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Group-like algebras and Hadamard matrices

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Abstract

We give a description in terms of square matrices of the family of group-like algebras with $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$. In the case that $S = \mathrm{id}$ and $\Bbbk \subseteq \mathbb{R}$, this translation take us to Hadamard matrices and, particularly, to examples of bi-Frobenius algebras satisfying $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$ and that are not Hopf algebras. Finally, we generalize some known results on separability and coseparability valid for finite-dimensional Hopf algebras to this special class of bi-Frobenius algebras with $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$, presenting a version of Maschke's theorem for this family.

Keywords: Bi-Frobenius algebra; Group-like algebra; Hadamard matrix

1. Introduction

Bi-Frobenius algebras generalize finite-dimensional Hopf algebras, in the following sense: a bi-Frobenius algebra has a structure of Frobenius algebra and co-Frobenius coalgebra, linked by a condition weaker than the pentagonal axiom, namely one assumes the existence of a (bijective) antimorphism of algebras and coalgebras $S : A \to A$. Bi-Frobenius algebras were introduced in 2000 by Y. Doi and M. Takeuchi [DT].

In general, S is not the convolution inverse of the identity. This is true in the particular situation of Hopf algebras. A natural question is whether this additional condition implies or not that the algebra is Hopf; in other words: does the fact that S is the convolution inverse of the identity forces A to be a bialgebra?

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In this paper, we show that this is not true by constructing a family of bi-Frobenius algebras where S is the convolution inverse of the identity that are not Hopf algebras.

Once we know there is an intermediate class between bi-Frobenius algebras and finite-dimensional Hopf algebras, it could be relevant to study which properties of finite-dimensional Hopf algebras still hold for this class. In this direction, we give some first steps, generalizing some known results concerning separability, semisimplicity and its dual notions.

In Section 2 we recall the definition and some of the basic properties of bi-Frobenius algebras.

In Section 3 we present the example of group-like algebras. Group-like algebras are the natural generalization of group algebras to the context of bi-Frobenius algebras. We also study additional conditions for a group-like algebra to satisfy $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$ and to be a Hopf algebra.

In Section 4, we describe, in terms of a family of square matrices, the algebra structure of a group-like algebra with $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$. As the coalgebra structure of a group-like algebra is trivial, this matrix approach will give us a complete description of the family of group-like algebras with $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$.

In Section 5 we briefly present Hadamard matrices. These matrices will be used in Section 6 to construct the counterexamples mentioned above.

In Section 6 we treat the cases in which $S = \mathrm{id}$ and $\Bbbk \subseteq \mathbb{R}$, showing how the family of square matrices mentioned before, gives rise, in this particular case, to a Hadamard matrix. We construct, via Hadamard matrices, a family, of unbounded dimension, of commutative group-like algebras with $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$ and that are not Hopf algebras.

In Section 7 we prove a version of Maschke's theorem for this special bi-Frobenius algebras.

Finally, in Section 8, we give an explicit description of a bi-Frobenius algebra satisfying $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$ that is not a Hopf algebra.

2. Bi-Frobenius algebras

All along this paper we use Sweedler's notation for the coproduct of a coalgebra.

Let A be an algebra over $\Bbbk$. We consider A^* with the structure of right A-module via $(f \leftharpoonup a)(x) = f(ax)$, $\forall f \in A^*$, $a, x \in A$. For every $f \in A^*$, the induced map $f \leftharpoonup : A \to A^*$ is a morphism of right A-modules.

If C is a coalgebra over $\Bbbk$, C has a natural structure of right C^*-module, via the right action $c \leftharpoonup f = \sum f(c_1)c_2$, $\forall c \in C$, $f \in C^*$. For every $c \in C$, the induced map $c \leftharpoonup : C^* \to C$ is a morphism of right C^*-modules.

Definition 1. Let $\Bbbk$ be a field. A *Frobenius algebra* is a pair (A, ϕ) where A is a $\Bbbk$-algebra and $\phi \in A^*$ is such that the morphism

$$\begin{aligned} \phi \leftharpoonup : A &\to A^*, \\ x &\mapsto \phi \leftharpoonup x \end{aligned}$$

is bijective.

Dually, a *co-Frobenius coalgebra* is a pair (C, t) where C is a $\Bbbk$-coalgebra and $t \in C$ is such that the morphism

$$\begin{aligned} t \leftharpoonup : C^* &\to C, \\ f &\mapsto t \leftharpoonup f \end{aligned}$$

is bijective.

Remark 1. Clearly, Frobenius algebras and co-Frobenius coalgebras are finite dimensional.

We recall now some results valid for the augmented and coaugmented cases.

Suppose A is a $\Bbbk$-algebra that admits an algebra morphism $\varepsilon : A \to \Bbbk$. In this case, the algebra is said to be *augmented* and we say that $s \in A$ is a right integral if $sx = \varepsilon(x)s$, $\forall x \in A$.

Proposition 1. *If* (A, ϕ, ε) *is an augmented Frobenius algebra and* $s \in A$ *is* (*the unique element*) *such that* $\phi \leftharpoonup s = \varepsilon$*, then* s *is an right integral in* A.

Dually, suppose C is a $\Bbbk$-coalgebra with a group-like element $1 \in C$, i.e. $1 \in C$ satisfies $\Delta(1) = 1 \otimes 1$. In this case, we say that the coalgebra is *coaugmented* and that $\psi \in C^*$ is a right cointegral for C if $\sum \psi(x_1)x_2 = \psi(x)1$, $\forall x \in C$.

Proposition 2. *If* $(C, t, 1)$ *is a coaugmented co-Frobenius algebra and* $\psi \in C^*$ *is* (*the unique element*) *such that* $t \leftharpoonup \psi = 1$*, then* ψ *is a right cointegral.*

The space of right integrals of an augmented Frobenius algebra A is one dimensional, since it is $I_r(A) = A^A \cong (A^*)^A = \Bbbk\varepsilon$.

Dually, the space of right cointegrals of a coaugmented co-Frobenius coalgebra is one dimensional.

Now we can give the definition of a bi-Frobenius algebra.

Definition 2. Let $\Bbbk$ be a field. The data $(A, m, 1, \Delta, \varepsilon, \phi, t)$ is said to be a *bi-Frobenius* algebra if the following conditions hold:

- $(A, m, 1, \phi)$ is a Frobenius algebra,
- $(A, \Delta, \varepsilon, t)$ is a co-Frobenius coalgebra,
- ε is an algebra map,
- $1 \in C$ is a group-like element,
- the map $S : A \to A$ defined by $S(x) = \sum \phi(t_1 x)t_2$ is an antimorphism of algebras and coalgebras.

In [DT] it is proved that S is necessarily bijective. S is called the *antipode* of the bi-Frobenius algebra A.

Notice that Propositions 1 and 2 imply that ϕ is a cointegral and t is an integral for A. Indeed,

$$S(1) = 1, \qquad \varepsilon \circ S = \varepsilon,$$

means that $\sum \phi(t_1)t_2 = 1$ and that $\phi(tx) = \varepsilon(x)$, $\forall x \in A$; in other words,

$$t \leftharpoonup \phi = 1 \quad \text{and} \quad \phi \leftharpoonup t = \varepsilon.$$

Definition 3. Let A be a bi-Frobenius algebra, s, ψ respectively a right integral and a right cointegral for A. Once can easily show that

$$s \leftharpoonup \psi = 1 \leftrightarrow \psi(x) = 1 \leftrightarrow \psi \leftharpoonup s = \varepsilon.$$

We say that (s, ψ) is a bi-Frobenius pair if one of these (equivalent) conditions holds.

Using the fact that the space of right integrals and the space of right cointegrals are one dimensional, it is easy to see that:

Remark 2.

- Given a non-zero right integral $s \in A$, there is one and only one $\psi \in A^*$ such that (s, ψ) is a bi-Frobenius pair.
- Given a non-zero right cointegral $\psi \in A^*$, there is one and only one $s \in A$ such that (s, ψ) is a bi-Frobenius pair.
- If (t, ϕ) is a bi-Frobenius pair then any other bi-Frobenius pair is of the form $(\lambda t, \frac{1}{\lambda}\phi)$, $\lambda \in \Bbbk$, $\lambda \neq 0$.
- If (s, ψ) is a bi-Frobenius pair, then $\sum \psi(s_1 x)s_2 = S(x)$.

Examples. It is well known that every finite-dimensional Hopf algebra is bi-Frobenius.

In the next section we present the coalgebraically trivial example of bi-Frobenius algebras. See [DT,F,WZ] for other examples.

Remark 3. If A is a bi-Frobenius algebra, in the finite-dimensional algebra $\mathrm{Hom}_{\Bbbk}(A, A)$ the condition $S * \mathrm{id} = u\varepsilon$ is equivalent to $\mathrm{id} * S = u\varepsilon$.

