PERGAMON Computers and Mathematics with Applications 46 (2003) 959–969

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

An Amplitude Spectral Tonometer

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(Received November 2001; accepted January 2002)

Abstract—Variations in harmonic amplitudes of one-dimensional recurrent signals of the types encountered in medical tonometry are analyzed over finite length records by single Fourier series. Two regularizational algorithms, one differentiation-invoking and the other differentiation-free, are advanced for the design of a universal amplitude spectral "tonometer" for such signals.

Keywords—Harmonic amplitudes, Variations, Recurrent functions, Finite time records, Inverse filtration, Regularization, Weighted Sobolev spaces.

1. INTRODUCTION

Harmonic analysis has always been a basic technique in the processing of cyclic signals coming from the natural, industrial, or human world. A strictly periodic function $f(t) \in \mathcal{P}$, with a period $\vartheta > 0$, is one for which

$$f(t) - f(t - k\vartheta) = 0, \qquad \forall t \in R = (-\infty, \infty) \quad \text{and} \quad \forall k \in N = \{1, 2, 3, \dots, \infty\}. \tag{1}$$

This happens, however, to be only an idealization that rarely exists in real life applications of oscillatory systems where intrinsic disturbances, fluctuations, or instabilities can rule out ideal periodicity. Moreover, satisfaction of the above condition by $f(t)$ in an approximate fashion invokes an identification of the following class of recurrent processes.

DEFINITION 1. RECURRENTIVITY. *(See [1].) The real $f(t)$, $R \to \aleph$, a normed vector space, is called k-recurrent of order p if it satisfies*

$$\max_{k \in N} \|f(t) - f(t - k\vartheta)\|_{L^p} \le \varepsilon_p, \tag{2}$$

with $0 < \varepsilon_p \ll 1$, $p \ge 1$, $\|f\|_{L^p}$ representing the L^p norm of $f(t)$ on R and $\vartheta = \vartheta(\varepsilon_p) > 0$. The space of such $f(t)$ functions will be denoted by R^p_k. When $k \to \infty$, i.e., for R^p_∞, $\vartheta \in \Xi(\varepsilon_p)$,

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 Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi: 10.1016/S0898-1221(03)00299-2

relatively dense set, becomes a translation number (almost period) in some sense of the theory of almost periodic functions.

We emphasize, however, that the above R_k^p class of real functions is strongly dependent on the small parameter ε_p. Clearly, it differs in concept from the conventional almost periodic (a.p.) class U of real functions, due to Bohr (see, e.g., [2,3]), which repeat their values approximately an infinity of times, and is in general less orderly than U.

Many physical phenomena involve almost periodic (also called semiperiodic) functions representing in actual fact an intermediate status between periodic and random processes. For example, in mechanical-vibration monitoring and diagnosis for machinery, semiperiodicity arises from rotation, revolution, and reciprocation of gears, pistons, and so on; in atmospheric science (e.g., weather forecasting), it arises from seasons, caused primarily by rotation and revolution of the earth; in radio astronomy, semiperiodicity arises from revolution of the moon, rotation and pulsation of the sun, rotation of Jupiter and revolution of its satellite Io, and so on; in biology, semiperiodicity in the form of biorhythms arise from both internal and external sources (e.g., cardiac pulses and circadian rhythms).

Among the actively investigated subclasses, which belong in the rather wide class of a.p. functions, one can list

(a) periodic signals with random amplitudes and/or random phases,
(b) damped oscillations, e.g., of the form $f(t) = Ae^{\xi t}\cos(2(\pi/T)t + \varphi)$, with A as amplitude, φ as phase, and ξ as the damping coefficient,
(c) quasi-periodic signals [4], which are combinations of different periodic functions with incommensurate periods, and
(d) cyclostationary processes [5] which are describable by functions having their mean and autocorrelation both periodic and with the same period.

