alpha\beta\delta} = E^{\alpha\beta\gamma}
\]

and ternary products \( E^{\cdots}E^{\cdots}E^{\cdots} \) not of this form are equal to 0. Thus, for example, \( E^{122}E^{112}E^{121} = E^{122} \) and \( E^{122}E^{121}E^{112} = 0 \). So, the ternary product is, in general, not commutative.
For any AST it is obvious that the adjacency matrices $A_0, ..., A_m$ span a vector space of dimension $m + 1$. We now show that these form an algebra under the ternary product defined in (1).

**Theorem 1.4.** Consider an AST defined on a set $\Omega$ with the relations $R_0, ..., R_m$ and the corresponding adjacency matrices $A_0, ..., A_m$, respectively. Then for $i, j, k \in \{0, 1, ..., m\}$ one has that

$$A_i A_j A_k = \sum_{l=0}^{m} p_{ijk}^l A_l.$$  

**Proof.** Using (1) the $(x, y, z)$ entry of $A_i A_j A_k$ is given by

$$\sum_w (A_i)_{wyz} (A_j)_{xzw} (A_k)_{xyz}.$$  

In (3) a term is nonzero if there is some $w \in \Omega$ such that $(w, y, z) \in R_i$, $(x, w, z) \in R_j$, and $(x, y, w) \in R_k$. Thus if $(x, y, z) \in R_l$ then by using the principal regularity condition one obtains that the number of nonzero terms in (3) is $p_{ijk}^l$. So, by varying $l$ over all the elements of the set $\{0, ..., m\}$ one gets that (3) is equal to the right-hand side of (2). Q.E.D.

The converse of Theorem 1.4 also holds which we state below; the proof is omitted as it is straightforward.

**Theorem 1.5.** Let $\{A_i : 0 \leq i \leq m\}$ be a family of $v \times v \times v$ matrices indexed by a set $\Omega$ with cardinality $v$ whose entries consist of 0's and 1's only, such that $\sum_{i=0}^{m} A_i = J$, where $J$ is the $v \times v \times v$ matrix with all the entries equal to 1 and for each $i, j, k \in \{0, 1, ..., m\}$,

$$A_i A_j A_k = \sum_{l=0}^{m} p_{ijk}^l A_l,$$

for some integers $p_{ijk}^l$. Also, assume that $A_i$, $0 \leq i \leq 3$, are the adjacency matrices of the trivial relations $R_i$, $0 \leq i \leq 3$, defined earlier. Then, one can construct an AST on the set $\Omega$ with $m + 1$ relations whose adjacency matrices are precisely the matrices $A_i$, $0 \leq i$.

**Remark 1.6.** (i) Given a set $\Omega$ it is possible to extend the definition of an AST to an association scheme on $n$-tuples consisting of a partition of the set $\Omega \times \cdots \times \Omega$ for any $n \geq 3$. Also, one can extend the definition of the ternary product defined in (1) to an $n$-ary product of $n$-dimensional arrays and obtain an $n$-ary algebra for the adjacency matrices for the matrices of this association scheme on $n$-tuples.
(ii) It is interesting to weaken somewhat the definition of an AST and consider those combinatorial structures in which the condition (i) of the definition of an AST does not necessarily hold. We shall investigate the properties of such structures in a later paper.

The organization of the rest of the paper is as follows. Section 2 contains a miscellany of properties (some quite elementary) of an AST and the related ternary algebra. In Sections 3 and 4 we give constructions of ASTs from 2-designs and 2-transitive groups, respectively, and also investigate the related ternary algebra. In Section 5 the relationship between ASTs and 2-graphs is investigated. Appendix I resolves a question raised in Section 3.

2. SOME PROPERTIES OF AN AST

Let $R_0, R_1, \ldots, R_m$ be the relations of an AST on a set $\Omega$, $|\Omega| = v$. Choose two distinct points $x, y \in \Omega$. For $1 \leq i \leq m$ define
\[ n_i^{(1)} = |\{w \in \Omega : w \neq \{x, y\}, (w, x, y) \in R_i\}|. \]

Similarly, one defines $n_i^{(2)}$ and $n_i^{(3)}$ by setting $(x, w, y) \in R_i$ and $(x, y, w) \in R_i$, respectively. From the definition of an AST it is clear that $n_i^{(1)}, n_i^{(2)},$ and $n_i^{(3)}$ do not depend on the choice of $x$ and $y$. We omit the proof of the next result which follows by a straightforward counting argument.

**Lemma 2.1.** Consider an AST with $m + 1$ relations defined on a set $\Omega$ with $v$ elements.

(i) For any relation $R_i$ of the AST one has that
\[ n_i^{(1)} = \sum_{k=0}^{m} p_{i2k}^2, \quad n_i^{(2)} = \sum_{k=0}^{m} p_{i1k}^1 \]

(ii) For any symmetric relation $R_i$ of the AST, one has that
\[ n_i^{(1)} = n_i^{(2)} = n_i^{(3)}. \]

Let $n_i$ denote the common value of the constant in (4). Then
\[ |R_i| = v(v - 1)n_i. \]

Recall that a 2-design $(X, \mathcal{B})$ with the parameters $b, v, k, \lambda$ is a family $\mathcal{B}$ of $k$-subsets of a set $X$ which are called blocks such that any 2-subset of $X$ lies in exactly $\lambda$ blocks, $|X| = v$, $|\mathcal{B}| = b$. Our next result shows that any
non-trivial symmetric AST with \( m + 1 \) relations generates in a natural way \( m - 3 \) families of 2-designs.

**Proposition 2.2.** Let \( R_i \) be a non-trivial symmetric relation of an AST defined on a set \( \Omega \). Let \( \mathcal{D} \) be the family of all 3-subsets \( \{x, y, z\} \) of \( \Omega \) with the property that \( (x, y, z) \in R_i \). Then \( \mathcal{D} \) is a 2-design on the set \( \Omega \).

**Proof.** Let \( x, y \in \Omega, x \neq y \). By the axiom (i) of the definition of an AST, the pair \( (x, y) \) occurs in a uniform number of triples \( (x, y, z) \) in \( R_i \). Since the given relation is symmetric, it follows from Lemma 2.1 that \( \{x, y\} \) lies in exactly \( n_i \) elements of the family \( \mathcal{D} \). Consequently, the elements of \( \mathcal{D} \) are the blocks of a 2-design on \( \Omega \). For this design it is clear that \( \lambda = n_i \) and \( k = 3 \). Q.E.D.

We remark that Proposition 2.2 can be easily extended to the case where, instead of taking only one relation \( R_i \), one considers a union of a class of symmetric relations of the AST.

An AST is called symmetric if every non-trivial relation \( R_i \) of the AST is symmetric. For a symmetric AST it follows that

\[
p_{ijk}^j = p_{i,j,k}^j
\]

for any permutation \( \lambda \mu \nu \) of \( ijk \).

**Proposition 2.3.** Consider a symmetric AST defined on a set \( \Omega \) with \( v \) elements with the relations \( R_i, 0 \leq i \leq m \), where \( R_0, ..., R_3 \) are the trivial relations. Then for \( i, j, k, l > 3 \), one has the identities

\[
n_i p_{jkl}^i = n_i p_{jlk}^i = n_j p_{ikl}^j = n_k p_{ijl}^k
\]

\[
n_i - 1 = \sum_{j=0}^{m} \sum_{k=0}^{m} p_{ijk}^i
\]

If \( i \neq l \), then

\[
n_i = \sum_{j=0}^{m} \sum_{k=0}^{m} p_{ijk}^i
\]

\[
\sum_{i=0}^{m} n_i = v - 2
\]

\[
\sum_{i, j, k=0}^{m} p_{ijk}^i = v - 2
\]

\[
\sum_{i, j, k=0}^{m} p_{ijk}^i = v - 2 - (m + 1).
\]
**Proof.** For each choice \((x, y, z) \in R_i\), there are \(p^i_{ijk}\) number of tetrahedra with \((x, y, z)\) as base such that if \(w\) is the vertex, then \((w, y, z) \in R_i, (x, w, z) \in R_j,\) and \((x, y, w) \in R_k\). Again, any such tetrahedron can also be regarded as a tetrahedron with base \((x, y, w) \in R_k\) such that \((x, y, z) \in R_i, (w, y, z) \in R_i\). So, by counting the tetrahedra in two ways, one obtains that

\[ n_i p^i_{ijk} = n_k p^k_{ijl}. \]

This proves one of the identities in (6) and the other identities in (6) are proved in a similar manner. The remaining identities can be obtained by similar counting arguments and we omit the details. Q.E.D.

