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# The impact problem for two strips

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**Abstract**

This paper considers the dynamic problem of contact interaction of two infinite strips arranged in parallel with an initial gap. Contact interaction of strips occurs due to impulse loading.

Equations of the 2-dimensional generalized theory of plates and the theory of elasticity are used to describe the dynamic behavior of strips. The contact problem is solved using the collocation method with iterative refining of the contact area sizes at each time step.


*Keywords:* Strip; Impact; Dynamic problem; Contact problem; Collocation method; Theory of elasticity

## 1. Introduction

One of the most challenging problems in mechanics of a strained solid is analysis of the strength of thin-wall elements of structures during their unilateral mechanical interaction (Blokh et al., 1977; Kantor, 1990; Mossakovsky et al., 1978). In this case, the law of pressure distribution, the dimensions and configurations of contact zones are unknown a priori, and should be found during problem solution. This involves challenging mathematical computations, especially when solving dynamic problems when interaction parameters have to be defined for each point of time.

Contact problems are dealt with by many authors (Galin, 1976; Goldsmith, 1960; Jäger, 2002; Johnson, 1989; Kantor, 1990; Mackerle, 1998; Muskhelishvili, 1966; Podgorny et al., 1989; Vorovich and Aleksandrov, 2001; Zelentsov, 2004). As a rule, in papers consider 2- and 3-dimensional static or quasistatic

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problems (Johnson, 1989; Kantor, 1990; Podgorny et al., 1989). Dynamic interaction processes have been investigated in less detail, and for this 2-dimensional models are used (Seimov, 1976).

In considering contact problems, a variety of assumptions are used. In some cases, it is supposed that a priori we know the dimensions of the contact area, and the character of distribution of contact pressures. In the conjugate method (Mossakovsky et al., 1978), one identifies the contact area and builds a solution for this area, and also finds a solution for the remaining part of the structure. Then the solutions obtained are joined on the contact area boundary.

In the semi-inverse method, Blokh et al. (1977) describe the character of distribution of contact pressures by an expression, which contains arbitrary constants to be determined during problem solution. Other approaches have been suggested, which are based on eliminating the unknown contact pressure by means of Winkler's link (Kantor, 1990); introducing the contact sublayer (Podgorny et al., 1989); asymptotic methods (Aleksandrov, 1968) and many others (Bezine and Fortune, 1984; Signorini, 1959).

To investigate the behavior of contacting bodies, the most frequently used methods are finite differences, the finite element, or boundary elements (Blokh et al., 1977; Iosilevich and Osipova, 1976; Kantor, 1990; Mackerle, 1998; Mossakovsky et al., 1978; Okamoto Noriaki and Nakazawa Masaru, 1979; Podgorny et al., 1989). Implementing analytical methods involves challenging mathematical computations, and they are frequently confined to considering the classical problem of a rigid die acting on an infinite or semi-infinite domain (Jäger, 2002; Lourier, 1955; Muskhelishvili, 1966; Zelentsov, 2004).

This paper suggests an analytical-numerical approach to the solution of a dynamic contact problem that combines the merits of analytical methods and the universality of numerical ones. The approach suggested is based on the collocations method (Kokhmaniuk et al., 1980; Yanyutin, 1993; Zhemochkin and Sinitsyn, 1962), which makes it possible to simplify significantly the solution of a dynamic contact problem. To describe the non-stationary deformation of interacting strips, the 2-dimensional generalized theory of plates (Ugrimov, 2002) and the theory of elasticity (Shupikov and Ugrimov, 1999) are used. No assumptions related to the character of contact interaction are made. This paper is the first stage in investigating processes occurring when joining sheet materials by explosions.

## 2. Equation of motion of a strip

A simply supported infinite strip of width $A$ and thickness $h$ is referenced in Cartesian rectangular coordinates linked to the upper external surface. The strip is made of a uniform isotropic material. It is assumed that impulse loads are applied to the upper and lower strip surfaces (Fig. 1):

$$q_x^{\text{top}} = q_x^{\text{bottom}} = 0, \quad q_z^{\text{top}} = P^{\text{top}} f^{\text{top}}(x,t), q_z^{\text{bottom}} = -P^{\text{bottom}} f^{\text{bottom}}(x,t), \tag{1}$$

where $q_x^{\text{top}}$, $q_z^{\text{top}}$, $q_x^{\text{bottom}}$, $q_z^{\text{bottom}}$ are the components of external loads applied to the external surfaces of the strip on the top and bottom respectively; $P^{\text{top}}$, $P^{\text{bottom}}$ are their intensities; and $f^{\text{top}}(x,t)$, $f^{\text{bottom}}(x,t)$ are functions defining the character of distribution of components along axis $Ox$ as well as the time-dependent change.

Strip strain is considered small and described by formulas (Parton and Perlin, 1981)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{2}$$

where $u$, $w$ are point displacements along coordinates axes $Ox$, $Oz$.

Stresses are computed using Hooke's law (Parton and Perlin, 1981)

$$p_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{zz}, \quad p_{zz} = (\lambda + 2\mu)\varepsilon_{zz} + \lambda\varepsilon_{xx}, \quad p_{xz} = 2\mu\varepsilon_{xz}, \tag{3}$$

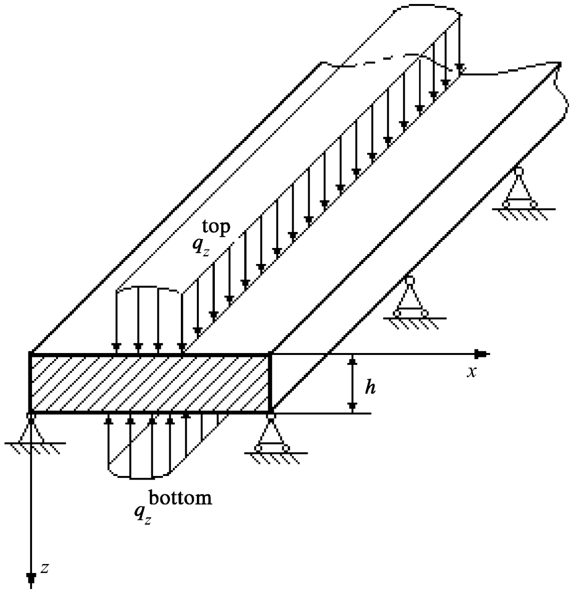


Fig. 1. Strip.

