A note on adjacency preservers on hermitian matrices over finite fields

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1. Introduction

Hermitian matrices $A$ and $B$ are adjacent if $\text{rk}(A - B) = 1$. Hua's fundamental theorem of the geometry of hermitian matrices (see e.g. [14, Theorem 6.4] or Theorem 2.1 below) characterizes all bijective maps $\Phi$ on hermitian matrices which preserve adjacency in both directions, i.e., matrices $A$ and $B$ are adjacent if and only if $\Phi(A)$ and $\Phi(B)$ are adjacent. Recently, important generalizations of Hua's theorem on hermitian matrices were obtained. In [8] “the assumption of both directions” was reduced to one direction only. In [5,6] the result of [8] was extended to arbitrary division rings. In [9] even the assumption of bijectivity was dropped for the case of complex hermitian matrices. In [7] the same problem as in [9] was solved for $2 \times 2$ hermitian matrices over division ring $D$ subject to modest restrictions. In particular, when $D$ is a finite field, it was assumed that its characteristic differs from two, and that the image of the preserver contains two nonadjacent matrices. It was mentioned as an open problem, whether adjacency preserving maps, whose image contains only pairwise adjacent matrices, exist in the case of finite fields. The main result of the present paper answers this question.

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negatively, provided that the involution on the field is proper. Moreover, it classifies all adjacency preserving maps on $n \times n$ hermitian matrices over arbitrary finite field (see Theorem 3.1 for details). In Section 4 we give examples to show that our results are no longer true when the involution is the identity map, i.e., when matrices are symmetric. Before we state and prove the main theorem of this paper, we need few preliminaries from linear algebra and graph theory.

2. Preliminaries

Throughout, $\mathbb{K}^n$ will be the space of all column-vectors of length $n$ over a (commutative) field $\mathbb{K}$. Given $x \in \mathbb{K}^n$ we can, and will do so, identify it with an $n \times 1$ matrix. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{K}^n$, and let $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ be the standard basis in the space $\mathcal{M}_n(\mathbb{K})$ of all $n \times n$ matrices with entries from $\mathbb{K}$. A map $^\ast : \mathbb{K} \to \mathbb{K}$ is called an involution if it satisfies: (i) $x^\ast = \bar{x}$, (ii) $xy = yx^\ast$, and (iii) $x^\ast = x$. Let $A^\text{tr}$ be the transpose of $A$. A matrix $A \in \mathcal{M}_n(\mathbb{K})$ is hermitian if $A^\ast := A^\text{tr} = A$, where $^\ast : \mathbb{K} \to \mathbb{K}$ is an involution applied entry-wise. Let $\mathcal{H}_n(\mathbb{K})$ be the set of all such matrices. The involution $^\ast$ is proper if it is not the identity map. In this case $\mathbb{K} = \mathbb{F} \oplus i\mathbb{F}$ is a direct sum of $\mathbb{F}$-vector spaces, where $\mathbb{F} := \{z \in \mathbb{K} \mid i^2 = -1\}$ is the fixed field of involution and $i \in \mathbb{K}$ satisfies $i^2 = -1$ when the characteristic of $\mathbb{K}$, char $\mathbb{K}$, is not 2, and $i = 1 + i$ when char $\mathbb{K} = 2$. Hence, if $\mathbb{K}$ is finite then $|\mathbb{K}|$ is a square, i.e., $\mathbb{K} = GF(q^2)$ is the Galois field of order $q^2$, where $q$ is a power of a prime. Inversely, any such field has a unique proper involution defined by $x \mapsto x^\text{tr}$ (see e.g. [3, the proof of Theorem 2]). It also follows from [13, Theorem 2.21] that the automorphisms of $GF(q^2)$ are precisely the maps $x \mapsto x^p^j$, $0 \leq j \leq 2k - 1$, where $q = p^k$, $p$ is a prime, and $k \geq 1$ is an integer. In particular all automorphisms commute with the involution $x \mapsto x^\text{tr}$.

When the involution $^\ast$ is proper, any hermitian matrix $A$ of rank $r$ can be written as $A = P(\sum_{j=1}^r \lambda_j E_{jj})P^\ast$, where $P \in \mathcal{M}_n(\mathbb{K})$ is invertible and $\lambda_j = \bar{\lambda}_j$ are all nonzero. When $\mathbb{K}$ is finite we can assume that $\lambda_j = 1$ for all $j$, so that $A = \sum_{j=1}^r x_j x_j^\ast$ for some linearly independent $x_1, \ldots, x_r \in \mathbb{K}^n$ (see e.g. [1, Theorem 4.1]).