**Corollary 2.4.** With the same hypothesis as in Proposition 2.3, one has that

\[ p^i_{jij} = p^i_{lij} = p^i_{jji}, \]

for all \(i, j > 3\).

**Remark 2.5.** Results 2.3 and 2.4 deal with a symmetric AST but it is possible to extend these results to the non-symmetric case although the identities take more complicated forms. We omit the details.

The following result gives a combinatorial interpretation of some special values of \(p^i_{ijk}\); we omit the proof which is straightforward.

**Proposition 2.6.** Let \(R_i\) be a non-trivial relation in any AST, so \(i \geq 4\). Then

\[ p^1_{iij} (\text{resp. } p^2_{iij}, p^3_{iij}) = \begin{cases} n_i & \text{if } R_i \text{ is symmetric with respect to (23) (resp. (13), (12))} \\ 0 & \text{otherwise.} \end{cases} \]

**Proposition 2.7.** Out of the 125 ternary products involving the five matrices \(K\) and \(I_i, 0 \leq i \leq 3\), only the following 15 products are nonzero:

\[
\begin{align*}
I_0 I_0 I_0 &= I_0, & I_0 I_1 I_1 &= I_1, & I_2 I_0 I_2 &= I_2, \\
I_3 I_3 I_0 &= I_3, & I_1 I_3 I_2 &= I_1, & I_3 I_2 I_1 &= I_2, \\
I_2 I_1 I_3 &= I_3, & I_1 I_2 I_3 &= (v - 1)I_0, & KI_3 I_2 &= K, \\
I_3 K I_1 &= K, & I_2 I_1 K &= K, & I_1 K K &= (v - 2)I_1, \\
K I_2 K &= (v - 2)I_2, & K K I_3 &= (v - 2)I_3, & K K K &= (v - 3)K.
\end{align*}
\]
The proof of 2.7 is straightforward and we give only an excerpt. Let $C$ be an arbitrary $v \times v \times v$ matrix and let $D = I_1I_2C$. Then

$$(D)_{xyz} = \sum (I_1)_{wyz} (I_2)_{xzw} (C)_{xyz},$$

which reduces to zero unless $v = z \neq w$ and $x = z \neq w$. That is,

$$(D)_{xxx} = \sum_{w \neq x} (C)_{xxw}$$

and $(D)_{xyz} = 0$ otherwise. In particular, $I_1 I_2 I_3 = (v - 1)I_0$ and $I_1 I_2 C = 0$ for $C = I_0, I_1, I_2$, and $K$. Similarly, if $E = I_1 I_1 C$, $(E)_{xyz}$ reduces to zero unless $y = z \neq w$ and $w = z \neq x$. These conditions are inconsistent, so $I_1 I_1 C = 0$.

**Corollary 2.8.** In an AST with the relations $R_0, R_1, \ldots, R_m$ if $R_0, \ldots, R_3$ are the trivial relations, then the adjacency matrices $A_4, \ldots, A_m$ generate a subalgebra of the ternary generated by $A_0, \ldots, A_m$.

**Proof.** Assume that $i, j, k \in \{4, \ldots, m\}$. We need to show that $A_iA_jA_k$ is in the vector space spanned by $A_4, \ldots, A_m$, that is, in the expansion (2), $p'_{ijk} = 0$ for $l = 0, \ldots, 3$. Substitute $K = A_4 + \cdots + A_m$ in the identity $KKK = (v - 3)K$. Applying the distributive law one obtains

$$\sum' A_iA_jA_k = \sum_{l=4}^m (v-3) A_l,$$

where $\sum'$ denotes $\sum_{i=4}^m \sum_{j=4}^m \sum_{k=4}^m$. Expanding $A_iA_jA_k$ using (2) and collecting coefficients of $A_l$, one finds

$$\sum' p'_{ijk} = 0, \quad l = 0, \ldots, 3.$$

Since each $p'_{ijk}$ is non-negative, this implies that $p'_{ijk} = 0$, $l = 0, \ldots, 3$. Q.E.D.

**Corollary 2.9.** For any element $C$ in the subalgebra described in Corollary 2.8, although $I_1, I_2, \ldots, I_3$ do not lie in this subalgebra, one has the identities

$$C I_3 I_2 = I_3 C I_1 = I_2 I_1 C = C.$$

(We call such pairs $(I_2, I_1)$ identity pairs and these are considered by us in detail in [12]).

**Remark 2.10.** Theorem 2.7 and expansion (2) show that for the coarsest AST on a set (in the sense of Example 1.1), only 15 of the 625 parameters $p'_{ijk}$ are non-zero, including 10 whose value is 1. Using the substitution
$K = A_4 + \cdots + A_m$, these facts can be related to a general AST, where they imply zero values or restricted values for many $p_{ijk}^l$ in which

$$\{i, j, k, l\} \cap \{0, 1, 2, 3\} \neq \emptyset.$$  

For instance, the proof of Corollary 2.8 mentions $4(m-3)^3$ cases in which $p_{ijk}^l = 0$, and if $0 \leq l \leq m$, $4 \leq j \leq m$, Corollary 2.9 applied to $C = A_j$ implies

$$p_{j32}^l = p_{3jl}^l = p_{21l}^l = \begin{cases} 1 & \text{if } j = l \\ 0 & \text{if } j \neq l \end{cases}$$

Other values of $p_{ijk}^l$ were evaluated in Proposition 2.6. Partly for these reasons the most interesting of the $(m + 1)^4$ parameters $p_{ijk}^l$ of a general AST are the $(m - 3)^4$ in which $\{i, j, k, l\} \subseteq \{4, \ldots, m\}$ and, since these are the parameters of the subalgebra described in Corollary 2.8, we are able in some of our investigation of ASTs to confine our calculations to the subalgebra. This is advantageous in computation because of the reduction in the dimension of the algebra and the number of $p_{ijk}^l$. The reduction is striking when $m = 5$ which is the case with many of the ASTs constructed in this paper.

Using expansions of the form given by (2), one obtains the following result whose proof we omit:

**Proposition 2.11.** The ternary algebra generated by an AST with $m + 1$ relations is associative if and only if

$$\sum_{l=0}^{m} p_{ijk}^l p_{irs}^l = \sum_{r=0}^{m} p_{jkr}^l p_{irs}^l = \sum_{\mu=0}^{m} p_{iju}^\mu p_{kr\mu}^\mu$$

for all $i, j, k, r, s, \lambda \in \{0, 1, \ldots, m\}$.

**Corollary 2.12.** In the ternary algebra generated by a symmetric AST with $m + 1$ relations, the following weak associative law holds

$$(A_i A_j A_i) A_i = A_i (A_j A_i A_i) A_i = A_i A_i (A_j A_i A_i)$$

for all $i, j \in \{0, 1, \ldots, m\}$. Consequently, $(A_i)^5$ is a well-defined ternary product for any symmetric AST.

To end this section, we now show how to generate families of association schemes (in the classical sense) from an arbitrary AST.

**Theorem 2.13.** Consider an AST on a set $\Omega$ with the relations $R_i$, $0 \leq i \leq m$, where $R_0, \ldots, R_3$ are the trivial relations. Then using this AST, one
can construct $|\Omega|$ association schemes (in the classical sense) on the sets $\Omega' = \Omega \setminus \{\alpha\}$, $\alpha \in \Omega$, by classifying in a natural way the sets $\Omega' \times \Omega'$.

Proof. Let $\alpha$ be a fixed point in $\Omega$. For $4 \leq i \leq m$ let

$$\Gamma_i = \{(b, c) : (x, b, c) \in R_i, b, c \in \Omega'\}.$$

Also, let

$$\Gamma_0 = \{(b, b) : b \in \Omega'\}.$$

Then, it can be shown that the classes $\Gamma_0^1, \Gamma_1^1, \Gamma_2^1, \ldots, \Gamma_m^1$ form the relations of an association scheme on the set $\Omega'$. Now, in a similar manner, each of the families

$$\Gamma_0^2 = \{(b, c) : (b, \alpha, c) \in R_i, b, c \in \Omega'\},$$

where $4 \leq i \leq m$ together with the trivial relation, and

$$\Gamma_0^3 = \{(b, c) : (b, c, \alpha) \in R_i, b, c \in \Omega'\},$$

where $4 \leq i \leq m$ together with the trivial relation, form association schemes in the classical sense. However, one notices that both these association schemes are identical with the association scheme constructed earlier except for a possible change of notation. Q.E.D.