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)}.$$

Here $E$, $\nu$ are the Young modulus and Poisson ratio respectively.

### 2.1. Generalized theory

The 2-dimensional generalized theory is based on representing the components of the displacement vector as finite power series with respect to the transverse coordinate (Ugrimov, 2002):

$$u(x,z,t) = \sum_{k=0}^{K} (z)^k u_k, \quad w(x,z,t) = \sum_{\ell=0}^{L} (z)^\ell w_\ell, \tag{4}$$

where $u_k$, $w_\ell$ are the sought for expansion coefficients as functions of arguments $x$, $t$; $K$, $L$ are parameters, which define the number of terms retained in expansions (4). The kinematic relations accepted in (4) at $K=1$, $L=0$ comply with the Timoshenko type hypotheses (Timoshenko, 1959).

The strip strains (2), with account of hypotheses (4), take the form

$$\varepsilon_{xx} = \sum_{k=0}^{K} (z)^k u_{k,x}, \quad \varepsilon_{zz} = \sum_{\ell=0}^{L} \ell \cdot (z)^{\ell-1} w_\ell, \quad \varepsilon_{xz} = \frac{1}{2}\left(\sum_{k=0}^{K} k \cdot (z)^{k-1} u_k + \sum_{\ell=0}^{L} (z)^\ell w_{\ell,x}\right),$$

and stresses (3) are defined by expressions

$$p_{xx} = (\lambda + 2\mu)\sum_{k=0}^{K} (z)^k u_{k,x} + \lambda \sum_{\ell=0}^{L} \ell \cdot (z)^{\ell-1} w_\ell, \quad p_{zz} = (\lambda + 2\mu)\sum_{\ell=0}^{L} \ell \cdot (z)^{\ell-1} w_\ell + \lambda \sum_{k=0}^{K} (z)^k u_{k,x},$$

$$p_{xz} = \mu\left(\sum_{k=0}^{K} k \cdot (z)^{k-1} u_k + \sum_{\ell=0}^{L} (z)^\ell w_{\ell,x}\right).$$

The generalized forces and moments in the strip are defined by formulas

$$N_{xx}^{k}=\int_{0}^{h}(z)^{k}p_{xx}\,\mathrm{d}z,\quad N_{zz}^{k}=\int_{0}^{h}(z)^{k}p_{zz}\,\mathrm{d}z,\quad N_{xz}^{k}=\int_{0}^{h}(z)^{k}p_{xz}\,\mathrm{d}z.$$

The equations of motion of the strip and the boundary conditions are obtained by using the Hamilton's variational principle (Washizu, 1987). In terms of forces and moments, the equations of motion take the following form

$$\begin{aligned}
&[N_{xx,x}^{0}-I_{x}^{0}]+q_{x}^{\mathrm{top}}+q_{x}^{\mathrm{bottom}}=0,\\
&[N_{xz,x}^{0}-I_{z}^{0}]+q_{z}^{\mathrm{top}}+q_{z}^{\mathrm{bottom}}=0,\\
&N_{xx,x}^{k}-kN_{xz}^{k-1}-I_{x}^{k}+(h)^{k}q_{x}^{\mathrm{bottom}}=0,\\
&N_{xz,x}^{\ell}-\ell N_{zz}^{\ell-1}-I_{z}^{\ell}+(h)^{\ell}q_{z}^{\mathrm{bottom}}=0,\quad k=\overline{1,K},\ \ell=\overline{1,L},
\end{aligned}\tag{5}$$

where

$$I_{x}^{r}=\frac{\rho}{r+1}\left(\sum_{k=0}^{K}\frac{(r+1)(h)^{r+k+1}}{k+r+1}u_{k,tt}\right),\quad r=\overline{0,K},$$

$$I_{z}^{p}=\frac{\rho}{p+1}\left(\sum_{\ell=0}^{L}\frac{(p+1)(h)^{p+\ell+1}}{\ell+p+1}w_{\ell,tt}\right),\quad p=\overline{0,L}.$$

Here $\rho$ is the density of the strip material.

Hence, non-stationary vibrations of a strip in terms of the 2-dimensional generalized theory are described by $K+L+2$ differential equations.

The boundary conditions for a simply supported strip have the form: at $x=0$ and $x=A$

$$N_{xx}^{k}=0,\quad w_{\ell}=0,\quad k=\overline{0,K},\quad \ell=\overline{0,L}.\tag{6}$$

The equations of motion (5) and the boundary conditions (6) can be written in terms of displacements

$$\Omega\cdot\overline{U}_{,tt}-\Lambda\cdot\overline{U}=\overline{Q},\quad 0\leqslant x\leqslant A,$$

$$\Gamma\cdot\overline{U}=0,\quad x=0,\ x=A,$$

where $\overline{U}$ is a vector whose components are the sought for functions

$$\overline{U}=(u_0,u_1,\ldots u_K,w_0,w_1,\ldots,w_L),$$

$\Lambda$, $\Omega$ and $\Gamma$ are square matrices of dimensionality $(K+L+2)\times(K+L+2)$; $\overline{Q}$ is a vector with dimensionality $(K+L+2)$ whose components are a function of the external loads components

$$\overline{Q}=(0,0,\ldots,0,q_{z}^{\mathrm{top}}+q_{z}^{\mathrm{bottom}},hq_{z}^{\mathrm{bottom}},\ldots,(h)^{L}q_{z}^{\mathrm{bottom}}).$$

The elements of matrices $\Omega$, $\Lambda$ and $\Gamma$ are given in Appendix A.