A map $\Phi : \mathcal{H}_n(\mathbb{K}) \to \mathcal{H}_n(\mathbb{K})$ preserves adjacency (resp. preserves adjacency in both directions) if $\text{rk}(A - B) = 1$ implies $\text{rk}(\Phi(A) - \Phi(B)) = 1$ (resp. $\text{rk}(A - B) = 1 \implies \text{rk}(\Phi(A) - \Phi(B)) = 1$). It is well known that if $\Phi$ preserves adjacency then $\text{rk}(\Phi(A) - \Phi(B)) \leq \text{rk}(A - B)$ for all $A, B$. To see this, write $A - B = \sum_{j=1}^r x_j x_j^\ast$ and set $A_j := A - \sum_{j=1}^r x_j x_j^\ast$, $j = 0, \ldots, r$. The adjacency of $A_j, A_{j+1}$ forces adjacency of $\Phi(A_j), \Phi(A_{j+1})$, yielding $\text{rk}(\Phi(A) - \Phi(B)) \leq r$.

The next theorem is called Hua’s fundamental theorem of geometry of hermitian matrices. It characterizes all bijective maps which preserve adjacency in both directions. Its proof can be found for example in [14, Theorem 6.4].

**Theorem 2.1.** Let $(\mathbb{K}, ^\ast)$ be a field with proper involution and let $n \geq 2$ be an integer. Then a bijective map $\Phi : \mathcal{H}_n(\mathbb{K}) \to \mathcal{H}_n(\mathbb{K})$ preserves adjacency in both direction if and only if it is of the form

$$\Phi(A) = \xi P A^\sigma P^\ast + B,$$

where $\xi \in \mathbb{F}$ is nonzero, $P \in \mathcal{M}_n(\mathbb{K})$ is invertible, $B \in \mathcal{H}_n(\mathbb{K})$, and $A^\sigma$ is a matrix obtained from $A$ by applying an automorphism $\sigma : \mathbb{K} \to \mathbb{K}$, which commutes with $^\ast$ (i.e., $\sigma(\bar{x}) = \sigma(x)$), entry-wise.

We will call maps of the form (1) standard. For any two adjacent matrices $A, B \in \mathcal{H}_n(\mathbb{K})$ the line $l(A, B)$ joining $A$ and $B$ is defined to be the set consisting of $A$, $B$ and all $C \in \mathcal{H}_n(\mathbb{K})$ which are adjacent to both $A$ and $B$. It was proved in [14] that $l(A, B) = \{A + \lambda (B - A) \mid \lambda \in \mathbb{F}\}$, where $\mathbb{F}$ is the fixed field of involution. Note that the matrices contained in the same line are pairwise adjacent. Note also that for adjacency preserving map $\Phi : \mathcal{H}_n(\mathbb{K}) \to \mathcal{H}_n(\mathbb{K})$ the restriction of $\Phi$ to arbitrary line is necessary injective. The lemma below is a special case of Lemma 2.1 in [8].

**Lemma 2.2.** Let $(\mathbb{K}, ^\ast)$ be a field with proper involution and let $l$ a line in $\mathcal{H}_n(\mathbb{K})$. Then either all matrices in $l$ have the same rank or there exists $B \in l$ such that $\text{rk} A = \text{rk} B + 1$ for all $A \in l \setminus \{B\}$. 

We will also need some results from graph theory. Throughout this paper all graphs will be finite, undirected, and simple (no multiple edges or loops are allowed). Recall that given a graph $\Gamma$ with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$ the **chromatic number** $\chi(\Gamma)$ is the smallest number for which there exists a proper coloring of $V(\Gamma)$, i.e., the smallest number $k$ for which there exists a map $c : V(\Gamma) \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E(\Gamma)$. A **homomorphism of graphs** $\Gamma_1$ and $\Gamma_2$ is a map $\Phi : V(\Gamma_1) \rightarrow V(\Gamma_2)$ such that $\{\Phi(u), \Phi(v)\} \in E(\Gamma_2)$ whenever $\{u, v\} \in E(\Gamma_1)$. In the sequel, we loosely write $\Phi : \Gamma_1 \rightarrow \Gamma_2$. The next lemma is proved in [10, Proposition 1.20].

**Lemma 2.3.** Let $\Phi : \Gamma_1 \rightarrow \Gamma_2$ be a homomorphism of graphs. Then $\chi(\Gamma_1) \leq \chi(\Gamma_2)$.

Recall that the **adjacency matrix** of a graph $\Gamma$ with the vertex set $V(\Gamma) = \{v_1, \ldots, v_t\}$ is the $t \times t$ binary matrix with 1 at position $(i, j)$ if $v_i$ and $v_j$ are adjacent and 0 otherwise. The well-known theorem below [4] gives a lower bound for the chromatic number, expressed with the eigenvalues of the adjacency matrix.

**Theorem 2.4.** For a graph $\Gamma$, whose edge set is nonempty,

$$\chi(\Gamma) \geq 1 + \frac{\lambda_{\text{max}}(\Gamma)}{\lambda_{\text{min}}(\Gamma)}$$

holds. Here, $\lambda_{\text{max}}(\Gamma)$ is a maximum eigenvalue and $\lambda_{\text{min}}(\Gamma)$ is a minimum eigenvalue of adjacency matrix of $\Gamma$.