Proposition 3. *Let A be a bi-Frobenius algebra over a field $\Bbbk$. If A is also a bialgebra then $S * \mathrm{id} = u\varepsilon$. Hence A is a Hopf algebra.*

Proof. As A is a bialgebra if and only if Δ is an algebra morphism, then

$$(S * \mathrm{id})(x) = \sum S(x_1)x_2 = \sum \phi(t_1 x_1)t_2 x_2 = \sum \phi\big((tx)_1\big)(tx)_2$$
$$= \phi(tx)1 = (\phi \leftharpoonup t)(x)1 = \varepsilon(x)1. \qquad \square$$

We will show in Section 5 that the converse is not true, namely that we can find bi-Frobenius algebras with the property $S * \mathrm{id} = u\varepsilon$ that are not Hopf algebras, in the sense that the multiplication and the comultiplication are not compatible.

Remark 4. Notice that, in order to prove Proposition 3, we have used in fact a condition that is weaker than the compatibility between product and coproduct in a bialgebra. Namely the condition

$$\varepsilon(x) \sum t_1 \otimes t_2 = \sum (tx)_1 \otimes (tx)_2 = \sum t_1 x_1 \otimes t_2 x_2 \tag{$*$}$$

implies that $S * \mathrm{id} = u\varepsilon$. It is relevant to know that condition $(*)$ is in fact weaker than the compatibility condition for a bialgebra. See the end of Section 6.

3. Group-like algebras

We present here a family of examples of bi-Frobenius algebras, introduced in [D] by Y. Doi. It generalizes the group algebras in the sense that the coalgebraic structure of the group-like algebras is trivial, as it is for group algebras.

We will present this example by adding, step by step, the minimal conditions required to have a bi-Frobenius structure, coalgebraically trivial, in a finite-dimensional k-vector space.

All along the rest of the paper we set

$$I = \{0, 1, 2, \ldots, n\}.$$

We will use 1 for both $1_{\Bbbk}$ and 1_A, since it will be clear from the context to which of them we are referring.

We start by giving the minimal data we need in order to have the simplest co-Frobenius coalgebra structure in a finite-dimensional vector space. We leave the proof of the next proposition to the reader.

Proposition 4. *Let $\Bbbk$ be a field and A be a finite-dimensional $\Bbbk$-vector space with basis $\mathcal{B} = \{b_0, b_1, b_2, \ldots, b_n\}$. Let $\varepsilon : \mathcal{B} \to \Bbbk$ be such that $\varepsilon(b_i) \neq 0$, $\forall i \in I$.*

(1) *If we define*

$$\Delta(b_i) = \frac{1}{\varepsilon(b_i)} b_i \otimes b_i, \quad \forall i \in I, \qquad t = b_0 + b_1 + b_2 + \cdots + b_n$$

and we extend Δ and ε to A by linearity, then $(A, \Delta, \varepsilon, t)$ is a co-Frobenius coalgebra.

(2) *Assume, moreover, that $\varepsilon(b_0) = 1$. Then, with respect to the structure Δ given above, b_0 is a group-like element of A and the preimage $\phi \in A^*$ of b_0 by $t \leftharpoonup$ satisfies $\phi(b_i) = \delta_{i,0}$, $\forall i \in I$.*

Assume, moreover, that we have an algebra structure in the vector space A given, in terms of the structure constants, by

$$b_i b_j = \sum_{k \in I} p_{ij}^k b_k, \quad \forall i, j \in I, \qquad 1 = b_0.$$

Notice that the fact that $b_0 = 1$ forces the cointegral $\phi \in A^*$ to be as defined in Proposition 4(2). Next proposition presents necessary and sufficient conditions on the coefficients p_{ij}^k in order to make the map $S : A \to A$ defined by the formula $S(x) = \sum \phi(t_1 x) t_2$ an antimorphism of algebras and of coalgebras. Notice that

$$S(b_j) = \sum_{i \in I} \frac{1}{\varepsilon(b_i)} \phi(b_i b_j) b_i = \sum_{i \in I} \frac{1}{\varepsilon(b_i)} p_{ij}^0 b_i.$$

If $\sigma : I \to I$ is a permutation, we say that S extends σ if $S(b_i) = b_{\sigma(i)}$, $\forall i \in I$.

Proposition 5. *Let $\Bbbk$ be a field and A be a finite-dimensional $\Bbbk$-algebra with basis $\mathcal{B} = \{b_0, b_1, \ldots, b_n\}$ and $b_0 = 1$. Let p_{ij}^k, $i, j, k \in I$, be the structure constants associated to the algebra A in the basis $\mathcal{B}$. Take $\varepsilon : \mathcal{B} \to \Bbbk$ to be a function that is never 0 and $\sigma : I \to I$ a permutation of I and endow A with the comultiplication defined before. Then:*

(1) *S extends σ if and only if $p_{ij}^0 = \varepsilon(b_i)\delta_{i,\sigma(j)}$.*

(2) *Suppose* (1) *holds. Then*:

(a) *S is an antimorphism of coalgebras if and only if $\varepsilon(b_{\sigma(i)}) = \varepsilon(b_i)$, $\forall i \in I$.*
(b) *S is an antimorphism of algebras if and only if*

$$\sigma(0) = 0 \quad \text{and} \quad p_{ij}^{k} = p_{\sigma(j)\sigma(i)}^{\sigma(k)}.$$

(3) *If S is an antimorphism of algebras and coalgebras that extends σ, then $\sigma^2 = \mathrm{id}$.*

Proof. (1) We know that $S(b_j) = \sum_{i\in I} \frac{1}{\varepsilon(b_i)} p_{ij}^0 b_i$. As $\mathcal{B}$ is linearly independent, the condition $S(b_j) = b_{\sigma(j)}$, $\forall j \in I$, is satisfied if and only if $\frac{1}{\varepsilon(b_i)} p_{ij}^0 = \delta_{i,\sigma(j)}$, which is equivalent to $p_{ij}^0 = \varepsilon(b_i)\delta_{i,\sigma(j)}$.

(2) We assume now that $S(b_i) = b_{\sigma(i)}$, $\forall i \in I$.

(a) We have that $\frac{1}{\varepsilon(b_{\sigma(i)})} b_{\sigma(i)} \otimes b_{\sigma(i)} = \Delta(b_{\sigma(i)}) = \Delta(S(b_i))$ and $(S \otimes S)(\Delta(b_i)) = \frac{1}{\varepsilon(b_i)} b_{\sigma(i)} \otimes b_{\sigma(i)}$. Then S is anticommutes with Δ if and only if $\varepsilon(b_{\sigma(i)}) = \varepsilon(b_i)$, $\forall i \in I$. Notice that this implies that S is counital, i.e. that $\varepsilon \circ S = \varepsilon$.
(b) S is an antimorphism of algebras if and only if $S(1) = 1$ and $S(xy) = S(y)S(x)$, $\forall x, y \in A$. But $S(1) = 1$ if and only if $S(b_0) = b_0$, which means, in terms of σ, that $b_{\sigma(0)} = b_0$, i.e. $\sigma(0) = 0$. On the other hand, $S(xy) = S(y)S(x)$, $\forall x, y \in A$ if and only if $S(b_i b_j) = S(b_j)S(b_i)$, $\forall i, j \in I$. Writing this equality in terms of the structure constants, we obtain $S(\sum_{k\in I} p_{ij}^k b_k) = b_{\sigma(j)} b_{\sigma(i)}$, $\forall i, j \in I$ and then $\sum_{k\in I} p_{ij}^k b_{\sigma(k)} = \sum_{k\in I} p_{\sigma(j)\sigma(i)}^k b_k$, $\forall i, j \in I$. Then $\sum_{k\in I} p_{ij}^k b_{\sigma(k)} = \sum_{k\in I} p_{\sigma(j)\sigma(i)}^{\sigma(k)} b_{\sigma(k)}$, $\forall i, j \in I$, i.e. $p_{ij}^k = p_{\sigma(j)\sigma(i)}^{\sigma(k)}$, $\forall i, j, k \in I$.

(3) Assume that S is an antimorphism of algebras and coalgebras that extends σ. Observe that, using (1) and (2), we have that $\forall i \in I$: $\varepsilon(b_{\sigma(i)}) = \varepsilon(b_{\sigma(i)})\delta_{\sigma(i),\sigma(i)} = p_{\sigma(i)i}^0 = p_{\sigma(i)\sigma^2(i)}^{\sigma(0)} = p_{\sigma(i)\sigma^2(i)}^0 = \varepsilon(b_{\sigma(i)})\delta_{\sigma(i)\sigma^3(i)}$, hence $\sigma^3(i) = \sigma(i)$, $\forall i \in I$, and, as σ is bijective, $\sigma^2(i) = i$, $\forall i \in I$. □

Now we are able to summarize all the conditions required in order to have in the vector space A a structure of bi-Frobenius algebra. The following theorem will follow directly from Propositions 4 and 5.