3. ASTs from 2-designs

We have shown in Proposition 2.2 that any non-trivial, symmetric relation of an AST generates a family of 2-designs. In this section we construct an AST from any 2-design with $\lambda = 1$ and also obtain several identities in the corresponding ternary algebra. At the end of the section we introduce the notion of partial 3-designs.

Theorem 3.1. Let $(X, \mathcal{B})$ be a 2-design with $\lambda = 1$ and $k \geq 4$. Then one can construct an AST with the relations $R_i$, $0 \leq i \leq 5$, where $R_0, R_1, R_2, R_3$ are the trivial relations described in Section 1 and further

$$R_4 = \{(x, y, z) : x, y, z \text{ are distinct, } \{x, y, z\} \text{ lies in some block of } \mathcal{B}\}$$

and

$$R_5 = \{(x, y, z) : x, y, z \text{ are distinct, } \{x, y, z\} \text{ does not lie in any block of } \mathcal{B}\}.$$
ASSOCIATION SCHEMES ON TRIPLES

Proof. Clearly, any 3-subset of $X$ lies in at most one block. We say that $R_4$ consists of the collinear triples and $R_5$ consists of non-collinear triples. Now, any two distinct points $x, y$ in $X$ are in $(k - 2)$ collinear 3-subsets and so in $6(k - 2)$ collinear triples of $R_4$. Again, any two distinct points $x, y$ are in $6(v - k)$ triples of $R_5$. Now it remains to verify the principal regularity condition. For $i, j, k \in \{0, 1, 2, 3, 4, 5\}$ we compute the value of $p_{ijk}^i$. Recall that for $(x, y, z) \in R_i$ we have

$$p_{ijk}^i = |\{w \in X : (w, y, z) \in R_j, (x, w, z) \in R_k, (x, y, w) \in R_i\}|.$$

The only two non-trivial cases to consider are when $l = 4, l = 5$. The other cases are trivial, for example $p_{111}^4 = 0$, $p_{111}^5 = 0$, etc.

Case 1. $l = 4$. Here $(x, y, z) \in R_4$, that is, $\{x, y, z\}$ lies in some unique block $B$, say. Let $B = \{x, y, z, u_1, ..., u_{k-3}\}$, $|X \setminus B| = v - k$. Now, if $w \in B$ then $\{x, y, z, w\}$ is a 4-set such that $\{x, y, w\}$, $\{y, z, w\}$, and $\{x, z, w\}$ are collinear 3-sets and so $(w, y, z)$, $(x, w, z)$, and $(x, y, w)$ all lie in $R_4$. Thus

$$p_{444}^4 \geq k - 3.$$ 

If $u \notin B$ then because $\{x, y\}$, $\{y, z\}$, and $\{z, x\}$ belong to the unique block $B$ only, it is not possible for $\{x, y, w\}$, $\{y, z, w\}$, and $\{z, x, w\}$ to be collinear for all the triples $(w, y, z)$, $(x, y, z)$, and $(x, y, w)$ in $R_4$. Thus

$$p_{555}^4 \geq v - k.$$ 

It is clear that no other possibilities for the element $w$ in the definition of $p_{ijk}^i$ can arise. Therefore one has that

$$p_{444}^4 = k - 3, \quad p_{555}^4 = v - k,$$

and one finds that $p_{214}^4 = p_{341}^4 = p_{432}^4 = 1$. We now claim that otherwise

$$p_{ijk}^4 = 0 \quad \text{if} \quad i, j, k \in \{0, 1, 2, 3, 4, 5\}.$$ 

For example, consider $p_{445}^4$. Here, we find the number of points $w$ such that $(x, y, z) \in R_4$, $(w, y, z) \in R_4$, $(x, w, z) \in R_4$, and $(x, y, w) \in R_5$. Now $(x, y, z) \in R_4$ implies that $\{x, y, z\}$ is contained in some block $B$ of $\mathcal{B}$. Again $(x, y, w) \in R_5$ implies that $\{x, y, w\}$ cannot be contained in any block of $\mathcal{B}$. However, $\lambda = 1$ implies that $\{x, y\}$ belongs to a unique block. Hence we get a contradiction unless $p_{445}^4 = 0$.

Case 2. $l = 5$. We calculate $p_{ij}^5$. Here, since $(x, y, z) \in R_5$, we have that there does not exist any block containing $\{x, y, z\}$. Now, let $B_1 =$
\{y, z, t_1, ..., t_{k-2}\}, B_2 = \{x, z, u_1, ..., u_{k-2}\}, \text{ and } B_3 = \{x, y, v_1, ..., v_{k-2}\} \text{ be the unique blocks containing } \{y, z\}, \{z, x\}, \{x, y\}, \text{ respectively. Here, the } t's, u's, \text{ and } v's \text{ are all distinct, since otherwise some pair would occur in two blocks contradicting the hypothesis that } \lambda = 1. \text{ Now, let } w \in X \text{ such that } w \text{ is distinct from each of } x, y, \text{ and } z. \text{ If } w \in \{t_1, ..., t_{k-2}\} \text{ then } (w, y, z) \in R_4, (x, w, z) \in R_5, \text{ and } (x, y, w) \in R_5; \text{ these choices of } w \text{ contribute to } p_{455}. \text{ Similarly, the choices } w \in \{u_1, ..., u_{k-2}\} \text{ and } w \in \{v_1, ..., v_{k-2}\} \text{ contribute respectively to } p_{545} \text{ and } p_{554}. \text{ Finally, } w \in X \setminus (B_1 \cup B_2 \cup B_3) \text{ gives } v - 3k + 3 \text{ choices of } w \text{ which contribute to } p_{555}. \text{ Thus all points } w \in X \text{ which are distinct from } x, y, \text{ and } z \text{ have now been accounted for and we therefore conclude that }

\begin{align*}
p_{455} - p_{545} &= p_{554} - k - 2 \\
p_{555} &= v - 3k + 3
\end{align*}

and \(p_{ijk} = 0\) otherwise, \(i, j, k \in \{0, 1, 2, 3, 4, 5\}\). Hence the principal regularity condition is satisfied. Therefore, we have constructed an AST with six relations in it.

Q.E.D.

Now we describe several identities in the ternary algebra associated with the AST constructed in Theorem 3.1. We keep the notation used in Theorem 3.1 and also, as usual, \(A_4\) and \(A_5\) denote the adjacency matrices of the relations \(R_4\) and \(R_5\). Although a few of the identities given below overlap with some of the identities given in Proposition 2.7, we include those here also for convenience.

**Theorem 3.2.** Consider the AST constructed in Theorem 3.1 from a 2-design with \(\lambda = 1\). Then the associated ternary algebra satisfies the following identities:

(i) \(I_1 A_4 A_4 = (k - 2) I_1\)

(ii) \(I_1 A_5 A_4 = 0\)

(iii) \(A_4 A_4 I_3 = (k - 2) I_3\)

(iv) \(A_4 A_4 A_4 = (k - 3) A_4\)

(v) \(A_4 I_2 A_4 = (k - 2) I_2\)

(vi) \(I_1 A_3 A_5 = (v - k) I_1\)

(vii) \(A_5 A_5 A_5 = (v - k) A_4 + (v - 3k + 3) A_5\)

(viii) \(A_5 A_4 A_5 = (k - 2) A_5\).

