Some Distribution Numbers of the Triangular Association Scheme

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In line with the two preceding papers here we compute some distribution numbers of the triangular association scheme (or Johnson-scheme) \( J(n, k) \). In the first section we give some general upper and lower bounds. In the second section we show that these bounds are met with equality in case of the existence of Steiner systems. Here we heavily use theorem 4 of [1]. In the third section we compute \( vt_t(J(n, k)) \) for \( I = \{2, 3, \ldots, k\} \) and all values of \( n, k \). Finally we conclude with an example where one of our (obvious) bounds is not attained with equality.

1. We obtain some general upper and lower bounds for the distribution numbers \( vt_t(J(n, k)) \). First we show:

\[
vt_t(J(n, k)) \leq \binom{n-t}{k-t} \quad \text{for all} \quad 1 \leq t \leq k \leq n, \quad I = \{1, 2, \ldots, t\}. \tag{1}
\]

**Proof of (1).** It is well known that for some specific \( t \)-subset of the given \( n \)-set \( N, T_0 \) with \( |T_0| = t \) say, the following set \( Y_0 \) is a \( \{t + 1, \ldots, k\} \)-design, where \( Y_0 \) is given as

\[
Y_0 = \{K \in J(n, k) \mid T_0 \subseteq K\}. \tag{1'}
\]

By theorem 3 of [1] (1) follows.

On the other hand we have that a vector \( e \in \mathbb{R}[J(v, k)] \) is in the space \( E_t \) for \( I = \{t+1, \ldots, k\} \) if and only if for all \( t \)-element subsets \( T \) of the \( n \)-set \( N \) the following relation holds for \( k \)-sets \( K \in J(n, k) \)

\[
\sum_{T \subseteq K} e(K) = 0 \quad \text{for all} \quad T \text{ with } |T| = t, \quad \text{summing over } K. \tag{2}
\]

Let \( \text{cov}(t, k, n) \) be the smallest number of \( k \)-sets needed to cover the given \( n \)-set \( N \) in such a way that every \( t \)-subset \( T \subseteq N \) is contained in at least one \( k \)-set \( K \subseteq N \). For fixed \( k, n \) any subset of \( J(n, k) \) with this property will be called a \( t \)-cover.

Now if \( e \in E_t \subset \mathbb{R}[J(n, k)] \) with \( I = \{t+1, \ldots, k\} \) and if \( e \) is a general vector, then it easily follows from (2) that in \( X_\lambda(e) = \{K \in J(n, k) \mid e(K) > 0\} \) there is at least one set \( K \) for every given set \( T \subseteq N, |T| = t \) such that \( T \subseteq K \). This shows that every distribution set \( X_\lambda(e) \) is a \( t \)-cover. Therefore we get the inequality

\[
vt_t(J(n, k)) \geq \text{cov}(t, k, n) \quad \text{for} \quad I = \{t+1, \ldots, k\}. \tag{3}
\]

We may then use the known bounds on \( \text{cov}(t, k, n) \) (see [5], [6]) to obtain actual numerical bounds on \( vt_t(J(n, k)) \) for \( I = \{t+1, \ldots, k\} \). For \( x \in \mathbb{R} \) let \( \lfloor x \rfloor \) be the smallest integer \( y \) such that \( y \geq x \). Then from [5], p. 1407, theorem 1 with \( \lambda = 1 \) we may deduce

\[
vt_t(J(n, k)) \geq \left\lfloor \frac{n \left\{ \frac{n-1}{k-1} \left\{ \ldots \left\{ \frac{n-t+1}{k-t+1} \right\} \ldots \right\} \right\}} {k} \right\rfloor, \quad I = \{t+1, \ldots, k\}. \tag{4}
\]

In particular if there exists a Steiner system \( S(t, k, v) \) we can conclude that

\[
vt_t(J(n, k)) \geq \frac{n(n-1) \ldots (n-t+1)}{k(k-1) \ldots (k-t+1)}. \tag{5}
\]

In some cases we may derive bounds stronger than (4) from (3), see [6].
2. On the other hand from the existence of a Steiner system we may derive bounds complementary to (1) and (5). Let \( ST \) be a Steiner system \( S(t, k, v) \), and put \( ST = c(ST) = \sum_{s \in ST} e_s \in \mathbb{R}[J(n, k)] \) the characteristic vector of \( ST \). Apparently the two sets \( ST \) and \( Y_0 \) in (1') together form a perfect pair in the sense of [4], and from the transitivity of the automorphism group \( \Sigma_a \) on \( J(n, k) \) we may deduce using theorems 3, 4 of [1] that

\[
v_{t_i}(J(n, k)) = \frac{n(n - 1) \ldots (n - k + 1)}{k(k - 1) \ldots (k - t + 1)} \quad \text{for} \ I = \{t + 1, \ldots, k\},
\]

where a distribution set is given by \( ST \).