The equations of motion and the boundary conditions are supplemented with the initial conditions. The initial conditions are taken to be null:

$$u_k=w_\ell=0,\quad u_{k,t}=w_{\ell,t}=0\ \text{at}\ t=0.\tag{7}$$

The method of solving 2-dimensional equations of motion of a strip is based on expanding the sought for functions $u_k$, $w_\ell(k=\overline{0,K},\ \ell=\overline{0,L})$ and external loads $q_z^{\mathrm{top}},q_z^{\mathrm{bottom}}$ into series with respect to functions satisfying conditions (6) on the support periphery

$$u_k = \sum_{m=1}^{M} \Phi_{xkm}(t) \cdot \cos\frac{m\pi x}{A},$$

$$\left[w_\ell,\, q_z^{\text{top}},\, q_z^{\text{bottom}}\right] = \sum_{m=1}^{M} [\Phi_{z\ell m}(t),\ q_{zm}^{\text{top}}(t),\ q_{zm}^{\text{bottom}}(t)] \cdot \sin\frac{m\pi x}{A}, \tag{8}$$

where $\Phi_{xkm}(t)$, $\Phi_{zkm}(t)$ are the sought for functions, and $M$ is the number of terms retained in the series.

As a result, the problem on non-stationary deformation of a strip for each value $m$ is reduced to integrating a system of ordinary differential equations with constant coefficients

$$\Omega \cdot \frac{d^2\overline{\Phi}_m}{dt^2} - \Lambda_m \cdot \overline{\Phi}_m = \overline{Q}_m, \tag{9}$$

where $\overline{\Phi}_m$ and $\overline{Q}_m$ are vectors of dimensionality $(K + L + 2)$

$$\overline{\Phi}_m = (\Phi_{xkm}, \Phi_{z\ell m}), \quad k = \overline{0,K}, \ \ \ell = \overline{0,L},$$

$$\overline{Q}_m = (0, \ldots, 0, q_{zm}^{\text{top}} + q_{zm}^{\text{bottom}}, hq_{zm}^{\text{bottom}}, \ldots, (h)^L q_{zm}^{\text{bottom}}).$$

Matrix $\Lambda_m$ is derived from $\Lambda$ by substituting the values of partial derivatives into the expressions for matrix elements (Appendix A).

With account of (8), initial conditions (7) take the form

$$\Phi_m = \frac{d\Phi_m}{dt} = 0 \quad \text{at } t = 0.$$

To integrate the system obtained, a unique one-step method is used, which is based on the method of expanding a solution into Taylor's series (Bakhvalov, 1975).

### 2.2. Theory of elasticity

The behavior of each strip is described by Lame equations (Novatsky, 1975; Parton and Perlin, 1981), which, in case of flat strain, have the form

$$\begin{aligned} &\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial z}\right) = \rho\frac{\partial^2 u}{\partial t^2}, \\ &\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{10}$$

System (10) is solved jointly with conditions (1) for the external surfaces of the strip

$$\begin{aligned} &p_{xz}(x, 0, t) = 0, \quad p_{zz}(x, 0, t) = -q_z^{\text{top}}; \\ &p_{xz}(x, h, t) = 0, \quad p_{zz}(x, h, t) = q_z^{\text{bottom}}; \end{aligned} \tag{11}$$

the boundary conditions

$$p_{xx}(0, z, t) = w(0, z, t) = 0, \quad p_{xx}(A, z, t) = w(A, z, t) = 0, \tag{12}$$

and the initial conditions

$$u(x, z, 0) = w(x, z, 0) = \frac{\partial u(x, z, 0)}{\partial t} = \frac{\partial w(x, z, 0)}{\partial t} = 0. \tag{13}$$

Stresses are computed by formulas (3), and strains are computed by formulas (2).

The method of solving problems (10)–(13) is based on an approach described in paper (Shupikov and Ugrimov, 1999). Displacements and external loads are expanded into series with respect to functions satisfying boundary conditions (12)

$$u = \sum_{m=1}^{M} \Theta_{xm}(z,t) \cdot \cos\frac{m\pi x}{A},$$

$$[w, q_z^{\mathrm{top}}, q_z^{\mathrm{bottom}}] = \sum_{m=1}^{M} [\Theta_{zm}(z,t), q_{zm}^{\mathrm{top}}(t), q_{zm}^{\mathrm{bottom}}(t)] \cdot \sin\frac{m\pi z}{A}.$$

Further, partial derivatives with respect to $z$ of functions $\Theta_{xm}(z,t)$ and $\Theta_{zm}(z,t)$ are replaced with their finite-difference representations. For this, a regular grid is built in each strip

$$z^s = s\tau, \quad s = \overline{0,S}, \ \ \tau = \frac{h}{S}.$$

To approximate the partial derivatives, a 3-point template is used

$$\frac{\partial \Theta_{km}^s}{\partial z} = \frac{\Theta_{km}^{s+1} - \Theta_{km}^{s-1}}{2\tau}, \quad \frac{\partial^2 \Theta_{km}^s}{\partial z^2} = \frac{\Theta_{km}^{s+1} - 2\Theta_{km}^s + \Theta_{km}^{s-1}}{\tau^2},$$

where $\Theta_{km}^s = \Theta_{km}^s(t) = \Theta_{km}(z^s, t), \quad k = x,z.$

As a result of these transforms, the original problem (10)–(13) is reduced to integrating a system of equations for each value $m$. Eq. (10) takes the form

$$\begin{aligned}
&\frac{\mu}{\tau^2}\Theta_{xm}^{s-1} - \left[\frac{2\mu}{\tau^2} + (\lambda+2\mu)\frac{m^2\pi^2}{A^2}\right]\Theta_{xm}^s + \frac{\mu}{\tau^2}\Theta_{xm}^{s+1} + \frac{\lambda+\mu}{2\tau}\frac{m\pi}{A}(\Theta_{zm}^{s+1} - \Theta_{zm}^{s-1}) = \rho\frac{\mathrm{d}^2\Theta_{xm}^s}{\mathrm{d}t^2},\\
&-\frac{\lambda+\mu}{2\tau}\frac{m\pi}{A}(\Theta_{xm}^{s+1} - \Theta_{xm}^{s-1}) + \frac{\lambda+2\mu}{\tau^2}\Theta_{zm}^{s-1} - \left[\frac{2\lambda+4\mu}{\tau^2} + \mu\frac{m^2\pi^2}{A^2}\right]\Theta_{zm}^s\\
&\quad + \frac{\lambda+2\mu}{\tau^2}\Theta_{zm}^{s+1} = \rho\frac{\mathrm{d}^2\Theta_{zm}^s}{\mathrm{d}t^2}, \quad s = \overline{0,S}.
\end{aligned} \tag{14}$$