Theorem 1. *Let $\Bbbk$ be a field and A be a finite-dimensional $\Bbbk$-vector space with basis $\mathcal{B} = \{b_0, b_1, \ldots, b_n\}$. Let $p : A \otimes A \to A$ be a linear map with structure constants $\{p_{ij}^k \mid i, j, k \in I\}$ with respect to $\mathcal{B}$. Let $\varepsilon : \mathcal{B} \to \Bbbk$ be a function and $\sigma : I \to I$ be a permutation, where $I = \{0, 1, 2, \ldots, n\}$. If we have that*

(GL1) *p is associative*,	(GL2) *p has neutral element $b_0 = 1$*,
(GL3) $\varepsilon(b_i) \neq 0$, $\forall i \in I$,	(GL4) $\varepsilon(b_{\sigma(i)}) = \varepsilon(b_i)$, $\forall i \in I$,
(GL5) $p_{ij}^0 = \delta_{i,\sigma(j)}\varepsilon(b_i)$,	(GL6) $p_{ij}^k = p_{\sigma(j)\sigma(i)}^{\sigma(k)}$,
(GL7) $\sigma(0) = 0$,	(GL8) *ε extends to a morphism of algebras $\varepsilon : A \to \Bbbk$*,

then A has a structure of bi-Frobenius algebra with Δ, ϕ and t defined as before.

Proof. Notice that the conditions (GL1) to (GL8) guarantee, following Propositions 4 and 5, that A is a Frobenius algebra, A is a co-Frobenius coalgebra and that the induced map S is an antimorphism of algebras and coalgebras. Notice that $\varepsilon : A \to \Bbbk$ is a morphism of algebras (condition (GL8)). Moreover, if (GL8) holds, then $\varepsilon(1) = 1$, hence $\Delta(1) = \Delta(b_0) = \frac{1}{\varepsilon(b_0)} b_0 \otimes b_0 = 1 \otimes 1$ and then 1 is a group-like element for A. $\square$

Now we present the definition of a group-like algebra (see [D]).

Definition 4. A $\Bbbk$-*group-like algebra* is a 5-tuple $(A, \mathcal{B}, p, \varepsilon, \sigma)$ such that A is a $\Bbbk$-algebra with basis $\mathcal{B} = \{b_0, b_1, \ldots, b_n\}$, $p : A \otimes A \to A$ is a linear map, $\varepsilon : \mathcal{B} \to k$ is a function and $\sigma : I \to I$ is a bijection, such that conditions (GL1) to (GL8) are satisfied.

Corollary 1. *Every group-like algebra is a cocommutative bi-Frobenius algebra with* $S^2 = \mathrm{id}$.

Next we will exhibit the additional conditions needed in order to have a Hopf algebra structure in A.

Proposition 6. *Let* $\mathbf{A} = (A, \mathcal{B}, p, \varepsilon, \sigma)$ *be a* $\Bbbk$*-group-like algebra. Then*:

(1) $S * \mathrm{id} = u\varepsilon$ *if and only if* $\mathcal{B}$ *is closed under the operation of taking inverses with respect to* p.
(2) *The following conditions for* $\mathbf{A}$ *are equivalent*:
  (i) $\mathbf{A}$ *is a Hopf algebra,*
  (ii) $\mathbf{A}$ *is a bialgebra,*
  (iii) $(\mathcal{B}, p, 1)$ *is a monoid,*
  (iv) $(\mathcal{B}, p, 1)$ *is a group.*
(3) *In any of the previous situations we have that* $\varepsilon(b_i) = 1$, $\forall i \in I$.

Proof. (1) If $S * \mathrm{id} = u\varepsilon$, then $(S * \mathrm{id})(b_i) = \varepsilon(b_i) b_0$, $\forall i \in I$. Then we have

$$\frac{1}{\varepsilon(b_i)} b_{\sigma(i)} b_i = \varepsilon(b_i) b_0,$$

and $\frac{1}{\varepsilon(b_i)} p^0_{\sigma(i)i} = \varepsilon(b_i)$. Then $p^0_{\sigma(i)i} = (\varepsilon(b_i))^2$. But conditions (GL5) and (GL4) give $p^0_{\sigma(i)i} = \varepsilon(b_{\sigma(i)}) = \varepsilon(b_i)$, so using condition (GL3) we get that $\forall i \in I$: $\varepsilon(b_i) = 1$. Then we have that $b_{\sigma(i)} b_i = b_0 = 1$ (and also that $b_i b_{\sigma(i)} = b_0$, since A is finite dimensional), which means $b_i^{-1} = b_{\sigma(i)}$.

For the converse direction, assume b_i has an inverse in $\mathcal{B}$. From the condition (GL5) we deduce that the inverse of b_i has to be $b_{\sigma(i)}$. From the equality $b_{\sigma(i)} b_i = b_0$ and conditions (GL8) and (GL4) we get $\varepsilon(b_i)^2 = \varepsilon(b_0) = 1$, hence $\varepsilon(b_i) \in \{1, -1\}$, which implies $\frac{1}{\varepsilon(b_i)} = \varepsilon(b_i)$. We get that $\frac{1}{\varepsilon(b_i)} b_{\sigma(i)} b_i = \varepsilon(b_i) b_0 = \varepsilon(b_i) 1$, $\forall i \in I$, and hence, the equality $S * \mathrm{id} = u\varepsilon$ holds in the basis $\mathcal{B}$ and then in $\mathbf{A}$.

(2) The fact that $\mathbf{A}$ is a Hopf algebra if and only if it is a bialgebra, follows directly from Proposition 3. Hence (i) $\leftrightarrow$ (ii). Suppose now that $\mathbf{A}$ is a bialgebra. We know, by Proposition 3 that $S * \mathrm{id} = u\varepsilon$. So, by the proof of (1), we get that $\varepsilon(b_i) = 1$, $\forall i \in I$.

As $\mathbf{A}$ is a bialgebra, $\forall i, j \in I$, we have that $\Delta(b_i b_j) = \Delta(b_i)\Delta(b_j)$, which means, in terms of the structure constants,

$$\sum_{k \in I} p^k_{ij} (b_k \otimes b_k) = \sum_{k,l \in I} p^k_{ij} p^l_{ij} (b_k \otimes b_l), \quad \forall i, j \in I.$$

Then, $p_{ij}^k p_{ij}^l = 0$ for each pair (k,l) such that $k \neq l$, which means that given $i, j \in I$ there is $k_0 \in I$ such that $p_{ij}^k = 0$, $\forall k \neq k_0$. Moreover, using the bialgebra condition above for $k = l = k_0$, we get that $p_{ij}^{k_0} = (p_{ij}^{k_0})^2$, so that $p_{ij}^{k_0} \in \{0, 1\}$. If $p_{ij}^{k_0} = 0$, we would have $b_i.b_j = 0$ and so, using condition (GL8), $\varepsilon(b_i)\varepsilon(b_j) = 0$ which contradicts condition (GL3).

We conclude that $p_{ij}^{k_0} = 1$ and then $b_i.b_j = b_{k_0}$.

Conversely, suppose $\mathcal{B}$ is closed under p. Then for each pair $(i, j) \in I \times I$ there is $k \in I$ such that $b_i.b_j = b_k$. Applying Δ we obtain that

$$\begin{aligned}\Delta(b_i b_j) = \Delta(b_k) = \frac{1}{\varepsilon(b_k)} b_k \otimes b_k &= \frac{1}{\varepsilon(b_i)\varepsilon(b_j)}(b_i b_j \otimes b_i b_j) \\ &= \left(\frac{1}{\varepsilon(b_i)} b_i \otimes b_i\right)\left(\frac{1}{\varepsilon(b_j)} b_j \otimes b_j\right) = \Delta(b_i)\Delta(b_j)\end{aligned}$$

and then we have proved (ii) $\leftrightarrow$ (iii). (Notice we have used condition (GL8).) Finally, as $\mathbf{A}$ is a Hopf algebra if and only if $\mathbf{A}$ is a bialgebra and $S * \mathrm{id} = u\varepsilon$, and these two conditions hold if and only if $(\mathcal{B}, p, 1)$ is a monoid closed under taking inverses, i.e. if $(\mathcal{B}, p, 1)$ is a group, we have proved (i) $\leftrightarrow$ (iv).

(3) It is clear from the proof of (1) and (2) that in both cases $\varepsilon(b_i) = 1$, $\forall i \in I$. $\square$

In particular, we have the following result.

Corollary 2. *If a group-like algebra $\mathbf{A} = (A, \mathcal{B}, p, \varepsilon, \sigma)$ is Hopf, then $(\mathcal{B}, p, 1)$ is a group, σ is the inverse in $\mathcal{B}$ and $\mathbf{A}$ is the group algebra $\Bbbk\mathcal{B}$.*

4. The matricial approach

In the next theorem we give a description in terms of square matrices of the family of group-like algebras with the property that $S * \mathrm{id} = u\varepsilon$. In order to do this, we need some previous definitions. From now on, we denote by w both a vector in $\Bbbk^{n+1}$ and its transpose. Also $\{e_0, e_1, \ldots, e_n\}$ is the canonical basis of $\Bbbk^{n+1}$.

Definition 5. Let $\sigma : I \to I$ be a permutation and $v = (1, 1, \ldots, 1) \in \Bbbk^{n+1}$.

- A (I, σ)-*system* is a family of invertible matrices $\mathcal{F} = \{F_i \mid i \in I\} \subseteq M_{n+1}(\Bbbk)$ satisfying:

$$F_{\sigma(i)} = F_i^t = F_i^{-1}, \quad \forall i \in I, \qquad F_i v = v, \quad \forall i \in I, \qquad F_i e_0 = e_i, \quad \forall i \in I.$$

- If $\mathcal{F}$ and $\mathcal{G}$ are two systems, we say that they are *compatible* if $F_i G_j = G_j F_i$, $\forall i, j \in I$.