For vertices $u, v \in V(\Gamma)$ let $d(u, v)$ denote the length of the shortest path between them (i.e., the number of edges in the path). The diameter of a graph $\Gamma$ is defined by $\text{diam}(\Gamma) := \max(d(u, v) \mid (u, v) \in V(\Gamma) \times V(\Gamma))$. A connected graph $\Gamma$ is **distance-regular** if there exist numbers $p_{ij}^k$ ($i, j, k \in \{0, 1, \ldots, \text{diam}(\Gamma)\}$), called the intersection numbers, such that for any pair of vertices $(u, v)$ with $d(u, v) = k$ the number of $w \in V(\Gamma)$ with $d(u, w) = l$ and $d(w, v) = j$ equals $p_{ij}^l$. The proof of the next theorem can be found for example in [2, pp. 128–129].

**Theorem 2.5.** Eigenvalues (without their multiplicities) of the adjacency matrix of a distance-regular graph $\Gamma$ are equal to those of the tridiagonal $(d + 1) \times (d + 1)$ matrix

$$
\begin{pmatrix}
q^0 & b_0 \\
c_1 & a_1 & b_1 \\
& \ddots & \ddots \\
& & c_{d-1} & b_{d-1} \\
& & & c_d & a_d
\end{pmatrix},
$$

where $c_i = p_{1,i-1}^i$, $a_i = p_{1,i}^i$, $b_i = p_{1,i+1}^i$, and $d = \text{diam}(\Gamma)$.

Let $(\mathbb{K}, \cdot)$ be a finite field with (unique) proper involution, i.e., $\mathbb{K} = GF(q^2)$ where $q$ is a power of a prime, and $x = x^q$. To shorten writing, we will denote the set of all $n \times n$ hermitian matrices over $(GF(q^2), x \mapsto x^q)$ by $\mathcal{H}_n(q^2)$. Similarly, $\mathcal{M}_n(q^2)$ will stand for the set of all $n \times n$ matrices over $GF(q^2)$. Recall that a hermitian forms graph is a graph $\Gamma$ with $V(\Gamma) = \mathcal{H}_n(q^2)$ and $E(\Gamma) = \{(A, B) \mid A, B \in \mathcal{H}_n(q^2), \text{rk}(A - B) = 1\}$. It is well known that this graph is connected, and $d(A, B) = \text{rk}(A - B)$ (see e.g. [14]). Moreover, the hermitian forms graph is distance-regular of diameter $\text{diam}(\Gamma) = n$, and

$$
c_i = \frac{q^{i-1}(q^i - (-1)^i)}{q + 1}, \quad a_i = \frac{q^{2i} - q^{i-1}(q^i - (-1)^i) - 1}{q + 1}, \quad b_i = \frac{q^{2n} - q^{2i}}{q + 1}
$$

(2)
hold for all \( i \) (see e.g. [12] or [11]). We remark to the reader interested in graph theory that the line \( l(A, B) \), of two adjacent matrices \( A \) and \( B \), represents a maximum clique in the hermitian forms graph, that is, the largest subgraph with pairwise adjacent vertices. In the sequel we will slightly abuse the notation and use \( \mathcal{H}_n(q^2) \) also to denote the hermitian forms graph.

A hermitian variety of a matrix \( H \in \mathcal{H}_n(q^2) \) is defined by \( \mathcal{V}_H = \{ (x) \mid x^\text{T} H x = 0, \ x \neq 0 \} \). Here, \( (x) \) is the 1-dimensional subspace in \( GF(q^2) \), generated by the vector \( x \). The next theorem is proved in [1, Theorem 8.1 and its corollary], where it is written in a slightly different way.

**Theorem 2.6.** Let \( H \in \mathcal{H}_n(q^2) \) be of rank \( r \). Then the cardinality of its variety is

\[
|\mathcal{V}_H| = \frac{q^{2n-1} + (-1)^r (q - 1)q^{2n-r-1} - 1}{q^2 - 1}.
\]

### 3. Main results

We now state the main theorem of this paper, which says that the bijectivity in Theorem 2.1 is redundant if the field \( \mathbb{K} \) is finite. Moreover, “the assumption of both directions” is reduced to one direction only. In finite fields we may also assume that in (1), \( \xi = 1 \).

**Theorem 3.1.** Let \( k \geq 1 \), \( n \geq 2 \) be integers, and \( q = p^k \) a power of a prime \( p \). Then, \( \Phi : \mathcal{H}_n(q^2) \rightarrow \mathcal{H}_n(q^2) \) preserves adjacency if and only if it is of the form

\[
\Phi(A) = PA^\sigma P^* + B,
\]

where \( P \in \mathcal{M}_n(q^2) \) is invertible, \( B \in \mathcal{H}_n(q^2) \) is a hermitian matrix, and the field automorphism \( \sigma : GF(q^2) \rightarrow GF(q^2) \) is defined by \( \sigma(x) = x^{q^k} \) for some integer \( 0 < j < 2k - 1 \).