Generally speaking, numerical methods based on the conventional one-dimensional Fourier series are not directly applicable [6] to the analysis of a.p. or recurrent functions. Double Fourier series are more powerful in some respects [4,7], particularly in their ability to represent the features of two-dimensional tilings and to reveal some of their peculiar quasi-periodic (Penrose) structures [7]. Although tiling with Penrose's pieces (darts and kits) cannot be periodic, the tiling possesses unexpected symmetry. Motivated by such facts, Wirth was able recently [4] to successfully analyze quasi-periodic structures by a (generalized) Fourier series in the form of a restriction (trace) on their (periodic) two-dimensional Fourier series extension. His analysis is similar to extracting one-dimensional tiling information from a two-dimensional tiling of the plane. As an alternative, Haidar [6] has proposed a generalized Fourier series representation for a class of almost periodic functions in the form of a discrete Fourier-continuous wavelet inverse transformation.

A sketch of the oscilloscope traces of data from such an observed process, which can always be made positive definite by addition of a step output, is given in Figure 1. Over an averaging time T in the m^{th} semiperiodic finite length record $f_m(t)$, $m \in N \cup \{0\}$, some cuts, peaks, or dips may occasionally take place recurrently or at random, and they normally characterize the dynamic behavior of the observed object. These sharp and usually well-separated variations can contribute significantly and preferentially to variations in certain characteristic harmonic amplitudes.

Traditional medical tonometry as a means of pattern recognition deals with evaluating transfer functions for human beings to inputs in the form of pressure waveforms on blood vessels. Amplitude spectral and phase spectral plots are cataloged for various categories of patients. Certainly, information about changes in indicative harmonic amplitudes to be sifted and labeled from such recurrent data trains can have diverse useful applications not only in medical tonometry or sphygocardiography [8], but also in several other areas ranging from industrial control engineering, image processing to seismography or gravimetry.

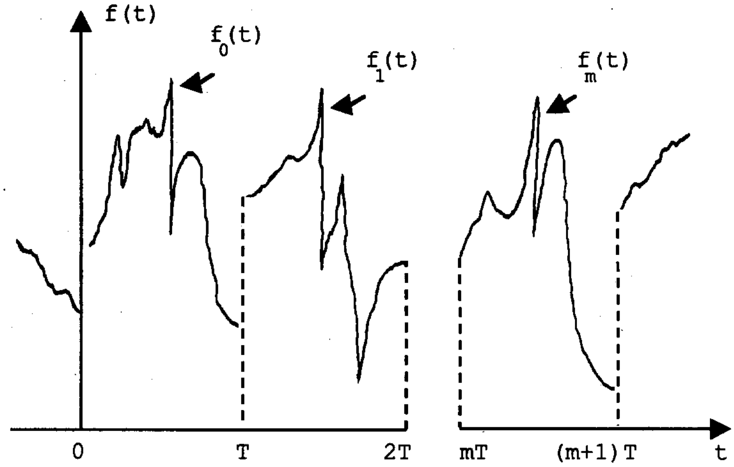

Figure 1. Sketch to illustrate a typical recurrent data train.

Our aim in this communication is to analyze finite length records of a class $\mathbb{T} \subset \wp_k^p \subset R_k^p$ (to be introduced in Section 2) of $f(t)$ by the single Fourier series and not by any other trigonometric series. We report on two regularizational algorithms, one is time-localizational and differentiation-invoking while the other is differentiation-free, which are advanced for the design of the circuit for a universal amplitude spectral analyzer (tonometer) for such signals.

2. DIFFERENTIATION-INVOKING ANALYSIS

The analysis of recurrent signals over equal intervals $Q_m = [\theta_m, \nu_m] = [mT, (m+1)T]$, i.e., $\theta_m - \nu_m = T$, $\forall m$, is directly faced with the problem of defining exactly how large T should be. Here, the Shannon sampling period is irrelevant, and is most likely to be too short, but the answer appears to lie with the theory of a.p. functions. In this theory, with P standing for the finite upper limit of $|f(t)|$, $\forall t \in R$, T is chosen so as to satisfy the rather abstract constraint

$$\left| \frac{1}{T} \int_0^T f(t)\, dt - M\{f\} \right| < \varepsilon, \qquad 0 \leq \varepsilon \ll 1,$$

which subsequently yields the fairly practical condition

$$T \geq T_0 = \frac{4Pl(\varepsilon/2)}{\varepsilon}. \tag{3}$$

T_0 may, however, practically be straightforwardly defined according to (19) of Proposition 2.