Conditions (11) and (13) are defined by expressions

$$\begin{aligned}
&\frac{\Theta_{xm}^1 - \Theta_{xm}^{-1}}{2\tau} + \frac{m\pi}{A}\Theta_{zm}^0 = 0, \quad -\lambda\frac{m\pi}{A}\Theta_{xm}^0 + (\lambda+2\mu)\frac{\Theta_{zm}^1 - \Theta_{zm}^{-1}}{2\tau} = -q_{zm}^{\mathrm{top}},\\
&\frac{\Theta_{xm}^{S+1} - \Theta_{xm}^{S-1}}{2\tau} + \frac{m\pi}{A}\Theta_{zm}^S = 0, \quad -\lambda\frac{m\pi}{A}\Theta_{xm}^S + (\lambda+2\mu)\frac{\Theta_{zm}^{S+1} - \Theta_{zm}^{S-1}}{2\tau} = -q_{zm}^{\mathrm{bottom}};
\end{aligned} \tag{15}$$

$$\Theta_{xm}^s(0) = \Theta_{zm}^s(0) = \frac{\mathrm{d}\Theta_{xm}^s(0)}{\mathrm{d}t} = \frac{\mathrm{d}\Theta_{zm}^s(0)}{\mathrm{d}t} = 0. \tag{16}$$

Boundary conditions (12) are met identically.

Conditions (15) make it possible to eliminate the values of unknown functions in peripheral points $\Theta_{xm}^{-1}$, $\Theta_{xm}^{S+1}$, $\Theta_{zm}^{-1}$, $\Theta_{zm}^{S+1}$ from system (14).

As a result, the solution of the problem on the strip dynamic response is reduced to integrating a system of ordinary differential equations with constant coefficients for each value $m$

$$\widetilde{\Omega} \cdot \frac{\mathrm{d}^2\overline{\Theta_m}}{\mathrm{d}t^2} - \widetilde{\Lambda}_m \cdot \overline{\Theta_m} = \overline{\widetilde{Q}_m}, \tag{17}$$

where $\overline{\Theta}_m$ and $\overline{\overline{Q}}_m$ are vectors of dimensionality $2(S+1)$ having the following components

$$\overline{\Theta}_m = (\Theta^0_{xm}, \Theta^0_{zm}, \Theta^1_{xm}, \Theta^1_{zm}, \ldots, \Theta^S_{xm}, \Theta^S_{zm}),$$

$$\overline{\overline{Q}}_m = \left(0, \frac{2}{\tau} q^{\text{top}}_{zm}, 0, \ldots, 0, 0, \frac{2}{\tau} q^{\text{bottom}}_{zm}\right).$$

$\widetilde{\Omega}$, $\widetilde{\Lambda}_m$ are square matrices of dimensionality $2(S+1) \times 2(S+1)$. The matrix elements are given in Appendix B.

The system is integrated similar to that of the generalized theory.

## 3. Contact problem

Two infinite simply supported strips are arranged in parallel with an initial gap $\delta$ (Fig. 2). The geometrical dimensions of the strips are as follows: $A$ is width; and $h^1$ and $h^2$ are the thickness of the upper and lower strips respectively. By convention, superscript 1 will designate all the parameters relating to the upper strip, and 2 will designate those for the lower strip.

Interaction of strips occurs in an a priori unknown area of width $\Sigma$ as a result of impulse loading of the upper strip with load (1).

Condition

$$-q_z^{1\text{bottom}} = q_z^{2\text{top}} = F(x,t), \quad x \in \Sigma. \tag{18}$$

is satisfied in the contact area. Here $F(x,t)$ is the unknown contact pressure. It is assumed that there is no friction in the contact area. Beyond the contact zone $q_z^{1\text{bottom}} = q_z^{2\text{top}} = 0$.

Hence, the original contact problem is reduced to integrating a system of differential equations (9) or (17), which describe the vibrations of strips with account of loads (1) and (18).

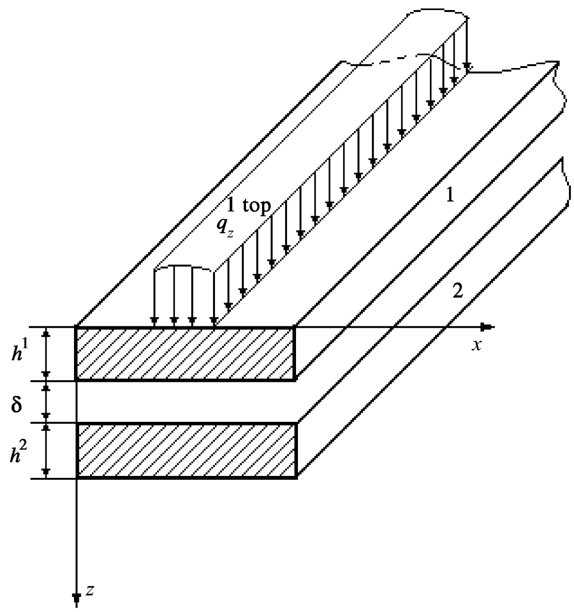


Fig. 2. Contact problem.

The unknown contact pressure and the contact area are found from condition

$$w^1(x,h^1,t) - w^2(x,0,t) - \delta \leqslant 0. \tag{19}$$

In the contact area, inequality (19) transforms to an equality.

To determine the contact area and the contact pressures, the collocations method is used (Kokhmaniuk et al., 1980; Yanyutin, 1993; Zhemochkin and Sinitsyn, 1962), according to which each strip is divided widthwise into $N$ sections

$$x_k = k\Delta x, \quad \Delta x = \frac{A}{N}, \ \ k = \overline{0,N}.$$

In each section, the contact pressure is considered to be constant

$$F(x,t) = F_k(t), \quad x \in [x_{k-1}, x_k],$$

thus allowing replacing a continuous contact pressure distribution step-function (Fig. 3):

$$F(x,t) = \sum_{k=1}^{N} F_k(t)(H(x_k - x) - H(x_{k-1} - x)),$$

where $H(x)$ is the Heaviside function.