We remark that \( \Phi : \mathcal{H}_n(q^2) \rightarrow \mathcal{H}_n(q^2) \) preserves adjacency if and only if it is an endomorphism of the hermitian forms graph. So in particular Theorem 3.1 says that any endomorphism of the hermitian forms graph is necessarily bijective, that is, an automorphism. Before we prove Theorem 3.1 we need few more lemmas.

**Lemma 3.2.** Let a hermitian matrix \( A \) be invertible and a vector \( x \) such that \( x^\text{T} (A^{-1}) x \neq 0 \), i.e., \( (x) \notin \mathcal{V}_{(A^{-1})} \). Then \( A - (x^\text{T} (A^{-1}) x)^{-1} \cdot xx^* \) is singular.

**Proof.** Note that \( x^* A^{-1} x = (x^* A^{-1} x)^{\text{T}} = x^\text{T} (A^{-1}) x \). Consequently, the vector \( A^{-1} x \) lies in the kernel of the matrix \( A - (x^\text{T} (A^{-1}) x)^{-1} \cdot xx^* \). \( \square \)

**Lemma 3.3.** Let \( A, B \in \mathcal{H}_2(q^2) \) and \( 2 = \text{rk} A = \text{rk} (B - A) \). There exist at most \( q + 1 \) matrices \( C = A + xx^* \) that are adjacent to \( B \), and such that \( (x) \in \mathcal{V}_{(A^{-1})} \).

**Proof.** By Theorem 2.6 applied at \( n = 2 = r \), \( |\mathcal{V}_{(A^{-1})}| = |\mathcal{V}_A| = q + 1 \), that is, \( \mathcal{V}_{(A^{-1})} = \{ (y_1), \ldots, (y_{q+1}) \} \).

It now suffices to show the following: whenever \( x_1, x_2 \in \{ y_i \} \setminus \{ 0 \} \), \( 1 \leq i \leq q + 1 \), are such that both matrices \( A + x_1 x_1^* \) and \( A + x_2 x_2^* \) are adjacent to \( B \) then \( A + x_1 x_1^* \neq A + x_2 x_2^* \) for some pair of such vectors \( x_1 \) and \( x_2 \). Then, the matrices \( A + x_1 x_1^* \) and \( A + x_2 x_2^* \) must be adjacent, since \( x_1 \) and \( x_2 \) are linearly dependent. Consequently, both \( A \) and \( B \) lie in the line \( l(A + x_1 x_1^*, A + x_2 x_2^*) \). Recall that all matrices from a line are pairwise adjacent. However by assumption, \( \text{rk}(A - B) = 2 \), which is a contradiction. \( \square \)

**Lemma 3.4.** Let \( A, B \in \mathcal{H}_2(q^2) \) and \( 2 = \text{rk} A = \text{rk} B = \text{rk}(B - A) \). Then there exist at most \( q + 1 \) rank-one matrices \( C \in \mathcal{H}_2(q^2) \) adjacent to both \( A \) and \( B \).
Proof. Since \( \operatorname{rk} A = 2 = \operatorname{rk} B \) and the field is finite there exist vectors \( x, y, w, v \in \mathbb{F}^2 = GF(q^2)^2 \) with 
\[
A = xx^* + yy^*, \quad B = wv^* + vv^*,
\]
and such that \( x \) and \( y \) as well as \( w \) and \( v \) are linearly independent. Let \( C \) be an arbitrary rank-one matrix which is adjacent to both \( A \) and \( B \). Since vectors \( x \) and \( y \) span \( \mathbb{F}^2 \), \( C \) can be written as \( C = zz^* \) where \( z = xx + yy \) for some scalars \( x \) and \( y \), at least one of them nonzero. Similarly, there exist scalars \( w \) and \( v \) such that
\[
wv + vv = z = xx + yy.
\]
Choose invertible \( P \) such that \( Pw = e_1 \) and \( Pv = e_2 \). Since \( C \) and \( B \) are adjacent, we deduce that 
\[
\det(P(C - B)P^*) = 0,
\]
which implies that \( wv + vv = 1 \) by a straightforward calculation. Similarly we obtain the equation \( xx + yy = 1 \). Consequently,
\[
u_1^T \bar{u}_1 = u_2^T \bar{u}_2
\]
for \( u_1 := (x, y)^T \) and \( u_2 := (w, v)^T \). Choose scalars \( a, b, c, d \) such that \( x = aw + bv \) and \( y = cw + dv \). Then, by (4), \( (w - xa - yc)w + (v - xb - yd)v = 0 \). Since \( w \) and \( v \) are linearly independent, we deduce that \( u_2 = Q^T u_1 \), where
\[
Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
Therefore, (5) reduces into
\[
0 = (Q^T u_1)^T (Q^T u_1) - u_1^T \bar{u}_1 = u_1^T (Q Q^* - I) u_1,
\]
which is equivalent to
\[
\langle (x, y)^T \rangle = (u_1) \in V_{Q Q^* - I}.
\]
Since \( \operatorname{rk}(A - B) = 2 \), it follows that \( \det(P(A - B)P^*) \neq 0 \). By a straightforward calculation we see that \( \det(P(A - B)P^*) = \det(Q Q^* - I) \). Hence, \( \operatorname{rk}(Q Q^* - I) = 2 \). Theorem 2.6 further implies that \( |V_{Q Q^* - I}| = q + 1 \), that is, \( V_{Q Q^* - I} = \{ (t_1), \ldots, (t_{q + 1}) \} \).