**Proof.** We include the proof of (iv) and omit the proofs of the other identities which are quite similar.
(iv) The \((x, y, z)\) entry of \(A_4 A_4 A_4\) is

\[
\sum_w (A_4)_{w, w, z} (A_4)_{x, w, z} (A_4)_{x, y, z}.
\]

Here, for a nonzero term we must have that \(\{w, y, z\} \in B_1\), \(\{x, w, z\} \in B_2\), and \(\{x, y, w\} \in B_3\), where \(B_1, B_2,\) and \(B_3\) are three blocks of the given 2-design from which the AST has been constructed. Now, if \(\{x, y, z\}\) belongs to some block \(B\) of the 2-design, then, since \(\lambda = 1\), it is clear that we must have \(B = B_1 = B_2 = B_3\). Therefore the number of choices for \(w\) is the cardinality of the set \(B \setminus \{x, y, z\}\) and so the result follows. Now, consider the case when \(\{x, y, z\}\) does not lie in any block of the given 2-design. Then it follows that \(B_1, B_2,\) and \(B_3\) must be all distinct from each other; for example, if \(B_1 = B_2\) then \(\{x, y, z\} \in B_1\), a contradiction to our hypothesis. Further if there is a \(w\) such that \(\{w, y, z\} \in B_1, \{x, w, z\} \in B_2\) and \(\{x, y, w\} \in B_3\) then we have, for example, \(\{w, z\}\) lies both in \(B_1\) and \(B_2\), a contradiction to the fact that \(\lambda = 1\). Hence this case cannot arise.

Q.E.D.

Remark 3.3. For the AST constructed in Theorem 3.1, the values of \(n_4\) and \(n_5\) can be easily inferred from Theorem 3.2 (also compare with 2.3). One has that \(n_4 = k - 2\) and \(n_5 = v - k\).

Remark 3.4. In the ternary algebra discussed in Theorem 3.2, one sees that the matrices \(A_4\) and \(A_5\) generate a subalgebra which is commutative. In fact, the identities,

\[
A_4 A_4 A_5 = A_4 A_5 A_4 = A_5 A_4 A_5 = (k - 2) A_5
\]

\[
A_4 A_5 A_5 = A_4 A_5 A_4 = A_5 A_4 A_4 = 0,
\]

together with the identities (iv) and (vii) of Theorem 3.2 prove this assertion. Also, one notices that this subalgebra is not a minimal subalgebra, since \(A_4\) by itself generates a subalgebra which is properly contained in it. The subalgebra generated by \(A_4\) and \(A_5\) is non-associative. For one has that

\[
(A_5 A_5 A_5) A_4 A_4 = [(v - k) A_4 + (v - 3k + 3) A_5] A_4 A_4 = (v - k)(k - 3) A_4
\]

\[
A_5 (A_5 A_5 A_4) A_4 = A_5 [(k - 2) A_5] A_4 = (k - 2)^2 A_5
\]

\[
A_5 A_5 (A_5 A_4 A_4) = A_5 A_5 0 = 0.
\]

Theorem 3.5 (Converse of Theorem 3.1). Let \(\mathcal{A} = \{R_0, ..., R_5\}\) be an AST on a set \(\Omega\) of \(v\) elements having symmetric \(R_4\) and \(R_5\) and with parameters such that

\[
n_4 = k - 2, \quad p_{444}^4 = k - 3,
\]
where \(3 < k < v\). Then there is a \(2-(v, k, \lambda)\) design with \(\lambda = 1\), from which \(\mathcal{A}\) can be obtained by applying the construction formulated in Theorem 3.1.

**Proof.** Let \(x, y\) be two distinct points of \(\mathcal{A}\). Let

\[ Z = \{ z : (x, y, z) \in R_4 \} . \]

Define

\[ B = \{ x, y \} \cup Z \]

as the block generated by \(x\) and \(y\).

**Claim.** For any \(u, v \in B\), the block generated by \(u\) and \(v\) coincides with \(B\).

If \(\{u, v\} = \{x, y\}\), the claim is true by the definition of \(B\). Let \(z_i\) be an arbitrary element of \(Z\). Then \((x, y, z_i) \in R_4\) and there are \(p_4^{44} = k - 3\) elements \(w\) such that

(a) \((w, y, z_i) \in R_4\)
(b) \((x, w, z_i) \in R_4\)
(c) \((x, y, w) \in R_4\).

For each such \(w\), one has that \(w \in Z\) by (c), and \(w \neq x, y\) by (a) and (b). It follows that these elements \(w\) are precisely the elements of \(Z \setminus \{z_i\}\). By (a), (b) and the symmetry of \(R_4\), it follows that \((x, z_i, z_j) \in R_4\) and \((y, z_i, z_j) \in R_4\) for \(z_j \in Z, j \neq i\). Here, the index \(i\) is arbitrary and we have proved the claim for every \(\{u, v\}\) which includes \(x\) or \(y\). Now, let \(\{u, v\} \subseteq Z\), say \(u = z_i, v = z_j\). Then \((x, z_i, z_j) \in R_4\). Again, using the fact that \(p_4^{44} = k - 3\), we find that \(p_4^{44}\) counts precisely the elements of \(\{y\} \cup Z \setminus \{z_i, z_j\}\). In particular, \((z_i, z_j, z_k) \in R_4\) for every \(\{z_i, z_j, z_k\} \subseteq Z\) and the claim is proved for \(\{u, v\} \subseteq Z\). This completes the proof of the claim.

Now, form a block design \(\mathcal{B}\) with point set \(\Omega\), using as blocks all the distinct blocks generated by the 2-subsets of \(\Omega\). Note that the blocks in \(\mathcal{B}\) are of uniform size \(k\). Further, each subset \(\{x, y\}\) of 2-elements of \(\Omega\) generates a unique block and is contained only in the block it generates. This implies that \(\mathcal{B}\) is a 2-design with \(\lambda = 1\).

It remains to show that if \(\mathcal{A}'\) is the AST obtained from \(\mathcal{B}\) by the construction of Theorem 3.1, then \(\mathcal{A}' = \mathcal{A}\). Let \(R_4'\) be the set of collinear triples in \(\mathcal{A}'\). For any triple \((x, y, z) \in R_4\), the block \(B\) generated by any two of \(x, y, z\) contains \((x, y, z)\) as a collinear triple. That is, \((x, y, z)\) is an element of \(R_4\), implying \(R_4 \subseteq R_4'\). Since \(\mathcal{A}\) and \(\mathcal{A}'\) have the same parameters \(v\) and \(n_4, |R_4'| = |R_4|\). Thus \(R_4 = R_4'\), which suffices to show that \(\mathcal{A}' = \mathcal{A}\). Q.E.D.
In view of Theorem 3.5 it is easy to see that any two non-isomorphic 2-designs with \( \lambda = 1 \) with the same parameters give non-isomorphic ASTs.

The designs considered so far have \( \lambda = 1 \) and it is a natural question to ask whether one could also construct ASTs from 2-designs with values of \( \lambda \) greater than 1 in a manner similar to the construction given in Theorem 3.1. We have found that it is not possible to do so in general (see Appendix I for details) but give below an example when it does work for a particular design with \( \lambda = 2 \).

**Example 3.6.** Consider the 2-design on the six elements \( 1, \ldots, 6 \) with the blocks

\[
123 \quad 124 \quad 135 \quad 146 \quad 156 \quad 236 \quad 245 \quad 256 \quad 345 \quad 346.
\]

It is well known that this is a 2-design with \( \lambda = 2 \). Let \( R_0, \ldots, R_3 \) be the trivial relations. Now, as in Theorem 3.1, let \( R_4 \) denote the set of all triples which are in some blocks of this design and \( R_5 \) denote the set of all triples which do not lie in any block. Then, one can verify that this does give us an AST with precisely six relations. The \( 6 \times 6 \times 6 \) adjacency matrices \( A_4 \) and \( A_5 \) corresponding to the only two non-trivial relations are given in Table III, using the notation explained in Example 1.1. The adjacency matrices of the trivial relations are omitted to save space, but their shapes can be easily visualized from Table I, for example. (We remark that with a slight change of notation this AST can also be obtained using Theorem 4.4 from the 2-transitive permutation group \( \text{PSL}(2, 5) \) acting on the projective line of six points.)

**Definition 3.7.** A 2-design \( (\Omega, \mathcal{D}) \) is said to be a partial 3-design if there is a symmetric AST with the relations \( R_i, 0 \leq i \leq m \), on the set \( \Omega \) and integers \( \lambda_4, \ldots, \lambda_m \) such that for \( i = 4, \ldots, m \) if \( (x, y, z) \in R_i \) then there are exactly \( \lambda_i \) blocks of the design containing the 3-set \( \{x, y, z\} \).