Further, for each section, conditions (19) are written as equalities

$$w^1(\zeta_k,h^1,t) - w^2(\zeta_k,0,t) - \delta = 0, \quad k = \overline{1,N},$$

where $\xi_k = \frac{x_k+x_{k-1}}{2}$, making it possible to form an inhomogeneous system of linear algebraic equations with respect to function $F_k(t)$

$$A\overline{F} = \overline{B}. \tag{20}$$

Here $A$ is an $N \times N$ matrix; $\overline{F}$ is a vector whose components are functions $F_k(t)$, $\overline{F} = (F_1, F_2, \ldots, F_N)$.

If the solution of this system yields only negative values of $F_k$, there is no contact between strips, and $F_k$ is assumed to equal zero.

If in some sections $F_k > 0$, and in other ones $F_k \leqslant 0$, this is indicative of a possible contact. The contact area is defined by the multitude of sections where $F_k > 0$.

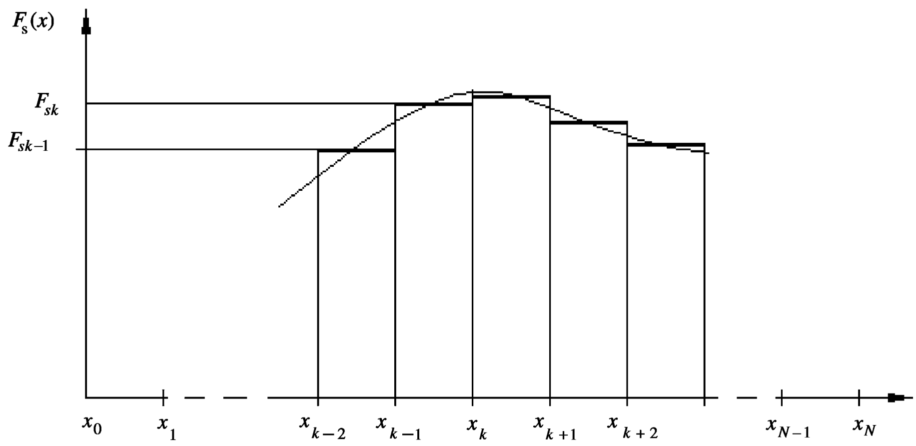


Fig. 3. Collocation sections.

The contact area found is refined with an iterative procedure comprising in essence the following:

- for sections where $F_k \leqslant 0$, the values of contact pressures are assumed to equal zero $F_k = 0$;
- for sections where $F_k > 0$, the system of Eqs. (20) is built and solved;
- if all anew obtained solutions $F_k > 0$, then they are the sought for values of unknown contact pressures $F_k$; and the sections, for which $F_k > 0$, comprise the refined contact area;
- if some anew found $F_k \leqslant 0$, the procedure is repeated.

Searching for the contact pressures and the contact area by using the given technique, is carried out at each step of integrating the system of differential equations (9) or (17) with respect to time.

## 4. Numerical results

Let us consider contact interaction of two steel strips ($A = 1$ m, and $h^1 = h^2 = 0.05$ m) arranged with a gap of 0.5 cm, and affected by an impulse load

$$q_z^{1\mathrm{top}} = P_0 H(t), \quad 0 \leqslant x \leqslant A,$$

where $H(t)$ is the Heaviside function, and $P_0 = 0.7$ MPa. The steel properties are as follows: $E = 2.1 \times 10^5$ MPa, $\nu = 0.3$, and $\rho = 7.85 \times 10^3$ kg/m$^3$.

In all computations given below, the stress-strained condition of the strips is described by the generalized theory with parameters $K = 7$, $L = 6$ and by the theory of elasticity. In so doing, 200 terms were retained in the Fourier series ($M = 200$), and 105 collocation sections ($N = 105$) were selected. In addition, when solving the theory of elasticity equations, in each strip a 40-point grid ($S = 40$) was used. During subsequent increase of parameters $K$, $L$, $M$, $N$, and $S$, the numeric results remained practically invariable, which points to the inner convergence of the method. The time interval investigated was within 0–17.5 ms.

Table 1 summarizes the maximum values of deflections $w^i$, stresses $p^i_{xx}$, $p^i_{zz}$, as well as the values of the contact force per strip unit length

$$P = \int_{\Sigma} F(x,t)\,\mathrm{d}x.$$

In the table, values obtained by the generalized theory are shown above the line, and those obtained by using the theory of elasticity are shown under the line. The maximum values of deflections $w^i$ and stresses $p^i_{xx}$ computed by these theories practically coincide with one another. Note that the conditions of contact between strips both in terms of the theory of elasticity and the generalized theory are strictly satisfied ($p_{zz}^{1\mathrm{bottom}} = p_{zz}^{2\mathrm{top}}$).

Table 1
Stress-strained state of strips during impact and the maximum contact force

| | Strip 1 | | Strip 2 | |
|---|---|---|---|---|
| | Top | Bottom | Top | Bottom |
| Deflection, $w^i$ (cm) | $\frac{0.5853}{0.5858}$ | $\frac{0.5852}{0.5857}$ | $\frac{0.2885}{0.2895}$ | $\frac{0.2885}{0.2895}$ |
| Stresses $p^i_{xx}$ (MPa) | $\frac{-385.1}{-385.2}$ | $\frac{381.3}{381.5}$ | $\frac{-212.6}{-214.3}$ | $\frac{212.2}{213.7}$ |
| Stresses $p^i_{zz}$ (MPa) | $\frac{-1.0}{-0.7}$ | $\frac{-91.5}{-80.1}$ | $\frac{-91.5}{-80.1}$ | $\frac{-0.3}{0}$ |
| Contact force $P$ (kN) | $\frac{49.62}{50.05}$ | | | |