We now proceed similarly as in the proof of Lemma 3.3. It suffices to show that whenever \( (x_1, y_1)^T, (x_2, y_2)^T \) are adjacent to both \( A \) and \( B \) then \( C_1 = C_2 \). So assume erroneously that \( C_1 \neq C_2 \). Since \( C_1 \neq C_2 \) for some pair of such vectors \( (x_1, y_1)^T \) and \( (x_2, y_2)^T \). Then, \( C_1 \) and \( C_2 \) are adjacent, since \( (x_1, y_1)^T \) and \( (x_2, y_2)^T \) are linearly independent. Consequently, both \( A \) and \( B \) lie in the line \( l(C_1, C_2) \), which is a contradiction, since \( A \) and \( B \) are not adjacent. \( \square \)

Lemma 3.5. If an adjacency preserving map \( \Phi : \mathcal{H}_2(q^2) \to \mathcal{H}_2(q^2) \) satisfies \( \Phi(0) = 0 \), and if \( \operatorname{rk} A = 2 = \operatorname{rk} \Phi(A) \) for some \( A \in \mathcal{H}_2(q^2) \), then every rank-two matrix \( B \in \mathcal{H}_2(q^2) \) satisfies \( \Phi(B) \neq 0 \).

Proof. If \( B \) and \( A \) are adjacent then \( \operatorname{rk}(\Phi(A) - \Phi(B)) = 1 \) and hence \( \Phi(B) \neq 0 \), since \( \operatorname{rk} \Phi(A) = 2 \). It remains to check the claim for rank-two matrices \( B \) for which \( \operatorname{rk}(B - A) = 2 \). We separate three cases.

Case 1. Let \( q \geq 4 \). Choose arbitrary \( B \) such that \( \operatorname{rk} B = 2 = \operatorname{rk}(B - A) \). Note that any matrix \( C \), which is adjacent to \( A \), is of the form \( C = A + \lambda x \lambda c^* \) for some vector \( x \) and \( c \). Since the hermitian forms graph \( \mathcal{H}_2(q^2) \) is distance-regular, there exist precisely \( p_{11}^2 = c_2 \) matrices \( C = A + \lambda x \lambda c^* \) that are adjacent to both \( A \) and \( B \). By (2), \( c_2 = q(q - 1) \). By Lemma 3.4, at most \( q + 1 \) of these \( q(q - 1) \) matrices \( C \) are of rank one. Moreover, by Lemma 3.3, at most \( q + 1 \) of these \( q(q - 1) \) matrices \( C \) are such that \( \{ x, c \} \in V_{(A - \lambda c)} \). Hence, there are at least \( q(q - 1) - (q + 1) - (q + 1) > 0 \) matrices \( C = A + x \lambda c^* \) of rank two which are adjacent to both \( A \) and \( B \) and such that \( \{ x, c \} \notin V_{(A - \lambda c)} \). Now, pick such \( C = A + x \lambda c^* \). By Lemma 3.2, the \( 2 \times 2 \) matrix \( M := A - \lambda x \lambda c^* \) is singular for \( \lambda := (x \lambda c^* (A - \lambda c) x \lambda c^*)^{-1} \). Moreover, \( \operatorname{rk} A = 2 = \operatorname{rk}(M - A) \). Hence, \( \operatorname{rk} M = 1 \). In particular, \( M \neq C \), since \( \operatorname{rk} C = 2 \). Therefore, \( A, M, C \) are pairwise adjacent. The same must be true for \( \Phi(A), \Phi(M), \Phi(C) \). In particular, this triple lies in the same line \( l \). However,
Φ(0) = 0 forces rk Φ(M) = 1, while by the assumption, rk Φ(A) = 2. So not all matrices in l have the same rank. By Lemma 2.2 there is but one possible exception with rank-one, so rk Φ(C) = 2. Recall that B and C are adjacent. Therefore, rk(Φ(C) − Φ(B)) = 1 and consequently Φ(B) ≠ 0.