Assume, further, s to be the sequential number for a set Γ of sequences of finite time records, when each sequence contains K_s members, with $K_s \leq \max_{k \in N}$, in the sense of Definition 1. We shall denote by m the finite time record number, and within each s-sequence $s \leq m \leq K_s - 1$, so as to define

$$\Gamma : \sum_{s \in \Gamma} \sum_{m=s}^{s+K_s-1} m = \sum_{m=0}^{\infty} m. \tag{4}$$

Clearly, then each sequence is defined over the corresponding set of intervals

$$\tilde{O}_s = \bigcup_{m=s}^{s+K_s-1} Q_m,$$

within a sampled representation

$$f(t) = \sum_{m=0}^{\infty} f_m(t) = \sum_{s \in \Gamma} \sum_{m=s}^{s+K_s-1} f_m(t). \tag{5}$$

DEFINITION 2. *$\wp_k^p$ is the space of all $f(t) \in R_k^p$ functions whose $\ln f_m(t)$ is, $\forall m$, expandable over Q_m (in an even periodic extension sense) in term-by-term differentiable cosinusoidal Fourier series.*

The n^{th} harmonic amplitude total variation over the Q_m interval relative to the preceding Q_{m-1} one in the norm bounded by ε_1 is going to be denoted as

$$E_{m-1}^n = E_n(Q_m \setminus Q_{m-1}) = A_m^n - A_{m-1}^n. \tag{6}$$

The following substitutions are also going to be needed in the sequel:

$$\alpha_m = \frac{1}{T}\int_{Q_m} f_m(t)\,dt, \tag{7}$$

$$\beta_m^n = \int_{Q_m} \left[\frac{f_m'(t)}{f_m(t)}\right] \cos n\frac{\pi}{T}t\,dt. \tag{8}$$

THEOREM 1. *(See [1].) If $f(t) \in \wp_k^p$, then $\forall m$ and $n \in N$, there holds*

$$E_{m-1}^n = (\alpha_m + \alpha_{m-1})\beta_m^n + (\alpha_m - 3\alpha_{m-1})\beta_{m-1}^n. \tag{9}$$

PROPOSITION 1. *(See [1].) Let $\varepsilon_1 \geq 0$, $\varepsilon_1 \ll 1$. If a function $f(t) \in \wp_k^p$, it is necessary and sufficient that*

$$\max_{k\in N} \|\alpha_m - \alpha_{m-1}\|_{p,\Gamma} \leq \varepsilon_p. \tag{10}$$

In $\wp_k^1$, for example, the above condition corresponds to

$$\sum_{s\in\Gamma} \sum_{m=s}^{s+K_s-1} \|[\alpha_m - \alpha_{m-1}]\| \leq \varepsilon_1, \qquad \forall s \in \Gamma.$$

DEFINITION 3. *(See [1].) The space of $f(t)$ which satisfies (10) in the strong sense of $\alpha_m - \alpha_{m-1} = 0$, $\forall m \in N$, is called a tonometric space and will be denoted by $\mathbf{T}$.*

DEFINITION 4. *(See [1].) A function $f(t) \in \wp_\infty^p$, i.e., satisfying (10) when $k = \infty$, is called almost tonometric (a.t.).*

Let us introduce the following notation:

$$X_m^n = \alpha_m\beta_m^n, \qquad \forall n, m,$$
$$\varpi_{m-1} = \frac{\alpha_m}{\alpha_{m-1}}, \tag{11}$$
$$a_m = \frac{\alpha_m}{\alpha_m + \alpha_{m+1}},$$

in order to put (9) in the form

$$\alpha_m\beta_m^n = \varpi_{m-1}\left[(3\varpi_{m-1} - 1)(\varpi_{m-1}+1)^{-1}\right]\alpha_{m-1}\beta_{m-1}^n + (\varpi_{m-1}+1)^{-1}E_{m-1}^n,$$

which is the same as

$$X_m^n = A_m X_{m-1}^n + g_{m-1}^n, \qquad m \in N, \tag{12}$$

with

$$A_m = \varpi_{m-1}\left[\frac{3\varpi_{m-1} - 1}{\varpi_{m-1}+1}\right], \qquad g_{m-1}^n = (\varpi_{m-1}+1)^{-1}E_{m-1}^n.$$