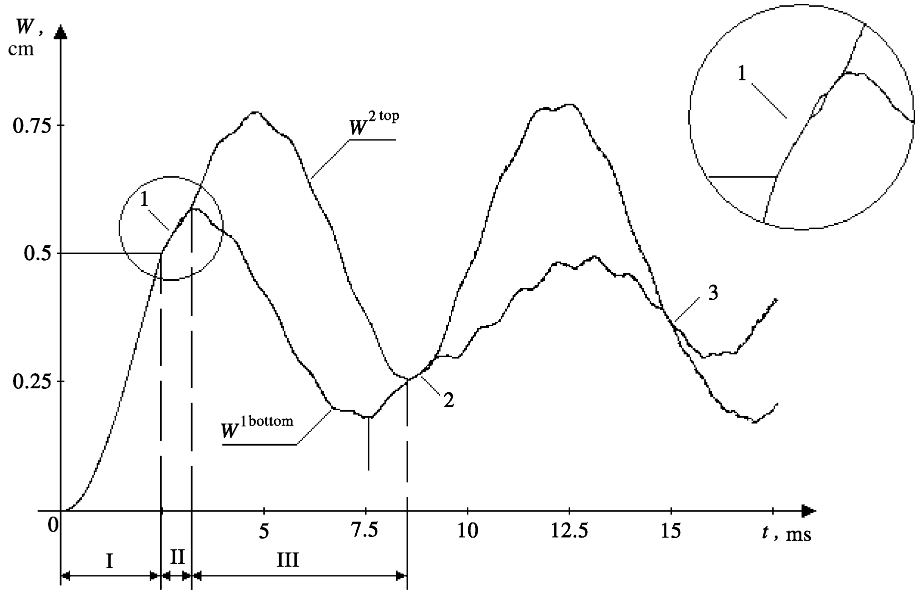


Fig. 4. Deflection of strips.

Fig. 4 shows the deflection in the middle of the surfaces of contacting strips vs. time. Hereinafter the solid line will designate results obtained by the theory of elasticity, and the dotted line will refer to those obtained by the generalized theory. It is clear that the results of calculating deflections by the theories considered practically coincide.

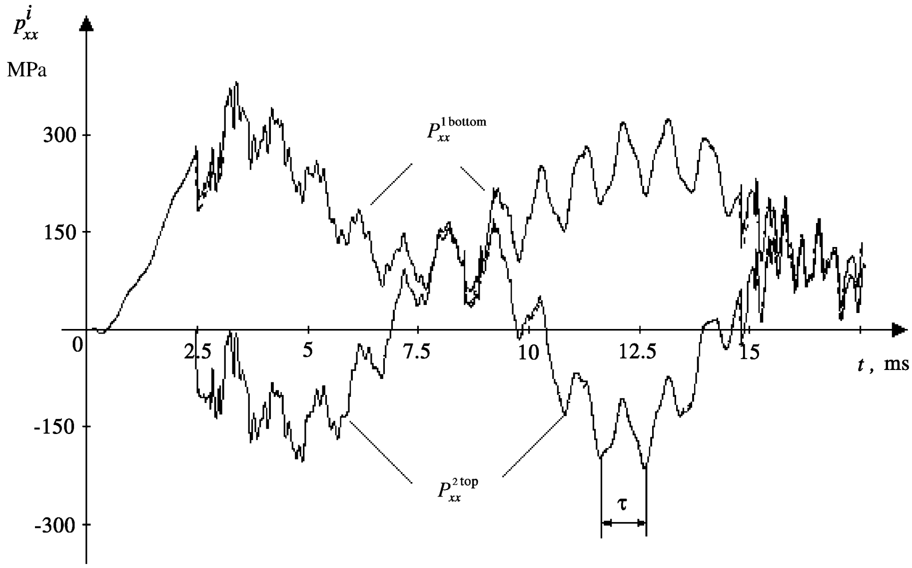


Fig. 5. Stresses $p_{xx}^{i}$ in strips vs. time.

The figure shows three time intervals (I, II, and III) corresponding to the characteristic stages of strip deformation.

First stage (I). The upper strip performs forced vibrations, and the lower one located at the distance of 0.5 cm from the upper strip is at rest.

Second stage (II). The strips come into contact interaction and perform joint motion. In so doing, the effect of recoil of one strip from the other one followed by subsequent impact is observed (expanded view 1 in Fig. 4).

Third stage (III). The strips perform independent vibrations. The upper strip performs forced vibrations under the effect of the external impulse load, and the lower strip performs free vibrations until the next impact.

Fig. 4 also shows that, in the time interval being investigated, there are three areas 1 to 3 of contact interaction of strips. After each interaction the strips perform independent motion.

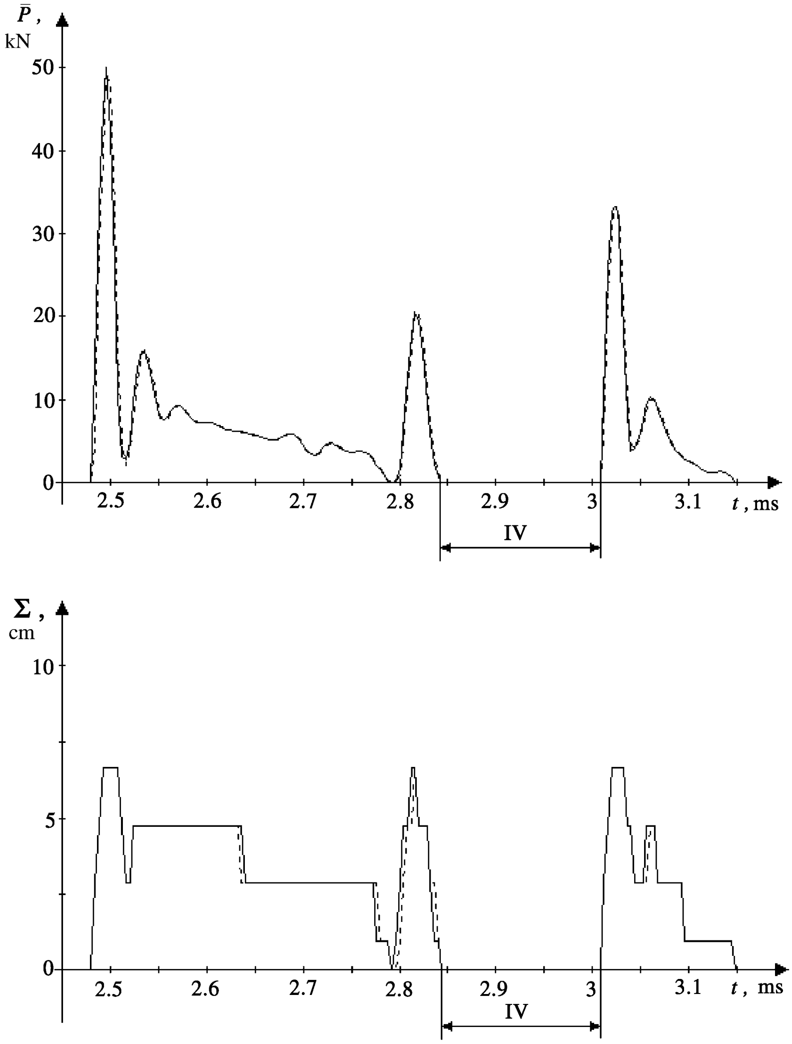


Fig. 6. Contact force and contact area width.