Case 2. Let q = 2. Assume erroneously that Φ(B) = 0 for some rank-two matrix B. There exists a rank-two matrix D which is adjacent to B (actually there are p_{12}^2 = 3 such matrices). Pick a matrix M among p_{11}^2 = c_2 = 2 rank-one matrices which are adjacent to D. Then, rk(Φ(D) − Φ(M)) = 1 = rk(Φ(D) − Φ(B)) = rk(Φ(D) − 0). Hence, Φ(D) is in the line l(0, Φ(M)) = 1 = 1, where by the assumption, rk Φ(C) = 2.

Case 3. Finally, let q = 3. There exists an invertible P such that A = P(E_{11} + E_{22})P^t. Hence, we may assume that A = E_{11} + E_{22} (otherwise consider the map Ψ(Φ) = Φ(P ∪ P^t)). Recall that GF(3^2) = {0, 1, 1} ⊕ i{0, 1, −1} where i^2 = −1 and i^3 = −i. Hence, x̅ = 1 if and only if x ∈ {1, −1, i, −i} and x̅ = −1 if and only if x ∈ {1 + i, −1 + i, 1 − i, −1 − i}.

Firstly, we will show that if rk B = 2 = rk(B − A), and B is not among the next 8 matrices

\[
\begin{bmatrix}
0 & x & x \\
y & 1 & \lambda \\
x & 0 & \mu
\end{bmatrix}
\]

then Φ(B) ≠ 0. Let B = λE_{11} + μE_{22} + xE_{12} + x̅E_{21} be such a matrix. Note that λ, μ ∈ F = {0, 1, −1}. The situation x̅ = 1, λ = 0 = μ cannot occur since rk(B − A) = 2. The same holds if x̅ = −1 and λ, μ = 0, 1 (order is not prescribed). Any other possibility ((6) not included) is contained in the first three columns of the table below.

<table>
<thead>
<tr>
<th>x̅</th>
<th>λ</th>
<th>μ</th>
<th>y</th>
<th>α</th>
<th>β</th>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>μ</td>
<td>−μ</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>λ</td>
<td>−λ</td>
</tr>
<tr>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We define matrix C = αE_{11} + βE_{22} + yE_{12} + y̅E_{21} as described in the last three columns of the table above. Then, C ≠ E_{11} + E_{22} = A and rk C = 2. Moreover, C is adjacent to both A and B, and the matrix A − (C − A) is of rank one. We prove this only for the first situation, i.e., when x̅ = 1 and λ, μ ≠ 0. In this case, det(C − A) = (0 − 1) · (0 − 1) − iαx̅x = 0. Hence, C and A are adjacent. Since rk B = 2 we deduce that x̅ ≠ λ, μ ≠ 0, i.e., λ, μ = −1. Consequently, det(B − C) = 0. Since C is adjacent to A and rk(B − A) = 2 we deduce that C ≠ B, i.e., B and C are adjacent. Clearly, A − (C − A) = 0. Hence, rk(A − (C − A) = 1, since det(A − (C − A)) = 0.

Note that Φ(A), Φ(C), Φ(A − (C − A)) are pairwise adjacent. Since rk Φ(A) = 2 and rk Φ(A − (C − A)) = 1, Lemma 2.2 implies that rk Φ(C) = 2. If Φ(B) = 0 we are in contradiction with rk(Φ(C) − Φ(B)) = 1. Hence, if a rank-two matrix B is not among the 8 matrices from (6) then Φ(B) ≠ 0.

Now, assume erroneously that Φ(B_0) = 0 for some B_0 from (6). The rank-two matrix B_0 has p_{11}^0 = b_0 = 20 adjacent matrices, where p_{12}^0 = c_2 = 6 of them are of rank one. In particular, there exists a rank-two matrix D_1, adjacent to B_0, such that rk D_2 = 2, where D_2 := B_0 − (D_1 − B_0). Note that B_0, D_1, D_2 are pairwise adjacent. Now, D_1 has 6 adjacent rank-one matrices. The same holds for D_2. Denote these matrices with M_1, ..., M_6 and N_1, ..., N_6, respectively. Further, denote G_i = D_1 − (M_i − D_1) and H_i = D_2 − (N_i − D_2) for all i. Note that matrices D_1, M_i, G_i are pairwise adjacent. The same holds for matrices D_2, N_i, H_i (see Fig. 1 below). Moreover, rk G_i = 2 = rk H_j by Lemma 2.2. Note also that G_i, H_i ≠ B_0. If, in fact, for example B_0 = G_i then M_i ∈ l(B_0), D_1 = [G_i, D_1, D_2], i.e., M_i = D_2, a contradiction, since rk D_2 = 2. If G_i = H_j then this matrix is an element of l(D_1, D_2) = [B_0, D_1, D_2], i.e., B_0 = G_i = H_j, a contradiction. Hence, G_i ≠ H_j for all i, j.