(We remark that one could broaden Definition 3.7 to nonsymmetric ASTs but the present definition is appropriate for our purposes). One notices that Definition 3.7 is analogous to the well-known concept of a PBIBD (partially balanced incomplete block design) as a 1-design which is partially balanced for 2-subsets. Whenever a 2-design yields an AST by the type of construction used in Theorem 3.1, then one sees that this 2-design is a partial 3-design in the sense of Definition 3.7. Thus a natural question arises that which 2-designs are partial 3-designs? We have settled this question in Theorem 3.1 for the particular case when \( \lambda = 1 \) but the general question remains open.
4. ASTs from 2-Transitive Groups

We show that any 2-transitive permutation group gives rise to an AST in a natural way. As a particular case we then consider in detail the AST constructed from the group \( \text{PSL}(2, q) \) in its 2-transitive representation on the projective line of \( q + 1 \) points. We also discuss briefly the construction for some other 2-transitive groups like the affine group, Higman–Sims group and the Suzuki group.

<table>
<thead>
<tr>
<th></th>
<th>( A_4 )</th>
<th>( A_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>000000</td>
<td>000000</td>
<td></td>
</tr>
<tr>
<td>001100</td>
<td>000011</td>
<td></td>
</tr>
<tr>
<td>010010</td>
<td>000101</td>
<td></td>
</tr>
<tr>
<td>010001</td>
<td>001010</td>
<td></td>
</tr>
<tr>
<td>010001</td>
<td>010100</td>
<td></td>
</tr>
<tr>
<td>000110</td>
<td>011000</td>
<td></td>
</tr>
<tr>
<td>001100</td>
<td>000011</td>
<td></td>
</tr>
<tr>
<td>000000</td>
<td>000000</td>
<td></td>
</tr>
<tr>
<td>100001</td>
<td>000110</td>
<td></td>
</tr>
<tr>
<td>100010</td>
<td>001001</td>
<td></td>
</tr>
<tr>
<td>000101</td>
<td>101000</td>
<td></td>
</tr>
<tr>
<td>010101</td>
<td>100100</td>
<td></td>
</tr>
<tr>
<td>010010</td>
<td>000101</td>
<td></td>
</tr>
<tr>
<td>100001</td>
<td>000110</td>
<td></td>
</tr>
<tr>
<td>000000</td>
<td>000000</td>
<td></td>
</tr>
<tr>
<td>000011</td>
<td>110000</td>
<td></td>
</tr>
<tr>
<td>100100</td>
<td>010001</td>
<td></td>
</tr>
<tr>
<td>010100</td>
<td>100010</td>
<td></td>
</tr>
<tr>
<td>010001</td>
<td>001010</td>
<td></td>
</tr>
<tr>
<td>100010</td>
<td>001001</td>
<td></td>
</tr>
<tr>
<td>000011</td>
<td>110000</td>
<td></td>
</tr>
<tr>
<td>100100</td>
<td>100001</td>
<td></td>
</tr>
<tr>
<td>100100</td>
<td>010010</td>
<td></td>
</tr>
<tr>
<td>001100</td>
<td>010100</td>
<td></td>
</tr>
<tr>
<td>000101</td>
<td>101000</td>
<td></td>
</tr>
<tr>
<td>100100</td>
<td>010001</td>
<td></td>
</tr>
<tr>
<td>011000</td>
<td>100001</td>
<td></td>
</tr>
<tr>
<td>000000</td>
<td>000000</td>
<td></td>
</tr>
<tr>
<td>110000</td>
<td>001100</td>
<td></td>
</tr>
<tr>
<td>000110</td>
<td>011000</td>
<td></td>
</tr>
<tr>
<td>001010</td>
<td>100100</td>
<td></td>
</tr>
<tr>
<td>010100</td>
<td>100010</td>
<td></td>
</tr>
<tr>
<td>101000</td>
<td>010010</td>
<td></td>
</tr>
<tr>
<td>110000</td>
<td>001100</td>
<td></td>
</tr>
<tr>
<td>000000</td>
<td>000000</td>
<td></td>
</tr>
</tbody>
</table>
THEOREM 4.1. Let \( G \) be a 2-transitive permutation group acting on a set \( \Omega \). Then the orbits of the natural action of \( G \) on \( \Omega \times \Omega \times \Omega \) form the classes of an AST on \( \Omega \).

Proof. Let the orbits of \( G \) on \( \Omega \times \Omega \times \Omega \) be \( R_i, \, 0 \leq i \leq m \), where \( R_0 = \{(x, x, x) : x \in \Omega \} \). Since \( G \) is 2-transitive on \( \Omega \), given \((x, y), (u, v)\) in \( \Omega \times \Omega \) there exists \( g \in G \) such that \((x, y)^g = (u, v)\). Thus the pairs \((x, y), (u, v)\) are alike. Further each pair \((x, y)\) is in a uniform number of triples \((x, y, z)\) of the class \( R_i \). Now let \( A_i \) be the incidence matrix corresponding to the class \( R_i \). If \(|\Omega| = v\), then \( A_i \) is a \( v \times v \times v \) matrix indexed by \( \Omega \) such that the \((x, y, z)\) entry of \( R_i \) is given by

\[
(A_i)_{x, y, z} = \begin{cases} 
1 & \text{if } (x, y, z) \in R_i \\
0 & \text{otherwise.}
\end{cases}
\]

We observe that \( \sum_{i=0}^{m} A_i = J \). Also, if \( \sigma \) is a permutation of \( \{1, 2, 3\} \) then \( A_i^\sigma \) is the adjacency matrix of some class \( R_k \) for \( k \in \{0, 1, \ldots, m\} \). Now, let \((x, y, z)\) and \((x', y', z')\) be two triples belonging to the class \( R_i \). Let

\[
A = \{w \in \Omega : (w, y, z) \in R_i, (x, w, z) \in R_j, (x, y, w) \in R_k\}
\]

and

\[
B = \{w' \in \Omega : (w', y', z') \in R_i, (x', w', z') \in R_j, (x', y', w') \in R_k\}.
\]

Then, one sees that there is a 1-1 correspondence between the sets \( A \) and \( B \) and so the regularity condition of the definition of an AST is satisfied. Hence the relations \( R_0, \ldots, R_m \) form an association scheme on triples.

\[\text{Q.E.D.}\]

Before applying Theorem 4.1 to construct ASTs from 2-transitive permutation groups we first describe two technical lemmas. We omit the proofs which are quite straightforward.

**Lemma 4.2.** Let \( G \) be a 2-transitive permutation group on a set \( \Omega \) and let \( \Delta \) be a \( G \)-orbit on \( \Omega \times \Omega \times \Omega \). For \( \alpha, \beta \) in \( \Omega \), \( \alpha \neq \beta \), let

\[
\Delta(\alpha, \beta) = \{\gamma : (\alpha, \beta, \gamma) \in \Delta\}.
\]

Then \( \Delta(\alpha, \beta) \) is a \( G_{x,\beta} \)-orbit on \( \Omega \setminus \{\alpha, \beta\} \). Furthermore, the map

\[
\Delta \to \Delta(\alpha, \beta)
\]

is a one-to-one correspondence between the \( G \)-orbits of \( \Omega \times \Omega \times \Omega \) and the \( G_{x,\beta} \) orbits on \( \Omega \setminus \{\alpha, \beta\} \).
LEMMA 4.3. Let $\Omega$ be the projective line $GF(q) \cup \{\infty\}$. Consider the action of $PSL(2, q)$ on $\Omega \times \Omega$. Under this action the triples $(x, \beta, \gamma)$, $x, \beta, \gamma$ all distinct from each other, fall into two (one) orbits if $q$ is odd (resp. even). Each such orbit contains a triple of the form $(\infty, 0, \eta)$ as a representative. If $q$ is odd then define $\Theta_1$ as the orbit in which $\eta$ is a nonzero square, and $\Theta_2$ as the orbit in which $\eta$ is nonsquare or zero. Then for a triple $(x, \beta, \gamma)$ we have that it lies in $\Theta_1$ or $\Theta_2$ according as the nonzero product $(x - \beta)(\beta - \gamma)(\gamma - \alpha)$ is a square or not, provided $x, \beta, \gamma$ are all distinct from $\infty$. Further, the triples $(x, \beta, \infty)$, $(x, \infty, \gamma)$, and $(\infty, \beta, \gamma)$ lie in $\Theta_1$, or $\Theta_2$ according as $\beta - \alpha$, $x - \gamma$, $\gamma - \beta$, respectively, is a nonzero square or not.