Lemma 1. *If $\mathcal{F}$ and $\mathcal{G}$ are compatible (I, σ)-systems, then*

$$F_0 = G_0 = \mathrm{Id} \quad \textit{and} \quad (F_i)_{kj} = (G_j)_{ki}.$$

Proof. Observe that $F_0 e_i = F_0 G_i e_0 = G_i F_0 e_0 = G_i e_0 = e_i$.

On the other hand, we have that $F_i e_j = F_i G_j e_0 = G_j F_i e_0 = G_j e_i$. Therefore

$$(F_i)_{kj} = (F_i e_j)_k = (G_j e_i)_k = (G_j)_{ki}$$

and the proof is finished. $\square$

Notice that we can consider, without loss of generality, group-like algebras whose supporting vector space is $\mathbb{k}^{n+1}$. We will do so from now on, letting $I = \{0, 1, 2, \ldots, n\}$, $\mathcal{C} = \{e_0, e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{k}^{n+1}$ and for each $i, j, k \in I$, $p_{ij}^k \in \mathbb{k}$.

What follows is the framework needed in order to formulate next theorem. We define

- $p : \mathbb{k}^{n+1} \otimes \mathbb{k}^{n+1} \to \mathbb{k}^{n+1}$, $p(e_i \otimes e_j) = \sum_{k \in I} p_{ij}^k e_k$,
- $\mathbf{1} : \mathbb{k}^{n+1} \to \mathbb{k}$, $\mathbf{1}(e_i) = 1$, $\forall i \in I$,
- $\mathcal{H} = \{H_i \mid i \in I\}$, where $H_i \in M_{n+1}(\mathbb{k})$ is defined by $(H_i)_{kj} = p_{ij}^k$, $\forall i, j, k \in I$,
- $\mathcal{V} = \{V_i \mid i \in I\}$, where $V_i \in M_{n+1}(\mathbb{k})$ is defined by $(V_j)_{ki} = p_{ij}^k$, $\forall i, j, k \in I$.

Remark 5. Notice that:

$$\begin{cases} H_i \text{ is the matrix associated to the map } e_i.\_ : A \to A \text{ in the canonical basis,} \\ V_j \text{ is the matrix associated to the map } \_.e_j : A \to A \text{ in the canonical basis.} \end{cases}$$

Theorem 2. *In the context defined above, if $\sigma : I \to I$ is an arbitrary permutation, we have that $(k^{n+1}, \mathcal{C}, p, \mathbf{1}, \sigma)$ is a group-like algebra with $S * \mathrm{id} = u\varepsilon$ if and only if $\mathcal{H}$ and $\mathcal{V}$ are compatible (I, σ)-systems and $\sigma^2 = \mathrm{id}$.*

Proof. Assume first that $(\mathbb{k}^{n+1}, \mathcal{C}, p, \mathbf{1}, \sigma)$ is a group-like algebra. We know that $\sigma^2 = \mathrm{id}$ (Corollary 2). By Remark 5, it is clear that $H_i e_0 = V_i e_0 = e_i$, $\forall i \in I$, and also that $H_{\sigma(i)} = H_i^{-1}$ and $V_{\sigma(i)} = V_i^{-1}$ (since p is associative and $S * \mathrm{id} = u\varepsilon$ implies that $e_{\sigma(i)} e_i = e_0 = e_i e_{\sigma(i)}$, $\forall i \in I$).

As we have that

$$\phi\big(e_{\sigma(k)}(e_i e_j)\big) = \phi\big((e_{\sigma(k)} e_i) e_j\big), \quad \forall, i, j, k \in I,$$

we deduce that $\sum_{l \in I} p_{\sigma(k)l}^0 p_{ij}^l = \sum_{l \in I} p_{\sigma(k)i}^l p_{lj}^0$, which implies, by condition (GL5) (and the fact that $\sigma^2 = \mathrm{id}$), that

$$p_{ij}^k = p_{\sigma(k)i}^{\sigma(j)}, \quad \forall i, j, k \in I.$$

Using condition (GL6), we get that $p_{ij}^k = p_{\sigma(i)k}^j$, $\forall i, j, k \in I$, which means that $H_i^t = H_{\sigma(i)}$. Similarly, we prove that $V_i^t = V_{\sigma(i)}$.

Now,

$$H_i v = \left(\sum_{j \in I} p_{ij}^0, \sum_{j \in I} p_{ij}^1, \ldots, \sum_{j \in I} p_{ij}^n\right) = \left(\sum_{j \in I} p_{\sigma(i)0}^j, \sum_{j \in I} p_{\sigma(i)1}^j, \ldots, \sum_{j \in I} p_{\sigma(i)n}^j\right) = v,$$

where last equality follows from $\varepsilon(b_{\sigma(i)} b_0) = \varepsilon(b_{\sigma(i)}) \varepsilon(b_0) = 1$ (condition (GL8)). In a similar way we prove that $V_i v = v$, $\forall i \in I$.

We have already proved that the families $\mathcal{H}$ and $\mathcal{V}$ are (I,σ)-systems. We still have to prove that they are compatible. For this, we use the associativity of p, i.e. $(e_ie_k)e_j = e_i(e_ke_j)$. If we put this equality in terms of the associated matrices we get

$$H_iV_j = V_jH_i$$

and we are done.

Let us prove the converse, so assume that $\mathcal{H}$ and $\mathcal{V}$ are (I,σ)-compatible systems. We have to check conditions (GL1) to (GL8) of Theorem 1 and also that $S * \mathrm{id} = u\varepsilon$.

The associativity follows from $V_jH_i = H_iV_j$, $\forall i, j \in I$, so we have (GL1).

By Lemma 1, we have that $H_0 = V_0 = \mathrm{Id}$ and this implies, by Remark 5, that $e_0 = 1$ is the neutral element of the algebra, i.e. (GL2).

Conditions (GL3) and (GL4) hold trivially, since $\varepsilon = \mathbf{1}$.

As $H_{\sigma(i)} = H_i^t$, we have that

$$p_{ij}^0 = (H_i)_{0j} = (H_{\sigma(i)})_{j0} = p_{\sigma(i)0}^j = (V_0)_{j\sigma(i)} = \delta_{\sigma(i),j},$$

so condition (GL5) holds (notice we have used $\sigma^2 = \mathrm{id}$ and $\varepsilon = \mathbf{1}$).

Now, using first that $H_j^t = H_{\sigma(j)}$, then that $V_k^t = V_{\sigma(k)}$, and finally that $H_i^t = H_{\sigma(i)}$, we deduce that

$$p_{\sigma(j)\sigma(i)}^{\sigma(k)} = p_{j\sigma(k)}^{\sigma(i)} = p_{\sigma(i)k}^j = p_{ij}^k,$$

i.e. condition (GL6).

Let us prove condition (GL7), i.e. $\sigma(0) = 0$. We have that $H_{\sigma(0)} = H_0^t = \mathrm{Id}$, but notice that $H_i \neq \mathrm{Id}$, $\forall i \neq 0$, since $H_ie_0 = e_i$, so we get $\sigma(0) = 0$.

To prove that $\varepsilon = \mathbf{1}$ is a morphism of algebras, it is enough to verify that $\forall i, j \in I$, $\varepsilon(e_ie_j) = \varepsilon(e_i)\varepsilon(e_j)$. But $H_{\sigma(i)} = H_i^t$ and $H_iv = v$, $\forall i \in I$, imply that

$$\sum_{k\in I} p_{ij}^k = \sum_{k\in I} p_{\sigma(i)k}^j = 1, \quad \forall i, j \in I,$$

i.e. $\varepsilon(e_ie_j) = \sum_{k\in I} p_{ij}^k = 1 = \varepsilon(e_i)\varepsilon(e_j)$.

It remains to prove that $S * \mathrm{id} = u\varepsilon$. This can be deduced directly from associativity and the fact that $H_{\sigma(i)} = H_i^{-1}$. □

Corollary 3. *Let (I,σ) be as in Definition* 5 *and $\mathcal{C}$ be the canonical basis of $\mathbb{k}^{n+1}$.*

(1) *There is a one to one correspondence between group-like algebras of the form $\mathbf{A} = (\mathbb{k}^{n+1}, \mathcal{C}, p, \varepsilon, \sigma)$ satisfying $S * \mathrm{id} = u\varepsilon$ and pairs of compatible (I,σ)-systems.*
(2) *Let $\mathbf{A}$ be as described in* (1) *and $(\mathcal{H}, \mathcal{V})$ its corresponding pair of compatible (I,σ)-systems. Then $\mathbf{A}$ is a Hopf algebra if and only if the family $\mathcal{H}$ is a group with the usual matrix product* (*and this happens if and only if the family $\mathcal{V}$ is a group with the usual matrix product*).