Table 2
Contact force on collocation section

| $t$, ms | Contact force $F_k \cdot \Delta x$, kN | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $F_{49} \cdot \Delta x$ | $F_{50} \cdot \Delta x$ | $F_{51} \cdot \Delta x$ | $F_{52} \cdot \Delta x$ | $F_{53} \cdot \Delta x$ | $F_{54} \cdot \Delta x$ | $F_{55} \cdot \Delta x$ | $F_{56} \cdot \Delta x$ | $F_{57} \cdot \Delta x$ |
| 2.480 | 0 | 0 | 0 | 0 | 0.38 | 0 | 0 | 0 | 0 |
| 2.484 | 0 | 0 | 0 | 4.30 | 5.37 | 4.30 | 0 | 0 | 0 |
| 2.488 | 0 | 0 | 5.32 | 7.56 | 7.96 | 7.56 | 5.32 | 0 | 0 |
| 2.492 | 0 | 2.67 | 7.25 | 8.07 | 8.70 | 8.07 | 7.25 | 2.67 | 0 |
| 2.496 | 0 | 4.79 | 7.50 | 8.41 | 8.65 | 8.41 | 7.50 | 4.79 | 0 |
| 2.500 | 0 | 4.87 | 6.09 | 6.74 | 6.91 | 6.74 | 6.09 | 4.87 | 0 |
| 2.504 | 0 | 3.12 | 3.88 | 3.90 | 3.91 | 3.90 | 3.88 | 3.12 | 0 |

Fig. 5 shows a comparison of stresses $p^i_{xx}$ obtained by the generalized theory and the theory of elasticity. The calculation results are given for the middle of the external surfaces of contacting strips. In Figure, it is clear that, after collision, the 1st, 2nd and higher frequencies are excited. The 1st and 2nd natural frequencies are equal 116.8 Hz and 1020.6 Hz correspondingly. The period of the 2nd natural frequency ($\tau = 0.98$ ms) is given in Fig. 5.

Changes in contact forces and dimensions of contact area $\Sigma$ at stage II are shown in Fig. 6. It is clear that a change in contact force value involves a change in dimensions of contact area $\Sigma$. Roman numeral IV in the figure shows the time interval during which one can observe the above-mentioned effect of recoil of one strip from the other one in the process of contact interaction at stage II (Fig. 4).

Besides, Table 2 also summarizes the changes in the contact area and values of contact force. It shows values $F_k(t) \cdot \Delta x$ for the initial phase of stage II obtained by the theory of elasticity. The results of generalized theory analysis are similar to the data given in the table, so they are not shown. The table shows that the dimensions of the contact area and the contact force in each collocation section are a function of time. The contact area for the problem being considered is located symmetrically relative to the middle of the strip.

## 5. Conclusions

The paper presents an approach to solving the dynamic problem of contact interaction of two parallel strips with an initial gap between them, which are affected by an impulse load applied to the external surface of one of the strips.

The approach suggested is based on the collocation method. Using this method allows to significantly simplify the process of searching for an a priori contact area and determining the contact pressures at each point in time.

The contact area is a multitude of collocation sections, for which positive values of contact pressures have been obtained. Its dimensions are found with accuracy equal to the length of the collocation section. The area dimensions are changed by joining new sections or excluding those, for which the contact condition is violated. Therefore, the contact area width vs. time function (Fig. 6) has a step character.

For the static loading case, methods have been developed, which allow determining the dimensions of the contact area and the values of contact pressures with greater accuracy than the approach used. However, for the case of dynamic contact problems, which are very involved from the mathematical viewpoint, the approach offered is justified.

The strip response is described by equations of the 2-dimensional generalized theory of plates and the theory of elasticity. The results of investigations obtained by using these theories practically coincide. There are some differences for the components of the stress tensor $p^i_{zz}$ and the contact force $P$. These differences are explained by the fact that the 2-dimensional theory does not take into account the presence of wave processes over the thickness of the strips, which occur during their dynamic contact interaction.

## Appendix A

The elements of the lower triangle of matrix $\Omega$ have the following form

$$\Omega_{1\,1} = -\Omega_{2+K\,2+K} = -h\rho, \quad \Omega_{1+k\,1} = -\frac{h^{k+1}\rho}{k+1}, \quad \Omega_{1+k\,1+r} = -\frac{h^{k+r+1}\rho}{k+r+1},$$
$$\Omega_{2+K+p\,2+K} = \frac{h^{p+1}\rho}{p+1}, \quad \Omega_{2+K+p\,2+K+\ell} = \frac{h^{p+\ell+1}\rho}{p+\ell+1}, \qquad k,r = \overline{1,K}, \quad \ell,p = \overline{1,L}.$$