Now, rk(Φ(D_1) − Φ(M_i)) = 1, rk(Φ(D_1) − 0) = rk(Φ(D_1) − Φ(B_0)) = 1 and rk(Φ(M_i) − 0) = (Φ(M_i) − Φ(0)) = 1. Therefore, l(Φ(D_1), Φ(M_i)) = {Φ(D_1), Φ(M_i)}. Since Φ(G_i) is adjacent to Φ(D_1) and Φ(M_i) it follows that Φ(G_i) = 0. We deduce that Φ(H_i) = 0 in the same way. Hence,
at least 13 matrices of rank two, namely $B_0, G_1, \ldots, G_6, H_1, \ldots, H_6$, are mapped to 0 by $\Phi$. Clearly, all cannot be contained in (6), a contradiction. □

**Proof of Theorem 3.1.** The “if part” is obvious. We will now prove the “only if part.” So assume that $\Phi : \mathcal{H}_n(q^2) \to \mathcal{H}_n(q^2)$ preserves adjacency. Let $\text{Im} \, \Phi$ denote the image of $\Phi$.

Assume first that the following holds: whenever $Y \in \mathcal{H}_n(q^2)$ is adjacent to some matrix $\Phi(A)$ then $Y \in \text{Im} \, \Phi$. We claim that $\Phi$ is surjective in this case. Let $B \in \mathcal{H}_n(q^2)$ be arbitrary. Fix some matrix $\Phi(A)$ in the image $\text{Im} \, \Phi$. Since the hermitian forms graph $\mathcal{H}_n(q^2)$ is connected, there exists a path with vertices $Y_0, \ldots, Y_r \in \mathcal{H}_n(q^2)$ which connects $Y_0 = \Phi(A)$ to $Y_r = B$. Since $Y_1$ and $\Phi(A)$ are adjacent, we deduce that $Y_1 \in \text{Im} \, \Phi$ by assumption. If we repeat these arguments $(r - 1)$ times we deduce that $B = Y_r \in \text{Im} \, \Phi$. Hence, $\Phi$ is surjective, so also bijective, since $\mathcal{H}_n(q^2)$ is finite. Moreover, as there are only finite pairs of adjacent matrices, $\Phi^{-1}$ preserves adjacency as well. By Theorem 2.1, $\Phi$ must be of the form (1), where $\sigma$ is an automorphism of $\text{GF}(q^2)$. We know from preliminaries that $\sigma(x) = x^{q^j}$ for some integer $0 \leq j \leq 2k - 1$. Note also that any $\xi \in \text{GF}(q^2)$ is of the form $\xi = x\xi = x^{q^j + 1}$ for some $x \in \text{GF}(q^2)$ (see [1, Section 2]). Consequently, we may replace the matrix $P$ in (1) with $xP$ and set $\xi = 1$.

Now, assume erroneously that there exist adjacent matrices $Y, \Phi(A) \in \mathcal{H}_n(q^2)$ such that $Y \notin \text{Im} \, \Phi$. Since $\Phi$ maps the set $\Omega_A := \{X \mid \text{rk}(A - X) = 1\}$ of matrices adjacent to $A$ into the set $\Omega_{\Phi(A)} \supseteq Y$ at least two matrices, say $B, C \in \Omega_A$ must be mapped into the same one by $\Phi$. Clearly then, $B$ and $C$ cannot be adjacent, so $\text{rk}(B - C) = d(B, C) = 2$. Choose an invertible $P$ such that $B - C = P(E_{11} + E_{22})P^*$ and define $\Psi : \mathcal{H}_2(q^2) \to \mathcal{H}_n(q^2)$ by $\Psi(X) := \Phi(P(X \oplus 0_{n-2})P^* + C) - \Phi(B)$. Clearly, $\Psi$ preserves adjacency, satisfies $\Psi(0) = 0$, and annihilates a rank-two matrix $E_{11} + E_{22} \in \mathcal{H}_2(q^2)$. By Lemma 3.5, $\text{rk} \, \Psi(X) \neq 2$ for all rank-two matrices $X$. Actually, as $\text{rk} \, \Psi(X) = \text{rk} \, (\Psi(X) - \Psi(0)) \leq \text{rk}(X - 0) = \text{rk}X = 2$ (see preliminaries) we deduce $\text{rk} \, \Psi(X) \leq 1$ whenever $\text{rk}X = 2$. Since matrices in $\mathcal{H}_2(q^2)$ are of rank-two at most, it follows that every matrix in $\text{Im} \, \Psi$ is of rank $\leq 1$, i.e., $\text{Im} \, \Psi$ is contained in the union

$$\mathcal{U} := \bigcup \left\{l(0, M) \mid M \in \mathcal{H}_n(q^2), \, \text{rk}M = 1 \right\}.$$ 