Proof. (1) We recall that in Proposition 6 we proved that in group-like algebras with $S * \mathrm{id} = u\varepsilon$ it holds that $\varepsilon = \mathbf{1}$. It follows from Theorem 2 that we can define correspondences

$$\big(\mathbb{k}^{n+1}, \mathcal{C}, p, \mathbf{1}, \sigma\big) \mapsto (\mathcal{H}, \mathcal{V}) \quad \text{with } (H_i)_{kj} = p_{ij}^k = (V_j)_{ki},$$

$$(\mathcal{H}, \mathcal{V}) \mapsto (\mathbb{k}^{n+1}, \mathcal{C}, p, \mathbf{1}, \sigma), \quad \text{where}$$

$$p : \mathbb{k}^{n+1} \otimes \mathbb{k}^{n+1} \to \mathbb{k}^{n+1} \text{ is given by } p_{ij}^k = (H_i)_{kj}$$

$$\text{(its structure constants with respect to } \mathcal{C}\text{).}$$

Lemma 1 guarantees that these correspondences induce a bijection (more specifically, that the first map is surjective).

(2) By Proposition 6, **A** is a Hopf algebra if and only if $\mathcal{B}$ is closed under p if and only if $\forall i, j \in I$, there is $k \in I$ such that $e_i e_j = e_k$ if and only if $\forall i, j \in I$, there is $k \in I$ such that $H_i H_j = H_k$ (or $V_i V_j = V_k$) and we are done. □

5. Hadamard matrices

The Hadamard maximal determinant problem is the following: find the matrices of a given size with entries $+1$ and -1 with the largest (in absolute value) possible determinant. Despite well over a century of work, beginning with Sylvester's results of 1867, the general question remains unanswered. However, it is known that Hadamard matrices are solutions to Hadamard's maximal n^2-determinant problem.

Definition 6. A Hadamard matrix of size n is a square matrix (of size $n \in \mathbb{N}$) with coefficients in $\{1, -1\}$ that is orthogonal (with the usual inner product of $\mathbb{C}^n$) and such that the first row and the first column are both the vector $v = (1, 1, \dots, 1)$.

Remark 6. Sometimes the condition concerning the first column and the first row (called the normalization condition) is not required in the definition of a Hadamard matrix. However, if we multiply by -1 any row or any column of an orthogonal matrix with coefficients in $\{-1, 1\}$, we still get an orthogonal matrix. In this sense, any "generalized" Hadamard matrix is equivalent to a normalized one.

It is easy to check that if **P** is a Hadamard matrix of size n then $n = 1, 2$ or a positive multiple of 4. Also, it is clear that defining

$$\mathbf{P}_1 = (1), \qquad \mathbf{P}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{P}_k = \begin{pmatrix} \mathbf{P}_{k-1} & \mathbf{P}_{k-1} \\ \mathbf{P}_{k-1} & -\mathbf{P}_{k-1} \end{pmatrix}$$

we obtain Hadamard matrices of size 2^k.

For the purposes of the applications to group-like algebras, it will be relevant whether there exist or not Hadamard matrices whose size is not a power of 2. The answer is negative: it has been proved that there are Hadamard matrices of unbounded size n where n is not a power of 2. Indeed, Paley's construction (see [W]) guarantees that there exist Hadamard matrices of size n, for every n divisible by 4 and of the form $2^e(p^m+1)$, with $m, e, p \in \mathbb{N}$, $m \neq 0$, p prime, $p \neq 2$. We give below an example of a Hadamard matrix of size 12 (corresponding to $p=5$, $e=m=1$):

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \mathbf{-1} & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & 1 & 1 & 1 & \mathbf{-1} & 1 \\
1 & 1 & \mathbf{-1} & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & 1 & 1 & 1 & \mathbf{-1} \\
1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & 1 & 1 & 1 \\
1 & 1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & 1 & 1 \\
1 & 1 & 1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & 1 \\
1 & 1 & 1 & 1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\
1 & \mathbf{-1} & 1 & 1 & 1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} \\
1 & \mathbf{-1} & \mathbf{-1} & 1 & 1 & 1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & 1 & \mathbf{-1} \\
1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & 1 & 1 & 1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & 1 \\
1 & 1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & 1 & 1 & 1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} \\
1 & \mathbf{-1} & 1 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & 1 & 1 & 1 & \mathbf{-1} & 1 & \mathbf{-1}
\end{pmatrix}.$$

6. The case $S = \mathrm{id}$

In this section, we consider the particular case of group-like algebras supported in $\mathbb{k}^{n+1}$ with $S * \mathrm{id} = u\varepsilon$, where $S = \mathrm{id}$. They are of the form

$$\mathbf{A} = \left(\mathbb{k}^{n+1}, \mathcal{C}, p, \mathbf{1}, \mathrm{id}\right), \quad \text{satisfying also that } e_i^2 = 1,\ \forall i \in I.$$

This special type of group-like algebras are commutative, since $\mathrm{id} : \mathbf{A} \to \mathbf{A}$ is an antimorphism of algebras.

This section is divided in three main parts:

- First we consider the correspondence described in Corollary 3 for this particular case. In other words, we describe all possible pairs of (I, id)-compatible systems.
- Second, we show that, in the case $\mathbb{k} \subseteq \mathbb{R}$, this family of all pairs of (I, id)-compatible systems is in one to one correspondence with the family of Hadamard matrices.
- Finally, we use some results on Hadamard matrices (mentioned in Section 4) to find examples of group-like algebras satisfying $S * \mathrm{id} = u\varepsilon$ that are not Hopf algebras.

Suppose we have a group-like algebra $\mathbf{A} = (\mathbb{k}^{n+1}, \mathcal{C}, p, \mathbf{1}, \mathrm{id})$ with $e_i^2 = 1$, $\forall i \in I$, and consider the bijection described in Corollary 3. Observe that, as $\mathbf{A}$ is commutative, we have that $p_{ij}^k = p_{ji}^k$, $\forall i, j, k \in I$ and therefore

$$H_i = V_i, \quad \forall i \in I,$$

meaning that the compatible pair associated to $\mathbf{A}$ is $(\mathcal{H}, \mathcal{H})$, so it can be thought as a single (I, id)-system, with the additional condition that

$$H_i H_j = H_j H_i, \quad \forall i, j \in I.$$

This can be summarized as follows:

Definition 7. Let $\mathcal{H}$ be a (I, id)-system. We say that $\mathcal{H}$ is *self-compatible* if $H_i H_j = H_j H_i$, $\forall i, j \in I$. Explicitly the set $\{H_0, H_1, \ldots, H_n\}$ satisfies

$$H_i = H_i^{-1} = H_i^t, \qquad H_i H_j = H_j H_i, \qquad H_i v = v, \qquad H_i e_0 = e_i,$$

where $i, j \in I$, $v = (1, 1, \ldots, 1)$.

Theorem 3. *Let $\Bbbk$ be a field. There is a one to one correspondence between group-like algebras of the form $(\Bbbk^{n+1}, \mathcal{C}, p, \mathbf{1}, \mathrm{id})$ satisfying also that $e_i^2 = 1$, $\forall i \in I$, and self-compatible (I, id)-systems.*

Next we describe, provided some minor restrictions on $\Bbbk$, a correspondence between (I, id)-self-compatible systems and Hadamard matrices.

In the rest of the paper all our systems will be taken over (I, id).

Theorem 4. *Let $\Bbbk$ be a field and $n \in \mathbb{N}$. Then*:

(1) *If $\Bbbk \subseteq \mathbb{R}$, the matrices of a self-compatible system $\mathcal{H} = \{H_0, H_1, \ldots, H_n\}$ admit a basis of common eigenvectors $\{v^0, v^1, \ldots, v^n\}$ such that the matrix*

$$\mathbf{P} = \left(v^0 \middle| v^1 \middle| \cdots \middle| v^n\right)$$

is a Hadamard matrix.
Moreover, we have that

$$H_i = \mathbf{P} D_i \mathbf{P}^{-1},$$

where D_i is a diagonal matrix whose diagonal is the ith-row of $\mathbf{P}$. (Explicitly $(D_i)_{rs} = \delta_{r,s} \mathbf{P}_{ir}$.)

(2) *Conversely, assume now that* $\operatorname{char} \Bbbk$ *does not divide $n + 1$. Then, given a Hadamard matrix $\mathbf{P}$, we can construct a self-compatible system $\mathcal{H} = \{H_i \mid i \in I\}$ such that the columns of $\mathbf{P}$ form a common basis of eigenvectors for the matrices of $\mathcal{H}$.*

Proof. (1) First notice that, as $H_i^2 = \mathrm{Id}$, each H_i is equivalent to a diagonal matrix with diagonal entries in $\{1, -1\}$. Denote as $\langle\, , \rangle$ the standard inner product in $\Bbbk^{n+1}$ (we recall that $\Bbbk \subseteq \mathbb{R}$). We have also that $H_i H_j = H_j H_i$, $\forall i, j \in I$, and that $H_i^t = H_i$, i.e. each H_i, $i \in I$, is symmetric. Therefore, we can obtain for the matrices in $\mathcal{H}$ a common basis $\mathcal{B} = \{v^0, v^1, \ldots, v^n\}$ of orthogonal eigenvectors.