The remaining matrix elements for lower triangle are equal to zero.
The elements of the lower triangle of matrix $\Lambda$ have the form

$$\Lambda_{1\,1} = -h(\lambda+2\mu)\frac{\partial^2}{\partial x^2}, \quad \Lambda_{1+k\,1} = -\frac{h^{k+1}}{k+1}(\lambda+2\mu)\frac{\partial^2}{\partial x^2},$$
$$\Lambda_{1+k\,1+r} = -\frac{h^{k+r+1}}{k+r+1}(\lambda+2\mu)\frac{\partial^2}{\partial x^2} + \frac{kr\mu h^{r+k-1}}{r+k-1}, \quad \Lambda_{2+K\,1+r} = h^r\mu\frac{\partial}{\partial x},$$
$$\Lambda_{2+K\,2+K} = h\mu\frac{\partial^2}{\partial x^2}, \quad \Lambda_{2+K+p\,1} = -h^p\lambda\frac{\partial}{\partial x}, \quad \Lambda_{2+K+p\,1+r} = -h^{p+r}\frac{\lambda p - \mu r}{p+r}\frac{\partial}{\partial x},$$
$$\Lambda_{2+K+p\,2+K} = \frac{h^{p+1}\mu}{p+1}\frac{\partial^2}{\partial x^2}, \quad \Lambda_{2+K+p\,2+K+\ell} = \frac{h^{p+\ell+1}\mu}{p+\ell+1}\frac{\partial^2}{\partial x^2} - \frac{p\ell(\lambda+2\mu)h^{p+\ell-1}}{p+\ell-1},$$
$$k,r = \overline{1,K}, \quad \ell,p = \overline{1,L}.$$

The elements of matric $\Gamma$ have the form

$$\Gamma_{1\,1} = h(\lambda+2\mu)\frac{\partial}{\partial x}, \quad \Gamma_{1\,1+r} = \frac{h^{r+1}}{r+1}(\lambda+2\mu)\frac{\partial}{\partial x}, \quad \Gamma_{1\,2+K+\ell} = h^\ell\lambda,$$
$$\Gamma_{1+k\,1} = \frac{h^{k+1}}{k+1}(\lambda+2\mu)\frac{\partial}{\partial x}, \quad \Gamma_{1+k\,1+r} = \frac{h^{k+r+1}}{k+r+1}(\lambda+2\mu)\frac{\partial}{\partial x},$$
$$\Gamma_{1+k\,2+K+\ell} = h^{k+\ell}\lambda\frac{\ell}{k+\ell}, \quad \Gamma_{2+K\,2+K} = \Gamma_{2+K+p\,2+K+p} = 1.$$

The remaining elements of matrices $\Gamma$ are equal to zero.

## Appendix B

The elements of matric $\widetilde{\Omega}$ have the form

$$\widetilde{\Omega}_{ii} = \rho, \quad i = \overline{1,2(S+1)}$$

The remaining elements of matrices $\widetilde{\Omega}$ are equal to zero.

The elements of matrix $\widetilde{\Lambda}$ have the following form

$$\widetilde{\Lambda}_{11} = \frac{2\mu}{\tau^2} + \left(\lambda + 2\mu - \frac{\lambda(\lambda+\mu)}{\lambda+2\mu}\right)\frac{m^2\pi^2}{A^2}, \quad \widetilde{\Lambda}_{12} = -\frac{2\mu}{\tau}\frac{m\pi}{A}, \quad \widetilde{\Lambda}_{13} = -\frac{2\mu}{\tau^2}, \quad \widetilde{\Lambda}_{14} = 0,$$

$$\widetilde{\Lambda}_{21} = \frac{2\lambda}{\tau}\frac{m\pi}{A}, \quad \widetilde{\Lambda}_{22} = \frac{2(\lambda+2\mu)}{\tau^2} - \lambda\frac{m^2\pi^2}{A^2}, \quad \widetilde{\Lambda}_{23} = 0, \quad \widetilde{\Lambda}_{24} = -\frac{2(\lambda+2\mu)}{\tau^2},$$

$$\widetilde{\Lambda}_{1+2i\,2i-1} = -\frac{\mu}{\tau^2}, \quad \widetilde{\Lambda}_{1+2i\,2i} = \frac{(\lambda+\mu)}{2\tau}\frac{m\pi}{A}, \quad \widetilde{\Lambda}_{1+2i\,1+2i} = \frac{2\mu}{\tau^2} + (\lambda+2\mu)\frac{m^2\pi^2}{A^2}, \quad \widetilde{\Lambda}_{1+2i\,2+2i} = 0,$$

$$\widetilde{\Lambda}_{1+2i\,3+2i} = -\frac{\mu}{\tau^2}, \quad \widetilde{\Lambda}_{1+2i\,2i+4} = -\frac{(\lambda+\mu)}{2\tau}\frac{m\pi}{A}, \quad \widetilde{\Lambda}_{2+2i\,2i-1} = -\frac{(\lambda+\mu)}{2\tau}\frac{m\pi}{A},$$

$$\widetilde{\Lambda}_{2+2i\,2i} = -\frac{(\lambda+2\mu)}{\tau^2}, \quad \widetilde{\Lambda}_{2+2i\,1+2i} = 0, \quad \widetilde{\Lambda}_{1+2i\,2+2i} = \frac{2(\lambda+2\mu)}{\tau^2} + \mu\frac{m^2\pi^2}{A^2},$$

$$\widetilde{\Lambda}_{2+2i\,3+2i} = \frac{\lambda+\mu}{2\tau}\frac{m\pi}{A}, \quad \widetilde{\Lambda}_{1+2i\,4+2i} = -\frac{\lambda+2\mu}{\tau^2}, \quad \widetilde{\Lambda}_{2S+1\,2S-1} = -\frac{2\mu}{\tau^2}, \quad \widetilde{\Lambda}_{2S+1\,2S} = 0,$$

$$\widetilde{\Lambda}_{2S+1\,2S+1} = \frac{2\mu}{\tau^2} + \left(\lambda + 2\mu - \frac{\lambda(\lambda+\mu)}{\lambda+2\mu}\right)\frac{m^2\pi^2}{A^2}, \quad \widetilde{\Lambda}_{2S+1\,2S+2} = \frac{2\mu}{\tau}\frac{m\pi}{A}, \quad \widetilde{\Lambda}_{2S+2\,2S-1} = 0,$$

$$\widetilde{\Lambda}_{2S+2\,2S} = -\frac{2(\lambda+2\mu)}{\tau^2}, \quad \widetilde{\Lambda}_{2S+2\,2S+1} = -\frac{2\lambda}{\tau}\frac{m\pi}{A}, \quad \widetilde{\Lambda}_{2S+2\,2S+2} = \frac{2(\lambda+2\mu)}{\tau^2} - \lambda\frac{m^2\pi^2}{A^2}, \qquad i = \overline{1, S-1}.$$

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