We now construct an infinite family of ASTs using $PSL(2, q)$.

THEOREM 4.4. Consider the natural action of the group $G = PSL(2, q)$ acting on the projective line $\Omega = GF(q) \cup \{\infty\}$. This 2-transitive permutation group gives rise to an association scheme on triples whose relations are:

(i) $G$-orbits on $\Omega \times \Omega \times \Omega$ of the form $\{(x, \beta, \gamma): x, \beta, \gamma$ are all distinct$\}$. (If $q$ is odd there are exactly two such orbits and if $q$ is even, there is only one such orbit).

(ii) the four trivial relations given by $R_0, \ldots, R_3$.

Proof. By Lemma 4.2 the $G$-orbits on $\Omega \times \Omega \times \Omega$ are in one-to-one correspondence with the orbits of $G_{x, \beta}$ on $\Omega \setminus \{x, \beta\}$ for any two distinct points $x, \beta$ in $\Omega$. Choose $x \neq \infty$, the point at infinity, and set $\beta = 0$. Now, if $q$ is odd, the set $\Omega \setminus \{\infty, 0\}$ is a union of two $G_{\infty, 0}$-orbits; one orbit consisting of all elements which are quadratic residues and the other orbit consisting of all the quadratic nonresidues. If $q$ is even, then all the nonzero elements in $GF(q)$ are squares and so there is only one $G_{\infty, 0}$-orbit on $\Omega \setminus \{\infty, 0\}$. Thus by Theorem 4.1 we obtain the relations of a 3-dimensional association scheme. If $q$ is odd then by Lemma 4.3 the two relations of the type (i) described in the statement of the theorem have representatives of the type $(\infty, 0, \eta)$, where $\eta$ is a nonzero square or nonsquare. If $q$ is even then there is only one such relation. Q.E.D.

COROLLARY 4.5. Let $q$ be any odd prime power. Then one can construct an AST on a certain set $\Omega$, $|\Omega| = q + 1$, with precisely six relations.

We shall now list a number of identities in the ternary algebra associated with the AST constructed in Theorem 4.4. Our list of identities is not intended to be exhaustive; we omit the proof which is similar to the proof of Theorem 3.2.

THEOREM 4.6. Let $\{A_i: 0 \leq i \leq 5\}$ be the adjacency matrices corresponding to the relations $\{R_i: 0 \leq i \leq 5\}$ of the AST constructed from the group
PSL(2, q), where $R_0, R_1, R_2, R_3$ are the usual trivial relations associated with any AST and $R_4, R_5$ are the relations corresponding to the orbits $\Theta_1, \Theta_2$ defined in Lemma 4.3. Then one has the following identities:

(i) Each of the ternary products $A_1A_4A_4, A_1A_5A_5$ is equal to the expression

$$
2tA_1 \quad \text{if } q = 4t + 1 \\
0 \quad \text{otherwise},
$$

for some integer $t$. Similarly, each of the ternary products in the two groups $A_4A_2A_4, A_5A_2A_5$ and $A_4A_3A_3, A_5A_3A_3$ is equal to an expression obtained from the above expression by replacing $A_1$ by $A_2$ and $A_3$, respectively.

(ii) For some integer $t$, one has

$$
A_4A_5A_1 = \begin{cases} 
0 & \quad \text{if } q = 1 \pmod{4} \\
(2t + 1)A_1 & \quad \text{if } q = 4t + 3.
\end{cases}
$$

(iii) For some integer $t$, one has

$$
A_4A_4A_4 = \begin{cases} 
(t - 1)A_4 & \quad \text{if } q = 4t + 1 \\
(t + 1)A_5 & \quad \text{if } q = 4t + 3.
\end{cases}
$$

The ASTs constructed in 3.1 and 4.4 have precisely six relations; such an AST has just one more relation other than the trivial relations. Now, we describe an AST which can be constructed using Theorem 4.1 and which has, in general, more than six relations and also we obtain some identities in the related ternary algebra. We omit the proof.

**Proposition 4.7.** Consider the action of the affine group $AGL(1, q)$ acting as a 2-transitive permutation group on the finite field $GF(q)$. Then, one obtains an AST with $q + 2$ relations consisting of the orbits of $AGL(1, q)$ on $\Omega \times \Omega \times \Omega$ where $\Omega = GF(q)$. Here, each non-trivial relation has a unique (orbit) representative of the form

$$(0, 1, d), \quad d \in GF(q) \setminus \{0, 1\}.$$ 

If $A^a$ denotes the adjacency matrix of the relation containing the element $(0, 1, a)$ for any $a \in GF(q) \setminus \{0, 1\}$, then for $a, b, c \in GF(q) \setminus \{0, 1\}$, one has that

$$
A^aA^bA^c = \begin{cases} 
A^{bc} & \quad \text{if } c = a(a + b - 1)^{-1}, bc \neq 1 \\
0 & \quad \text{otherwise}.
\end{cases}
$$
Further, if \( b, c \in GF(q) \setminus \{0, 1\} \), such that \( bc = 1 \), then
\[
I_1 A^b A^c = I_1, \quad A^b I_2 A^c = I_2, \quad A^b A^c I_3 = I_3, \tag{12}
\]
and if \( bc \neq 1 \), then all the three ternary products in (12) are equal to 0.

By applying Theorem 4.1 to any 2-transitive permutation group one can construct an AST. This gives us a rich collection of examples of ASTs. We illustrate a few more ASTs constructed in this way.

**Proposition 4.8.** (i) Let \( G \) be the Higman–Sims group and consider its 2-transitive representation on 176 points. Then one obtains an AST with precisely seven relations.

(ii) Let \( G \) be the Suzuki group in its 2-transitive representation on \( q^2 + 1 \) points, where \( q = 2^m \) for some \( m \geq 1 \). Then one obtains an AST with precisely \( q + 5 \) relations.

**Proof.** (i) If \( K \) is the subgroup of \( G \) fixing the point \( \infty \) and another distinct point, then it is known (e.g., [5]) that \( K \) has exactly three orbits on the remaining 174 points of \( \Omega \) whose lengths are 12, 72, and 90. The result now follows using Lemma 4.2 and Theorem 4.1. (ii) can be proved by a similar reasoning.

5. ASTs and Two-Graphs

A two-graph \((\Omega, \Delta)\) is defined as a collection \(\Delta\) of 3-subsets of a set \(\Omega\) such that every 4-subset of \(\Omega\) contains an even number of members of \(\Delta\) \([4, 13, 14]\). A two-graph is said to be regular if each pair of elements of \(\Omega\) is contained in the same number of triples of \(\Delta\). In this section we show that some ASTs are equivalent to some two-graphs.

**Theorem 5.1.** Consider an AST defined on a set \(\Omega\) with the relations \(R_0, \ldots, R_m\), where \(R_0, \ldots, R_3\) denote, as usual, the trivial relations. Assume that \(m > 4\). Let \(J\) be a subset of \(\{4, 5, \ldots, m\}\) with the property that \(R_i\) is symmetric for each \(i \in J\). Define
\[
\Delta = \{\{a, b, c\}: (a, b, c) \in R_i \text{ for some } i \in J\}.
\]

Then \(\Delta\) is a regular two-graph on \(\Omega\) if and only if the condition (***) is satisfied:

(***) For any quadruple \(i, j, k, l\) from the set \(\Omega\) such that an odd number of these lie in the set \(J\), one has that \(p_{ijkl}^j = 0\).
Proof. \((\Rightarrow)\) Let \(i, j, k, l\) be a quadruple of indices such that an odd number of these lie in \(J\). Then, if \(p_{jkl}^i\) is non-zero it follows immediately that there is a subset \(\{x, y, z, w\}\) of \(\Omega\) such that \((x, y, z) \in R_i\), \((w, y, z) \in R_j\), \((x, w, z) \in R_j\), \((x, y, w) \in R_k\). So, an odd number of subsets of \(\{x, y, z, w\}\) are in elements of \(A\). This contradicts the fact that \(A\) is a two-graph on \(\Omega\).

\((\Leftarrow)\) If \(\{x, y, z, w\}\) is any 4-subset of \(\Omega\), using (**) it follows that only an even number of 3-subsets of \(\{x, y, z, w\}\) can belong to \(A\). Hence \(A\) is a two-graph on \(\Omega\). Regularity of the two-graph follows from condition (1) of the definition of an AST. Q.E.D.