As $v = (1, 1, \ldots, 1)$ is fixed for all H_i, $i \in I$, we can choose $v^0 = v$.

Let $w = (w_0, w_1, \ldots, w_n) \in \Bbbk^{n+1}$ be any common non-zero eigenvector.

Then, for each $i \in I$, $H_i w = w$ or $H_i w = -w$. But $H_i w = \sum_{k \in I} w_k (H_i e_k)$ and $(H_i e_k)_i = \langle e_i, H_i e_k \rangle = \langle H_i^t e_i, e_k \rangle = \langle H_i^{-1} e_i, e_k \rangle = \langle e_0, e_k \rangle = \delta_{0,k}$, hence $(H_i w)_i = w_0$ and therefore, as $(H_i w)_i = \pm w_i$,

$$\forall i \in I\colon \quad w_i = w_0 \quad \text{or} \quad w_i = -w_0.$$

Take w such that $w_0 = 1$. Writing $H_i w = \lambda_i w$ and computing the ith coordinate we deduce that $1 = w_0 = (H_i w)_i = \lambda_i w_i$ and therefore $\lambda_i = w_i$. In other words, for each i of H_i, the eigenvalue of H_i associated to w is w_i, i.e. $H_i w = w_i w$.

We can assume $(v^j)_0 = 1$ and still have an orthogonal basis of eigenvectors. Call $\mathbf{P}$ the matrix

$$\mathbf{P} = \left(v^0 \middle| v^1 \middle| \cdots \middle| v^n\right).$$

Then $\mathbf{P}$ has coefficients in $\{1, -1\}$ and

$$H_i = \mathbf{P} D_i \mathbf{P}^{-1},$$

where D_i is a diagonal matrix formed by the (ordered) eigenvalues of H_i. As the eigenvalue associated to v^j for H_i is $(v^j)_i$, we have that the ith-row of $\mathbf{P}$ gives all the eigenvalues, meaning the principal diagonal in D_i is the ith-row of $\mathbf{P}$.

Moreover, $\mathbf{P}$ is an orthogonal matrix with coefficients in $\{1, -1\}$ whose first row and first column are $v = (1, 1, \ldots, 1)$, i.e. $\mathbf{P}$ is a Hadamard matrix.

(2) Take a Hadamard matrix $\mathbf{P}$ of size $n + 1$.

It is clear that $\mathbf{P}$ is invertible since $\mathbf{P}\mathbf{P}^t = (n + 1)\mathrm{Id}$ and $\operatorname{char} \Bbbk$ does not divide $n + 1$. Let r_i be the ith-row of $\mathbf{P}$. Consider D_i the diagonal matrix where the principal diagonal is r_i and take

$$H_i = \mathbf{P} D_i \mathbf{P}^{-1}.$$

We have to show that $\mathcal{H} = \{H_i \mid i \in I\}$ is a self-compatible system. As $H_i = \frac{1}{n+1}\mathbf{P} D_i \mathbf{P}^t$, then

$$(H_i)^t = H_i,$$
$$H_i H_i^t = H_i^2 = \frac{1}{(n+1)^2}\mathbf{P} D_i \mathbf{P}^t \mathbf{P} D_i \mathbf{P}^t = \frac{1}{n+1}\mathbf{P}(D_i)^2\mathbf{P}^t = \frac{1}{n+1}\mathbf{P}\mathbf{P}^t = \mathrm{Id}.$$

Now, as $\mathbf{P} e_0 = \mathbf{P}^t e_0 = v$ and $\mathbf{P}^t v = (n + 1)e_0$, we have that, $\forall i \in I$,

$$H_i v = \frac{1}{n+1}\mathbf{P} D_i \mathbf{P}^t v = \frac{1}{n+1}\mathbf{P} D_i (n+1)e_0 = \mathbf{P} e_0 = v,$$
$$H_i e_0 = \frac{1}{n+1}\mathbf{P} D_i \mathbf{P}^t e_0 = \frac{1}{n+1}\mathbf{P} D_i v = \frac{1}{n+1}\mathbf{P} r_i = e_i.$$

We have proved that $\mathcal{H}$ is a system. It remains to prove that it is self-compatible, i.e. that the matrices in $\mathcal{H}$ commute with each other, but this follows from the fact that they are equivalent to a diagonal matrix via the same matrix $\mathbf{P}$. □

The fact that $\mathbf{P}$ and the D_i's have their entries in $\{0, 1, -1\} \subseteq \Bbbk$ gives the following result.

Corollary 4. *The entries of the matrices of a* (I, id)*-compatible system in* $\Bbbk \subseteq \mathbb{R}$ *are in fact in the prime field of* $\Bbbk$.

Remark 7. If $\mathbf{P}$ is a Hadamard matrix, the rows in $\mathbf{P}$ are self-inverses under the point-wise product.

On the other hand, the matrices in a (I, id)-compatible system are self-inverses under the usual matrix product.

Next theorem gives a stronger link between these two products.

Theorem 5. *Assume that* $\Bbbk \subseteq \mathbb{R}$.

Let $\mathcal{H}$ be a self-compatible system and $\mathbf{P}$ *a Hadamard matrix of common eigenvectors for $\mathcal{H}$. Then the matrices in $\mathcal{H}$ form a group* (*with the usual matrix product*) *if and only if the rows in* $\mathbf{P}$ *form a group* (*with the point-wise product*).

Proof. In view of Remark 7, we only have to prove that $\mathcal{H}$ is closed under the usual matrix product if and only if the set of rows in $\mathbf{P}$ is closed under the point-wise product. But this follows directly from

$$H_i H_j = H_k \quad \text{if and only if } D_i D_j = D_k \quad \text{if and only if } r_i r_j = r_k,$$

where r_i is the ith-row of $\mathbf{P}$. $\square$

Remark 8. The existence of an invertible Hadamard matrix of size $n+1$ implies that $\operatorname{char}\Bbbk$ does not divide $n+1$ (since the matrix is invertible) and in particular that $\operatorname{char}\Bbbk \neq 2$ (since the existence of a Hadamard matrix implies that $n+1$ is even).

Corollary 5. *If* $\operatorname{char}\Bbbk \neq 2$*, there are $\Bbbk$-bi-Frobenius algebras of unbounded dimension satisfying* $S * \mathrm{id} = u\varepsilon$ *that are not Hopf algebras.*

Proof. Take $p = \operatorname{char}\Bbbk \neq 2$ and $k \in \mathbb{N}$. As we observed before (cf. Section 5), it is known that there is a Hadamard matrix of size $n+1 = 2^e(p^m+1)$ bigger than k, where $e > 0$, $m \in \mathbb{N}$. It is clear that p does not divide $n+1$. So, we can take $\mathcal{H}$, the self-compatible system constructed from $\mathbf{P}$ as in Theorem 4. Let $\mathbf{A}$ be its associated group-like algebra as in Corollary 3. We claim that $\mathbf{A}$ is not a Hopf algebra.

If $\mathbf{A}$ were Hopf algebra, then, by Corollary 3, $\mathcal{H}$ is a group with the usual matrix-product and therefore, by Theorem 5, the rows of $\mathbf{P}$ would form a group. Then we would have an abelian finite group all whose elements haver order two and whose size $n+1$ is not a power of 2. This contradicts the structure theorem of finite abelian groups. Therefore $\mathbf{A}$ is not a Hopf algebra. $\square$

We finish this section by proving that the condition $(*)$ we considered in Remark 4 is weaker than the compatibility condition for bialgebras.

Proposition 7. *Let* $\mathbf{A} = (A, \mathcal{B}, p, \mathbf{1}, \sigma)$ *be a group-like algebra and* $t \neq 0$ *be a right integral for* $\mathbf{A}$. *Then* $S * \mathrm{id} = u\varepsilon$ *if and only if*

$$\forall x \in A, \quad \varepsilon(x) t_1 \otimes t_2 = \Delta(tx) = \sum t_1 x_1 \otimes t_2 x_2. \tag{$*$}$$

Proof. The converse implication was proved in Proposition 3.

For the direct implication, recall that any bi-Frobenius algebra A has the following property (cf. [D, Fact 1.2(4)]):

$$\sum x \bar{S}(t_2) \otimes t_1 = \sum \bar{S}(t_2) \otimes t_1 x, \quad \forall x \in A, \tag{$*$}$$

where $\bar{S}$ is the composite inverse of S. Indeed, this equality is equivalent, by applying the morphism $\mathrm{id} \otimes \psi \leftharpoonup : A \otimes A \to A \otimes A^*$ to $b\bar{S}(s_2)\psi(s_1 z) = \bar{S}(s_2)\psi(s_1 bz)$, $\forall z \in A$, or $b\bar{S}(Sz) = \bar{S}(S(bz))$, $\forall z \in A$. This last equality is obviously true.