On the other hand we get

\[
v_{t_i}(J(n, k)) = \binom{n-t}{k-t} \quad \text{for} \ I = \{1, 2, \ldots, k\},
\]

and a distribution set is given by \( Y_0 \), if \( S(t, k, v) \) exists.

We may thus use the existence theorems for Steiner systems to deduce many exact values for \( v_{t_i}(J(n, k)) \), where \( I = \{1, 2, \ldots, t\} \) or \( I = \{t + 1, \ldots, k\} \). We omit the detailed formulas.

3. In this section we show that

\[
v_{t_i}(J(n, k)) = q + 1 \quad \text{for} \ n = qk + r, \ 1 \leq r \leq k, \ q \geq 2, \ \text{and} \ I = \{2, 3, \ldots, k\}.
\]

This formula complements results of [3].

**PROOF OF (8).** Let \( X \) be a set of \( n \) elements, and choose two subsets \( X_1, X_2 \) of \( X \) with

\[ |X_1| = |X_2| = qk, \ |X_1 \cap X_2| = qk - r. \]

In each set \( X_i \) we choose a Steiner system \( ST_i \) of type \( S(1, k, qk) \) which is just a partition of the \( qk \)-set \( X_i \) into \( qk \)-sets. We choose \( ST_1 \) and \( ST_2 \) in such a way that the two partitions agree on a set of cardinality \( (q - 1)k \) contained in \( X_1 \cap X_2 \). We have accordingly

\[ |ST_1| = |ST_2| = q, \ |ST_1 \cap ST_2| = q - 1, \ |ST_1 \cup ST_2| = q + 1. \]

Correspondingly we have two copies of \( J(qk, k) \) which we shall denote by \( J(X_i, k) \) and we take the characteristic vectors of the Steiner systems to be \( ST_i \in \mathbb{R}[J(X_i, k)] \), and we let \( u_i \in \mathbb{R}[J(X_i, k)] \) be the all-one vectors. Then we may define the general vectors

\[
v_i = \binom{qk - 1}{k - 1} ST_i - u_i \in \mathbb{R}[J(X_i, k)],
\]

and via the inclusions \( J(X_i, k) \subseteq J(X, k) = J(n, k) \) we may regard them as vectors \( v_i \in \mathbb{R}[J(n, k)] \). By (2) it is easy to see that \( v_i \in E_i \subseteq \mathbb{R}[J(n, k)] \) with \( I = \{2, 3, \ldots, k\} \). The idea of the proof of (8) is to add some \( w \) to the sum of the vectors \( v_i \) in (9) such that \( v_1 + v_2 + w \) is a general vector in \( E_i \) and such that \( X_+(v_1 + v_2 + w) = ST_1 \cup ST_2 \). We proceed to the details.

First we define certain positive numbers depending on \( q \) and \( k \) whose actual expression in terms of binomial coefficients is not really required in the calculation below. For this reason we shall only give the Anzahl-definition in the following way.

\[
N_a = |\{S \in J(n, k) \mid S \ni x, \ S \notin X_1 \text{ and } S \subseteq X_2\}|,
\]

\[
N_b = |\{S \in J(n, k) \mid S \ni x, \ S \subset X_1 \text{ or } S \subset X_2, \text{ but } S \notin X_1 \cap X_2\}|,
\]

\[
N_c = |\{S \in J(n, k) \mid S \ni x, \ S \subset X_1 \cap X_2\}|,
\]
for some fixed element \( x \in X_1 \cap X_2 \). Similarly we define
\[
M_a = | \{ S \in J(n, k) | S \ni x, S \subset X_1 \text{ or } S \subset X_2 \} |,
\]
\[
M_b = | \{ S \in J(n, k) | S \ni x, S \cap X_1 \neq \emptyset \neq S \cap X_2 \} |,
\]
for some fixed \( x \notin X_1 \cap X_2 \).