Remark. The condition in Theorem 5.1 that \(R_i\) be symmetric for each \(i \in J\) could be replaced by the slightly weaker condition that the ternary relation obtained by taking the union of these \(R_i\) be symmetric. If \(J = \{4, \ldots, m\}\), the union of relations \(R_i\) is the set of all 3-subsets of \(\Omega\), which is the complete two-graph and is the non-trivial relation in the AST discussed in Remark 1.6. If \(J\) is empty we get the empty two-graph. For any \(J\) which gives a two-graph, the complement of \(J\) of course leads to the complementary two-graph.

Example 5.2. (i) Consider the AST constructed in Example 3.6 from a 2-design with \(\lambda = 2\). It is easily verified that the relations \(R_4\) and \(R_5\) both provide 2-graphs using the method of Theorem 5.1.

(ii) Consider the AST constructed in Theorem 4.1 from the group \(\text{PSL}(2, q)\). It can be shown (using Theorem 4.6) that \(R_4\) and \(R_5\) both give two-graphs if \(q \equiv 1 \pmod{4}\) but do not give two-graphs if \(q \equiv 3 \pmod{4}\).

Theorem 5.1 has the following converse.

Theorem 5.3. Let \(H\) be a regular two-graph on a set \(\Omega\). Define a symmetric ternary relation \(R_4\) consisting of the 3-subsets in \(H\), and let \(R_5\) be the relation consisting of the 3-subsets not in \(H\). Then \(R_4\) and \(R_5\) both satisfy the regularity condition necessary for the non-trivial relations of an AST.

Before proving Theorem 5.3 we need some background and also two lemmas on graphs.

It is well known that given a graph \(G = (\Omega, E)\) (where, as usual, \(\Omega\) is the set of vertices and \(E\) is the set of edges), one can construct a two-graph \(H = (\Omega, A)\) by taking \(A\) as the set of odd triples in \(G\) defined by

\[
A = \{\{x, y, z\} \subseteq \Omega: \text{an odd number of edges } xy, xz, yz \text{ are in } E\}.
\]

Any two-graph \(H\) is equivalent to the set of graphs, called the switching class of \(H\), from which it can be obtained by this construction. If \(G_i = (\Omega, E_i)\) is any graph, the switching class containing \(G_i\) is the class of graphs
\( G = (\Omega, E) \) whose edge set \( E \) can be obtained from \( E_1 \) by the switching operation of taking the symmetric difference of \( E_1 \) with the edge set of a complete bipartite graph on the vertex set \( \Omega \). If \( \alpha \) is any vertex of a two-graph \( H = (\Omega, \Delta) \), a convenient representative of the switching class of \( H \) is \( G_1 = (\Omega, E') \), where

\[
E' = \{xy : \{\alpha, x, y\} \in \Delta\}.
\]

Here, \( G_1 \) consists of an isolated vertex \( \alpha \) together with a graph \( G' = (\Omega', E') \), where \( \Omega' = \Omega \setminus \{\alpha\} \). We shall call the graph \( G' \) so defined the derived graph of \( H \) with respect to \( \alpha \). More formally,

\[
G' = (\Omega', E') \text{ is the derived graph of a two-graph } H
\]

if and only if \( \alpha \notin \Omega' \) and

\[
(\Omega' \cup \{\alpha\}, E') \text{ is in the switching class of } H.
\]

A strongly regular graph [2] could be defined to be an association scheme (in the classical sense) with exactly two non-trivial relations. In any association scheme (in the classical sense) whose underlying set is \( \Omega \), if \( R_i, R_j, \) and \( R_k \) are any three relations, then for a fixed \((x, y) \in R_k\) one defines

\[
p_{ij}^k = |\{z \in \Omega : (x, z) \in R_i, (z, y) \in R_j\}|,
\]

which is a number (called the intersection number) independent of the choice of \( x \) and \( y \). (The reader is urged not to confuse \( p_{ij}^k \) with the regularity condition symbol \( p_{ijk} \) for an AST.) Now, it is known [4] that if \( H \) is a regular two-graph then its derived graph with respect to any vertex is a strongly regular graph, say \( G' = (\Omega', E') \) with degree \( n_1 \) and intersection numbers \( p_{ij}^k \). We set \( n_2 \) to denote the degree of the complement of \( G' \). For \( x, y \in \Omega = \Omega' \cup \{\alpha\} \), the degree of \( \{x, y\} \) in \( H \) is the number of odd triples in \( G = (\Omega, E') \) which include \( x, y \). This number is

\[
p_{11}^1 + p_{22}^1 + 1 \quad \text{if } x, y \text{ are adjacent vertices of } G',
\]

\[
p_{12}^2 + p_{21}^2 = 2p_{12}^1 \quad \text{if } x, y \text{ are non-adjacent vertices of } G',
\]

\[
n_1 \quad \text{if } x \in \Omega', \ y = \alpha.
\]

Since \( H \) is regular we equate these expressions, obtaining equations equivalent to

\[
p_{12}^1 = p_{22}^1 = n_2/2, \quad p_{11}^1 = p_{12}^1 - n_1/2.
\]
These conditions are not independent; indeed any of the equations in (13) implies all of them in view of standard relations on the intersection numbers.

**Lemma 5.4.** A graph $G' = (\Omega', E')$ is the derived graph of a regular two-graph if and only if $G'$ is strongly regular with parameters which satisfy (13).

**Proof.** $(\Rightarrow)$ Using the known fact [4] that a derived graph of a regular two-graph is strongly regular, we have just proved the necessity of (13).

$(\Leftarrow)$ Taking $\Omega = \Omega' \cup \{\alpha\}$, and taking $\mathcal{A}$ as the set of odd triples in $(\Omega, E')$, we get a two-graph $(\Omega, \mathcal{A})$ which has $G'$ as a derived graph and which is regular by virtue of (13). Q.E.D.

In the next lemma it is shown that the regular two-graph constructed in Lemma 5.4 is (with its complement) also an AST. Needed in the proof of Theorem 5.3, it might be listed on its own as a construction method for ASTs but the "only if" part shows that the only ASTs it produces are those described in Theorem 5.3.

**Lemma 5.5.** Let $G' = (\Omega', E')$ be a strongly regular graph and let $\Omega = \Omega' \cup \{\alpha\}$, where $\alpha \notin \Omega'$. Let $R_4$ (resp., $R_5$) be the set of odd (resp., even) triples in $G = (\Omega, E')$. Then $R_4$ and $R_5$ are the non-trivial relations of a (symmetric) AST if and only if the parameters of $G'$ satisfy (13).

**Proof.** Observe that $(\Omega, R_4)$ is a two-graph. Let $(i, j, k, l)$ be a 4-tuple of 4's and 5's which includes an odd number of 4's, and take $\{x, y, z\} \in R_4$. Now if $p_{ijk}^4 > 0$ there is a 4-subset $\{w, x, y, z\} \subseteq \Omega$ which has an odd number of 3-subsets in $R_5$, contrary to the definition of two-graph. This implies

$$p_{445}^4 = p_{454}^4 = p_{544}^4 = p_{555}^5 = p_{445}^4 = p_{455}^5 = p_{545}^5 = p_{554}^5 = 0.$$

We proceed to calculate $p_{ijk}^4$ for odd triples $\{x, y, z\}$ in cases where $(i, j, k, l)$ consists of an even number of 4's and an even number of 5's. There are several cases according as $\{x, y, z\}$ is a subset of $\Omega'$ or includes $\alpha$, and according as $E'$ includes all of edges $xy, xz, yz$ or only one of them. One of the cases is portrayed schematically in Fig. 1 which shows three vertices $x, y, z \in \Omega'$, an isolated vertex $\alpha$, and eight sets into which the other $(v - 1) - 3$ vertices of $\Omega'$ are classified in Table IV by their adjacencies to $x, y, z$; and $a, b, \ldots, h$ denote the cardinalities of these eight sets. As Fig. 1 shows, $\{x, y, z\}$ is an odd triple, including a single edge $yz$. From the last four columns of the Table IV we find

$$p_{444}^4(x, y, z) = b + e, \quad p_{455}^4(x, y, z) = a + h + 1,$$
$$p_{545}^4(x, y, z) = d + g, \quad p_{554}^4(x, y, z) = c + f.$$
Figure 1 and the first four columns of the Table IV give equations such as

\[ a + d = p^2_{11}(x, y) = p^2_{11} \]
\[ b + f + 1 = p^2_{21}(x, y) = p^2_{12}, \quad \text{vertex } z \text{ accounting for the term } 1. \]