Applying the linear isomorphism $\tau \circ (S \otimes \mathrm{id})$ to $(*)$, we can see that this equality is equivalent to $\sum t_1 \otimes t_2 S(x) = \sum t_1 x \otimes t_2$, $\forall x \in A$. In particular, for a group-like algebra,

$$\sum t_1 \otimes t_2 b_{\sigma(j)} = \sum t_1 b_j \otimes t_2, \quad \forall j \in \{0, 1, \dots, n\},$$

and hence, if $S * \mathrm{id} = u\varepsilon$, we get, multiplying by $1 \otimes b_j$ at both sides of this last equality, $\sum t_1 \otimes t_2 = \sum t_1 b_j \otimes t_2 b_j$, $\forall j \in \{0, 1, 2, \dots, n\}$, and we are done (since $\varepsilon(b_j) = 1$, $\forall j \in \{0, 1, 2, \dots, n\}$). $\square$

The following result follows directly from Corollary 5 and Proposition 7.

Corollary 6. *If* $\operatorname{char} \Bbbk \neq 2$, *there are* $\Bbbk$*-bi-Frobenius algebras of unbounded dimension satisfying* $\Delta(tx) = \sum t_1 x_1 \otimes t_2 x_2$ *for a right integral* $t \neq 0$ *and that are not bialgebras.*

7. On separability and coseparability of bi-Frobenius algebras

The notion of separability is classical in ring theory. Every separable algebra is semisimple. Indeed, a $\Bbbk$-algebra A is separable if and only if for any field extension $E \supseteq \Bbbk$, the E-algebra $A^E = A \otimes_{\Bbbk} E$ is semisimple.

We use an alternative definition—see Definition 8—of a separable algebra and give also the (dual) notion of a coseparable coalgebra (see [DMI,T]).

In this section, if A is an algebra and C is a coalgebra, we consider

- A with the usual structure of A–A bimodule,
- $A \otimes A$ with the structure of A–A bimodule given by $a(x \otimes y)b = ax \otimes yb$,
- C with the usual structure of C–C bicomodule,
- $C \otimes C$ with the structure of C–C bicomodule given by $\sum (x \otimes y)_{-1} \otimes (x \otimes y)_0 \otimes (x \otimes y)_1 = \sum x_1 \otimes x_2 \otimes y_1 \otimes y_2$.

It is evident that the product $m : A \otimes A \to A$ is an epimorphism of A–A bimodules and that the coproduct $\Delta : C \to C \otimes C$ is a monomorphism of C–C bicomodules.

Definition 8.

(1) A $\Bbbk$-algebra A is said to be *separable* if the product $m : A \otimes A \to A$ splits in the category of A–A bimodules.
(2) A $\Bbbk$-coalgebra C is said to be *coseparable* if the coproduct $\Delta : C \to C \otimes C$ splits in the category of C–C bicomodules.

The following result is well known and can be found for example in [S].

Theorem 6. *Let H be a finite-dimensional Hopf algebra and $t \in H$ be a non-zero right or left integral. Then the following statements are equivalent.*

(i) *H is semisimple.*
(ii) *H is separable.*
(iii) $\varepsilon(t) \neq 0$.

We show that this result and its dual version can be generalized to the context of bi-Frobenius algebras satisfying $S * \mathrm{id} = u\varepsilon$.

Theorem 7. *Let $(A, m, 1, \Delta, \varepsilon, t, \phi, S)$ be a bi-Frobenius algebra such that $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$. Then*:

(1) *The following assertions are equivalent*:
  (a) *A is semisimple.*
  (b) *A is separable.*
  (c) $\varepsilon(t) \neq 0$.
(2) *The following assertions are equivalent*:
  (a′) *A is cosemisimple.*
  (b′) *A is coseparable.*
  (c′) $\phi(1) \neq 0$.

Proof. (1) For (a) implies (c), take $\ker(\varepsilon) \subseteq A$ as a right A-submodule. As A is semisimple, there is a right submodule $I \subseteq A$ of dimension 1 such that $\ker(\varepsilon) \oplus I = A$. We write $1 = a + s$, with $\varepsilon(a) = 0$ and $s \in I$, then $I = \Bbbk s$ and $\varepsilon(s) = 1$. If $x \in A$, we have $sx \in I$ and then $sx = \lambda s$, for some $\lambda \in \Bbbk$. Applying ε we deduce that $\varepsilon(x) = \lambda$ and therefore $sx = \varepsilon(x)s$, which means that s is a right integral for A. Hence, $t = \mu s \in I$, for some $\mu \in \Bbbk$, $\mu \neq 0$, which implies $\varepsilon(t) \neq 0$.

For (c) implies (b), take $\bar{S}$ the composite inverse of the antipode S, $s = \frac{1}{\varepsilon(t)} t$ and $\psi \in A^*$ such that (s, ψ) is a bi-Frobenius pair. Consider $\delta : A \to A \otimes A$, $\delta(x) = \sum x\bar{S}(s_2) \otimes s_1$. From the equality $\sum x\bar{S}(s_2)s_1 = x$, it follows that δ splits m. We have to prove that δ is a morphism of A–A bimodules; in other words that $\sum axb\bar{S}(s_2) \otimes s_1 = \sum ax\bar{S}(s_2) \otimes s_1 b$. But this follows from

$$\sum b\bar{S}(s_2) \otimes s_1 = \sum \bar{S}(s_2) \otimes s_1 b, \quad \forall b \in A$$

(cf. the proof of Proposition 7).

It is a well-known fact that any separable algebra is semisimple. We show it anyway, in order to give a complete proof. Take δ the map that splits the product and put $\delta(1) = \sum_{i=1}^{l} r_i \otimes r^i$. We have $\sum r_i r^i = 1$ and $\sum a r_i \otimes r^i = a\delta(1) = \delta(a) = \delta(1)a = \sum r_i \otimes r^i a$.

Given $N \subseteq M$ an inclusion of A-modules, consider a $\Bbbk$-linear map $p : M \to N$ such that $p_{|N} = \mathrm{id}_N$. The map

$$\pi : M \to N,$$
$$\pi(x) = \sum_{i=1}^{l} p(x r_i) r^i$$

is a morphism of A-modules that splits the projection. Indeed, for $x \in N$, $xr_i \in N$, $\forall i = 1, 2, \dots, l$ and therefore $\pi(x) = \sum xr_i r^i = x \sum r_i r^i = x$.

Moreover, π is a morphism of right A-modules, since

$$\pi(xa) = \sum p(xar_i)r^i = \sum p(xr_i)r^i a = \pi(x)a.$$

(2) The proof of this equivalence is obtained by dualizing the methods described above. We only present a sketch in order to obtain explicit expressions for the morphisms involved.

The proof that (a′) implies (c′) is obtained from (1), working in A^*.

In the proof that (c′) implies (b′), in order to split Δ in the category of A–A bicomodules, we take the map $*: A \otimes A \to A$ given by $x * y = \sum \psi(Sxy_1)y_2$, where $\psi \in A^*$ is a right cointegral such that $\psi(1) = 1$.

Finally, to prove that (b′) implies (c′) we take a map $*: A \otimes A \to A$ that splits Δ. If $N \subseteq M$ is a right A-subcomodule and $p: M \to N$ is a linear map that splits the inclusion, we consider $\pi: M \to N$, $\pi(x) = \sum p(m_0)_0 \otimes \varepsilon(p(m_0)_1 \otimes m_1)$. □

For the particular case of group-like algebras with $S * \mathrm{id} = u\varepsilon$, the following version of Maschke's theorem can be easily deduced.

Corollary 7. *Every group-like algebra is coseparable.*

A group-like algebra with $S * \mathrm{id} = u\varepsilon$ is separable if and only if $\operatorname{char} \Bbbk$ *does not divide its dimension.*

Proof. Coseparability follows from $A = \bigoplus_{i=0}^{n} \langle b_i \rangle_{\Bbbk}$ as a coalgebra, where each $\Bbbk$ vector space generated by b_i is clearly a simple subcoalgebra of A.

For separability, we use theorem 7 and the fact that $\varepsilon(t) = \sum_{i=0}^{n} \varepsilon(b_i) = n + 1 \in \Bbbk$. □

8. An explicit computation

To finish, we give the explicit algebra structure of the group-like algebra of dimension 12 obtained from the Hadamard matrix of size 12 thas has been presented in Section 5. Observe that here $\Bbbk$ can be any field such that $\operatorname{char} \Bbbk \notin \{2, 3\}$ (the prime divisors of 12).