Since $l(0, M) = \{\lambda M \mid \lambda \in \text{GF}(q^2)\}$, any two distinct lines from this union intersect in the zero matrix. Furthermore, nonzero matrices from different lines are not adjacent. Consequently, the subgraph $\Gamma'$ of $\mathcal{H}_n(q^2)$ that is induced by $\mathcal{U}$ (i.e., the graph with $\mathcal{U}$ as the vertex set and such that $M_1, M_2 \in \mathcal{U}$ form an edge if and only if $\text{rk}(M_1 - M_2) = 1$), has the chromatic number $\leq q = |\text{GF}(q^2)|$. Namely, we color the zero matrix with arbitrary fixed color, and then use other $q - 1$ colors for the rest of the matrices in particular line. For different lines the same $q - 1$ colors can be used.
Now, if we consider $\Psi$ as a graph homomorphism between $\mathcal{H}_2(q^2)$ and $\Gamma$ then we deduce that $\chi(\mathcal{H}_2(q^2)) \leq q$ by Lemma 2.3. However, we will show that $\chi(\mathcal{H}_2(q^2)) > q$, which is a desired contradiction. The eigenvalues of the adjacency matrix of the graph $\mathcal{H}_2(q^2)$ are $(q^3 - q^2 + q - 1)$, $(-q^2 + q - 1)$, and $(q - 1)$. They can be computed by putting the intersection numbers of Eq. (2) into the $3 \times 3$ matrix from Theorem 2.5. Hence, a maximum eigenvalue, $\lambda_{\text{max}}(\mathcal{H}_2(q^2))$, equals $q^3 - q^2 + q - 1$, while a minimum eigenvalue, $\lambda_{\text{min}}(\mathcal{H}_2(q^2))$, equals $-q^2 + q - 1$. Hence, by Theorem 2.4,

$$\chi(\mathcal{H}_2(q^2)) \geq 1 + \frac{q^3 - q^2 + q - 1}{(-q^2 + q - 1)} > q.$$  \hfill \square

4. Remarks

(a) When the field $\mathbb{K}$ is infinite, then there can exist nonstandard adjacency preserving maps (see [7]). For example, let $\mathbb{K} = \mathbb{R}(x, y)$ be the field of rational functions in indeterminates $x$ and $y$, with real coefficients, equipped with the proper involution $f(x, y) := f(y, x)$. For the field monomorphism $\sigma(f(x, y)) := f(x^2, y^2)$ it can be verified that $\det(I + (x + y)A^2 E_{11}) \neq 0$ for all $A \in \mathcal{H}_2(\mathbb{K})$, and $\Phi(A) := (I + (x + y)A^2 E_{11})^{-1}A^\sigma$ maps $\mathcal{H}_2(\mathbb{K})$ into $\mathcal{H}_2(\mathbb{K})$. Furthermore, it preserves adjacency (see [7, Theorem 2]). The map $\Phi$ is not of the form (3), i.e., a sum of a constant and an additive map, since $\Phi(0) = 0$ while $\Phi(2E_{11}) \neq 2\Phi(E_{11})$.

(b) In the case of infinite fields there can also exist adjacency preserving maps whose image contains only pairwise adjacent matrices, i.e., the whole image is contained in one line. As observed in [9], when $\mathbb{K} = \mathbb{C}$ is the field of complex numbers, the map $\Phi(A) = \text{Tr}(A)M$ is an example of such if $M$ is a rank-one hermitian matrix, and $\text{Tr}$ is the trace map, i.e., $\text{Tr}([a_{ij}]) = \sum_{i=1}^{n} a_{ii}$. In the introduction of [7] it is stated as an open problem, whether adjacency preserving maps, with image contained in one line, exist when $\mathbb{K}$ is finite. Theorem 3.1 answers this question negatively when the involution on $\mathbb{K}$ is proper.

(c) In symmetric matrices, i.e., when the involution is the identity map, there exist nonstandard adjacency preservers, whose image contains only pairwise adjacent matrices, even in finite fields. We list a few examples of such maps. For the field $\mathbb{Z}_p$ of integers modulo a prime $p$ it can be verified that the map $\Phi : S_2(\mathbb{Z}_p) \rightarrow S_2(\mathbb{Z}_p)$, $\Phi([a_{ij}]) = (a_{11} + a_{12} + a_{22})E_{11}$ preserves adjacency at least for $p = 2, 5, 11, 17, 23, 29$. The same holds for the map $\Phi([a_{ij}]) = (a_{11} + a_{12} - a_{22})E_{11}$ when $p = 2, 3, 7, 13, 17, 23$, and for the map $\Phi : S_2(GF(2^5)) \rightarrow S_2(GF(2^5))$, $\Phi([a_{ij}]) = (a_{11} + ia_{12} + a_{22})E_{11}$. Recall that $GF(2^5) = \{0, 1, i, 1 + i\}$ with $i^2 = 1 + i$.  

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References