We now consider two linear equations in three unknowns \( a, b, c \).
\[
N_a \cdot a + N_b \cdot b + N_c \cdot c = 0, \tag{10}
\]
\[
M_a \cdot a + M_b \cdot b = 0. \tag{10'}
\]
We can find a real solution of (10) and (10') which has the properties
\[
-\left( \frac{qk - 1}{k - 1} \right) + 2 < a < 0, \quad b, c, < 1 \tag{11}
\]
by multiplying any given nontrivial solution of (10) and (10') with a suitable small constant.

We now define the vector \( w \) by
\[
w(S) = \begin{cases} 
a & \text{if } S \notin X_1 \text{ and } S \notin X_2, 
b & \text{if } S \subset X_1 \text{ or } S \subset X_2, \text{ but } S \notin X_1 \cap X_2, 
c & \text{if } S \subset X_1 \cap X_2. \end{cases}
\]
For any point \( x \in X_1 \cap X_2 \) we then have by (10)
\[
\sum_{S \ni x} w(S) = N_a \cdot a + N_b \cdot b + N_c \cdot c = 0.
\]
Similarly for any point \( x \notin X_1 \cap X_2 \) we have by (10')
\[
\sum_{S \ni x} w(S) = M_a \cdot a + M_b \cdot b = 0.
\]
Thus we get \( w \in E_1 \) for \( I = \{2, 3, \ldots, k\} \) by (2). We now regard the vector
\[
v = v_1 + v_2 + w \in E_1 \subset \mathbb{R}[J(n, k)]. \tag{12}
\]

The negative entries in \( v_1 + v_2 \) are all \( \leq -1 \), while \( a, b, c < 1 \), so for any \( S \) with \( v_1(S) + v_2(S) < 0 \) we shall also have \( v(S) < 0 \). If on the other hand \( v_1(S) + v_2(S) = 0 \) then we have \( S \notin X_1 \) and \( S \notin X_2 \), so that \( v(S) = w(S) = a < 0 \) by (11). Finally if \( v_1(S) + v_2(S) > 0 \), then we must have
\[
v_1(S) + v_2(S) \in \left\{ \left( \frac{qk - 1}{k - 1} \right) - 2, 2 \left( \frac{qk - 1}{k - 1} \right) - 2 \right\}. \tag{13}
\]
Thus we get
\[
v(S) = v_1(S) + v_2(S) + w(S) \geq v_1(S) + v_2(S) + a,
\]
which is positive by (11) and (13). It follows that \( v \) in (12) is a distribution vector and that \( X_+(v) = ST_1 \cup ST_2 \). On the other hand from (4) we get \( vt(J(n, k)) \geq q + 1 \). This proves (8).

Note that for \( k = 2 \) (8) becomes
\[
vt_2(J(n, 2)) = \frac{n}{2}, \quad \text{if } n \equiv 0(2),
\]
\[
vt_2(J(n, 2)) = \frac{n + 1}{2}, \quad \text{if } n \equiv 1(2), \tag{14}
\]
a claim made in [2]. Minimal distribution sets according to the above proof are given by
\[
\{1, 2\}, \{3, 4\}, \ldots, \{n - 1, n\} \quad \text{for } n \equiv 0(2)
\]
\[
\{1, 2\}, \{3, 4\}, \ldots, \{n - 2, n - 1\}, \{n - 2, n\} \quad \text{for } n \equiv 1(2).
\]

4. We give an example of strict inequality in (3). Let \( t = 2, k = 3, n = 6 \). Then (4) and the cover \{123\}, \{145\}, \{126\}, \{245\}, \{346\} \{356\} show that \( \text{cov} (2, 3, 6) = 6 \).

On the other hand it is well known that in \( E_3 \subset \mathbb{R}[J(6, 3)] \) we have \( P_3(S) = -P_3(\bar{S}) \) where \( \bar{S} \) = complement of \( S \) in the set \( X = \{1, 2, 3, 4, 5, 6\} \). This obviously implies \( \nu t_3(J(6, 3)) = 10 \).

However we believe that this is the only such example for \( k = 3 \). This calculation will be given elsewhere.

REFERENCES


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