It has a linear basis $\{b_0, b_1, b_2, \dots, b_{11}\}$, where $b_0 = 1$, $b_i^2 = 1$, $\forall i \in \{1, 2, 3, \dots, 11\}$, it is commutative and

$$b_1 b_2 = \tfrac{1}{3}(b_3 - b_4 - b_5 + b_6 + b_7 - b_8 + b_9 + b_{10} + b_{11}),$$

$$b_1 b_3 = \tfrac{1}{3}(b_2 + b_4 + b_5 - b_6 + b_7 - b_8 + b_9 + b_{10} - b_{11}),$$

$$b_1 b_4 = \tfrac{1}{3}(-b_2 + b_3 + b_5 + b_6 + b_7 + b_8 + b_9 - b_{10} - b_{11}),$$

$$b_1 b_5 = \tfrac{1}{3}(-b_2 + b_3 + b_4 - b_6 - b_7 + b_8 + b_9 + b_{10} + b_{11}),$$

$$b_1 b_6 = \tfrac{1}{3}(b_2 - b_3 + b_4 - b_5 + b_7 + b_8 + b_9 - b_{10} + b_{11}),$$

$$b_1 b_7 = \tfrac{1}{3}(b_2 + b_3 + b_4 - b_5 + b_6 + b_8 - b_9 + b_{10} - b_{11}),$$

$$b_1 b_8 = \tfrac{1}{3}(-b_2 - b_3 + b_4 + b_5 + b_6 + b_7 - b_9 + b_{10} + b_{11}),$$

$$b_1 b_9 = \tfrac{1}{3}(b_2 + b_3 + b_4 + b_5 + b_6 - b_7 - b_8 - b_{10} + b_{11}),$$

$$b_1b_{10} = \tfrac{1}{3}(b_2 + b_3 - b_4 + b_5 - b_6 + b_7 + b_8 - b_9 + b_{11}),$$
$$b_1b_{11} = \tfrac{1}{3}(b_2 - b_3 - b_4 + b_5 + b_6 - b_7 + b_8 + b_9 + b_{10}),$$

$$b_2b_3 = \tfrac{1}{3}(b_1 + b_4 - b_5 - b_6 + b_7 + b_8 - b_9 + b_{10} + b_{11}),$$
$$b_2b_4 = \tfrac{1}{3}(-b_1 + b_3 + b_5 + b_6 - b_7 + b_8 - b_9 + b_{10} + b_{11}),$$
$$b_2b_5 = \tfrac{1}{3}(-b_1 - b_3 + b_4 + b_6 + b_7 + b_8 + b_9 + b_{10} - b_{11}),$$
$$b_2b_6 = \tfrac{1}{3}(b_1 - b_3 + b_4 + b_5 - b_7 - b_8 + b_9 + b_{10} + b_{11}),$$
$$b_2b_7 = \tfrac{1}{3}(b_1 + b_3 - b_4 + b_5 - b_6 + b_8 + b_9 + b_{10} - b_{11}),$$
$$b_2b_8 = \tfrac{1}{3}(-b_1 + b_3 + b_4 + b_5 - b_6 + b_7 + b_9 - b_{10} + b_{11}),$$
$$b_2b_9 = \tfrac{1}{3}(b_1 - b_3 - b_4 + b_5 + b_6 + b_7 + b_8 - b_{10} + b_{11}),$$
$$b_2b_{10} = \tfrac{1}{3}(b_1 + b_3 + b_4 + b_5 + b_6 + b_7 - b_8 - b_9 - b_{11}),$$
$$b_2b_{11} = \tfrac{1}{3}(b_1 + b_3 + b_4 - b_5 + b_6 - b_7 + b_8 + b_9 - b_{10}),$$

$$b_3b_4 = \tfrac{1}{3}(b_1 + b_2 + b_5 - b_6 - b_7 + b_8 + b_9 - b_{10} + b_{11}),$$
$$b_3b_5 = \tfrac{1}{3}(b_1 - b_2 + b_4 + b_6 + b_7 - b_8 + b_9 - b_{10} + b_{11}),$$
$$b_3b_6 = \tfrac{1}{3}(-b_1 - b_2 - b_4 + b_5 + b_7 + b_8 + b_9 + b_{10} + b_{11}),$$
$$b_3b_7 = \tfrac{1}{3}(b_1 + b_2 - b_4 + b_5 + b_6 - b_8 - b_9 + b_{10} + b_{11}),$$
$$b_3b_8 = \tfrac{1}{3}(-b_1 + b_2 + b_4 - b_5 + b_6 - b_7 + b_9 + b_{10} + b_{11}),$$
$$b_3b_9 = \tfrac{1}{3}(b_1 - b_2 + b_4 + b_5 + b_6 - b_7 + b_8 + b_{10} - b_{11}),$$
$$b_3b_{10} = \tfrac{1}{3}(b_1 + b_2 - b_4 - b_5 + b_6 + b_7 + b_8 + b_9 - b_{11}),$$
$$b_3b_{11} = \tfrac{1}{3}(-b_1 + b_2 + b_4 + b_5 + b_6 + b_7 + b_8 - b_9 - b_{10}),$$

$$b_4b_5 = \tfrac{1}{3}(b_1 + b_2 + b_3 + b_6 - b_7 - b_8 + b_9 + b_{10} - b_{11}),$$
$$b_4b_6 = \tfrac{1}{3}(b_1 + b_2 - b_3 + b_5 + b_7 + b_8 - b_9 + b_{10} - b_{11}),$$
$$b_4b_7 = \tfrac{1}{3}(b_1 - b_2 - b_3 - b_5 + b_6 + b_8 + b_9 + b_{10} + b_{11}),$$
$$b_4b_8 = \tfrac{1}{3}(b_1 + b_2 + b_3 - b_5 + b_6 + b_7 - b_9 - b_{10} + b_{11}),$$
$$b_4b_9 = \tfrac{1}{3}(b_1 - b_2 + b_3 + b_5 - b_6 + b_7 - b_8 + b_{10} + b_{11}),$$
$$b_4b_{10} = \tfrac{1}{3}(-b_1 + b_2 - b_3 + b_5 + b_6 + b_7 - b_8 + b_9 + b_{11}),$$
$$b_4b_{11} = \tfrac{1}{3}(-b_1 + b_2 + b_3 - b_5 - b_6 + b_7 + b_8 + b_9 + b_{10}),$$

$$b_5b_6 = \tfrac{1}{3}(-b_1 + b_2 + b_3 + b_4 + b_7 - b_8 - b_9 + b_{10} + b_{11}),$$
$$b_5b_7 = \tfrac{1}{3}(-b_1 + b_2 + b_3 - b_4 + b_6 + b_8 + b_9 - b_{10} + b_{11}),$$

$$b_5b_8 = \tfrac{1}{3}(b_1 + b_2 - b_3 - b_4 - b_6 + b_7 + b_9 + b_{10} + b_{11}),$$
$$b_5b_9 = \tfrac{1}{3}(b_1 + b_2 + b_3 + b_4 - b_6 + b_7 + b_8 - b_{10} - b_{11}),$$
$$b_5b_{10} = \tfrac{1}{3}(b_1 + b_2 - b_3 + b_4 + b_6 - b_7 + b_8 - b_9 + b_{11}),$$
$$b_5b_{11} = \tfrac{1}{3}(b_1 - b_2 + b_3 - b_4 + b_6 + b_7 + b_8 - b_9 + b_{10}),$$

$$b_6b_7 = \tfrac{1}{3}(b_1 - b_2 + b_3 + b_4 + b_5 + b_8 - b_9 - b_{10} + b_{11}),$$
$$b_6b_8 = \tfrac{1}{3}(b_1 - b_2 + b_3 + b_4 - b_5 + b_7 + b_9 + b_{10} - b_{11}),$$
$$b_6b_9 = \tfrac{1}{3}(b_1 + b_2 + b_3 - b_4 - b_5 - b_7 + b_8 + b_{10} + b_{11}),$$
$$b_6b_{10} = \tfrac{1}{3}(-b_1 + b_2 + b_3 + b_4 + b_5 - b_7 + b_8 + b_9 - b_{11}),$$
$$b_6b_{11} = \tfrac{1}{3}(b_1 + b_2 + b_3 - b_4 + b_5 + b_7 - b_8 + b_9 - b_{10}),$$

$$b_7b_8 = \tfrac{1}{3}(b_1 + b_2 - b_3 + b_4 + b_5 + b_6 + b_9 - b_{10} - b_{11}),$$
$$b_7b_9 = \tfrac{1}{3}(-b_1 + b_2 - b_3 + b_4 + b_5 - b_6 + b_8 + b_{10} + b_{11}),$$
$$b_7b_{10} = \tfrac{1}{3}(b_1 + b_2 + b_3 + b_4 - b_5 - b_6 - b_8 + b_9 + b_{11}),$$
$$b_7b_{11} = \tfrac{1}{3}(-b_1 - b_2 + b_3 + b_4 + b_5 + b_6 - b_8 + b_9 + b_{10}),$$

$$b_8b_9 = \tfrac{1}{3}(-b_1 + b_2 + b_3 - b_4 + b_5 + b_6 + b_7 + b_{10} - b_{11}),$$
$$b_8b_{10} = \tfrac{1}{3}(b_1 - b_2 + b_3 - b_4 + b_5 + b_6 - b_7 + b_9 + b_{11}),$$
$$b_8b_{11} = \tfrac{1}{3}(b_1 + b_2 + b_3 + b_4 + b_5 - b_6 - b_7 - b_9 + b_{10}),$$

$$b_9b_{10} = \tfrac{1}{3}(-b_1 - b_2 + b_3 + b_4 - b_5 + b_6 + b_7 + b_8 + b_{11}),$$
$$b_9b_{11} = \tfrac{1}{3}(b_1 + b_2 - b_3 + b_4 - b_5 + b_6 + b_7 - b_8 + b_{10}),$$

$$b_{10}b_{11} = \tfrac{1}{3}(b_1 - b_2 - b_3 + b_4 + b_5 - b_6 + b_7 + b_8 + b_9).$$

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