In this way we get equations

\[ a + b = p^1_{11}, \quad c + g = d + f = p^1_{12}, \]
\[ e + h + 1 = p^2_{22}, \quad a + e = a + d = p^2_{11}, \]
\[ c + e = d + e = b + f + 1 = b + g + 1 = p^2_{12}, \quad f + h = g + h = p^2_{22}. \]

**TABLE IV**

Classification of Vertices \( w \in \Omega \setminus \{x, y, z\} \)

<table>
<thead>
<tr>
<th>Relation size ijk to xyz</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
<td>a</td>
<td>455</td>
<td></td>
</tr>
<tr>
<td>211</td>
<td>b</td>
<td>444</td>
<td></td>
</tr>
<tr>
<td>121</td>
<td>c</td>
<td>554</td>
<td></td>
</tr>
<tr>
<td>112</td>
<td>d</td>
<td>545</td>
<td></td>
</tr>
<tr>
<td>122</td>
<td>e</td>
<td>444</td>
<td></td>
</tr>
<tr>
<td>212</td>
<td>f</td>
<td>554</td>
<td></td>
</tr>
<tr>
<td>221</td>
<td>g</td>
<td>545</td>
<td></td>
</tr>
<tr>
<td>222</td>
<td>h</td>
<td>455</td>
<td></td>
</tr>
<tr>
<td>w=α</td>
<td>l</td>
<td>455</td>
<td></td>
</tr>
</tbody>
</table>

*Note.* The last three columns represent indices \( i, j, k \in \{4, 5\} \) such that \((w, y, z) \in R_i, (x, w, z) \in R_j, (x, y, w) \in R_k.\)
It is easy to eliminate \( a, b, \ldots, h \) from the foregoing equations. Using standard identities and 5.3, we arrive at the results

\[
p_{444}^4(x, y, z) = p_{11}^1,
\]
\[
p_{455}^4(x, y, z) = p_{345}^4(x, y, z) = p_{554}^4(x, y, z) = p_{12}^1.
\]

Similar or simpler calculations can be carried out for the other cases of an odd triple \((x, y, z)\). In all cases the results are the same and we conclude that

\[
p_{444}^4 = p_{11}^1, \quad p_{455}^4 = p_{345}^4 = p_{554}^4 = p_{12}^1.
\]

Similar calculations for even triples in \( G \) give the results

\[
p_{445}^5 = p_{545}^5 = p_{555}^5 = p_{22}^2 = p_{444}^1 = p_{222}^2.
\]

Other parameters \( p'_{ijk} \) follow from Proposition 2.7 or are easily calculated. For completeness we mention

\[
p_{444}^1 = p_{424}^2 = p_{443}^3 = n_1, \quad p_{155}^1 = p_{525}^2 = p_{553}^3 = n_2.
\]

This completes the proof of Lemma 5.5. Q.E.D.

**Proof of Theorem 5.3.** By Lemma 5.4 the derived graph of the two-graph with respect to a vertex \( x \) is a strongly regular graph \( G' = (\Omega', E') \) whose parameters satisfy (13). The relations \( R_4 \) and \( R_5 \) described in the theorem are the same as those constructed from \( G' \) in Lemma 5.5 and shown there to be the non-trivial relations of an AST. Q.E.D.

**Remark 5.6.** Theorems 5.1 and 5.3 show that a regular two-graph is equivalent to an AST with exactly two non-trivial relations \( R_4, R_5 \) satisfying

\[
\text{(i) } \quad R_4 \text{ and } R_5 \text{ are symmetric,}
\]
\[
\text{(ii) } \quad p_{445}^4 = p_{454}^4 = p_{544}^4 = p_{554}^4 = p_{444}^5 = p_{455}^5 = p_{545}^5 = p_{555}^5 = 0.
\]  

If an AST has more than two non-trivial relations and satisfies the conditions of Theorem 5.1 for some set \( J \) of indices, then a regular two-graph can be produced and, by Theorem 5.3, so can an AST satisfying the description in the preceding paragraph, by a process of forming unions of ternary relations. The original AST can be regarded as a refinement of the one finally obtained. We have not tried to investigate ASTs whose ternary relations can be coalesced to give coarser ASTs but this paragraph notes one set of conditions which are sufficient for this to happen.

**Remark 5.7.** Since every regular two-graph is equivalent to a special case of an AST, it is possible to describe ASTs as a generalization of
regular two-graphs. It appears to be a very substantial generalization, the class of ASTs not corresponding to two-graphs including all ASTs with three or more non-trivial relations, all ASTs which are not symmetric and all ASTs violating condition (14). The classical association schemes described in Theorem 2.13 seem to be a natural and a proper generalization of the strongly regular graph that one obtains as the derived graph of a regular two-graph with respect to a vertex \( a \).

Remark 5.8. (i) The spectrum of eigenvalues which can be defined for every two-graph can in the case of regular two-graphs, be taken as the definition of a spectrum for the corresponding ASTs. It would be desirable to find an eigenvalue proof of Theorem 5.3 and interesting to extend the notion of eigenvalue to more general ASTs. Our ternary algebra is applicable to all ASTs, including those which are two-graphs, but it is not yet clear how powerful it will be as an algebraic and combinatorial tool.

(ii) The many two-graphs which are not regular do not correspond to the ASTs that we have discussed here but some of them probably correspond to the generalized ASTs mentioned in Remark 1.6(ii).

(iii) It has been observed that a regular two-graph is a 2-design (with block size 3). It is also a partial 3-design as introduced in Definition 3.7, but is quite a special case since partial 3-designs can in principle occur with any block size and with any symmetric AST.

APPENDIX I

We give here some technical details to show that it is not always possible to construct ASTs using the method of Theorem 3.1 from designs with values of \( \lambda \) greater than 1 (see Example 3.6 and the paragraph preceding it for the relevance of this question). For a design with parameters \( v = 8, k - 4, b - 14, r - 7, \lambda - 3 \), Table V lists four examples, I, II, III, IV. (This list is extracted from Gibbons [8, Appendix B].) Gibbons' search indicates that these are precisely the non-isomorphic 2-designs with these parameters.

Design IV is the unique 3-design in which each 3-subset of \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) occurs exactly once. The following facts about the designs I, II, and III are easily verified.

Design I

Four triples 1 2 8, 1 3 7, 2 3 6, 6 7 8 have no occurrences within blocks. Four triples 1 2 3, 1 7 8, 2 6 8, 3 6 7 occur two times within blocks.
The remaining \( (8)^3 - 8 = 48 \) triples occur once within blocks. The four triples which do not occur are obviously not the blocks of any 2-design on the set \( \{1, 2, 3, 4, 5, 6, 7, 8\} \). Using Proposition 2.2, this is sufficient to show that design I does not generate an AST by the methods of Theorem 3.1.

We remark that the same conclusion as above follows from the set of four triples which occur two times, and (with a little more work) from the residual set of 48 triples.

**Design II**

Here, the triples which occur zero times are

\[
128 \quad 137 \quad 236 \quad 257 \quad 358 \quad 678
\]

and the triples which occur two times are

\[
123 \quad 178 \quad 258 \quad 267 \quad 357 \quad 368.
\]

(There is a residual set of 44 triples each occurring one time.) None of these sets of triples comprises a 2-design on \( \{1, 2, 3, 4, 5, 6, 7, 8\} \).

**Design III**

Here the triples which occur zero times are

\[
128 \quad 137 \quad 156 \quad 236 \quad 257 \quad 358 \quad 678
\]

and the triples which occur two times are

\[
123 \quad 158 \quad 167 \quad 256 \quad 278 \quad 357 \quad 368.
\]
(There is a residual set of 42 triples each occurring 1 time). None of these sets of triples comprises a 2-region on \{1, 2, 3, 4, 5, 6, 7, 8\}.

Thus, using Proposition 2.2 it follows again that both the designs II and III fail to give an AST using the method of Theorem 3.1.

REFERENCES