THEOREM 2. *(See [1].) If $f(t) \in \mathbf{T}$, tonometric, then the pertaining E_m^n solution of the associated equation (12) must satisfy, $\forall n \in N$, the linear first kind Volterra difference equation*

$$\sum_{i=0}^{j} \frac{1}{2} E_i^n = X_{j+1}^n + \frac{1}{2} E_0^n - X_1^n. \tag{13}$$

The ill-posedness of (13) reveals itself as an instability of the solution E_m^n under small variations in the data, represented namely by θ_0, T, the α_ms, and the β_m^ns. In the present context, signal recovery (if need be) involves reconstructing the finite length records $f_m(t)$ from a given set of E_{m-1}^ns for a given set of the reference harmonic amplitudes A_m^ns. In practical terms, the last corollary means that the signal recovery problem is ill-posed [9].

From now on we shall drop the n superscript (to simplify notation) in a revisit to (13) in the form

$$\sum_{p}^{j} E_p = 2X_{j+1} - 2X_1 + E_0, \qquad X_0 = X_1 - \frac{1}{2} E_0, \tag{14}$$

with a unitary kernel which admits via differentiation the unique solution

$$E_j = 2X_{j+1}, \qquad E_0 = 2X_2. \tag{15}$$

The above first kind difference equation, with a weak Volterra property, will be shown to be 'on-line' regularizable via its localized autotransformability (see, e.g., [10]) into an equivalent second kind Volterra equation

$$\sigma \mathcal{E}_j + \sum_{p}^{j} E_p = 2X_{j+1} - 2X_1 + E_0, \qquad 0 < \sigma(\delta) \le \sigma_0, \tag{16}$$

corresponding to

$$\left\| {}^{\delta}X_{j+1} - X_j \right\|_{1,1,1;\Gamma} \le \delta.$$

THEOREM 3. *(See [1].) If in equation (16) $E_0 = 0$ and $X_j \in H^{2,1,w_\sigma}(\tilde{O}_s)$, $\forall s \in \Gamma$, with $w_\sigma^{ij} = e^{(i-j)/\sigma}$, where $\sigma(\delta)$ is such that as $\delta \to 0$, $\sigma(\delta) \to 0$ and $\lim_{\delta \to 0}[\delta/\sigma(\delta)] \to 0$, then the following holds.*

(i) *$\mathcal{E}_j \in H^{2,1,w_\sigma}(\tilde{O}_s)$ and is given by*

$$\mathcal{E}_j = 2X_{j+1} - 2 \sum_{i=j-3}^{j} e^{(i-j)/\sigma} X_{i+3}. \tag{17}$$

(ii)

$$\|\mathcal{E}_j - E_j\|_{2,1,w_\sigma;\Gamma} = 2\|X_j\|_{2,1,w^\sigma;\Gamma} = 2 \sum_{s \in \Gamma} \sum_{j=s}^{s+K_s-1} \sum_{i=j-3}^{j} e^{(i-j)/\sigma} |X_{i+3}| \le \kappa \left(\delta + \frac{\delta}{\sigma} \right),$$

where κ is some constant.

The practical implications of this sharp mathematical result can be quite diverse within the weighted Sobolev space $H^{2,1,w_\sigma}(\tilde{O}_s)$, $\forall s \in \Gamma$, with regularized exponential weighting. This theorem advances in fact a new "on-line" sequential regularization for the Kalman-type inverse filtration process, which invokes sweeping the neighborhood of every point involved in (16) from X_i to X_{i+3}, $\forall s \in \Gamma$.

Of outmost importance is the possible periodicity and asymptotic or *lp* stability (see, e.g., [11]) of solutions to the basic result of this work, i.e., to (16). The selection of the zero θ_0 and T for the segmentation process in taking the finite time records over Q_m may appear here to be of crucial significance. The reported theory takes in fact T directly into account. It also advances an inverse problem approach to the study of the dependence of E_m^n on variations of T.

PROPOSITION 2. *Consider* $\mathcal{E}_m^n = \mathcal{E}_m^n(T) = \Psi_n(\sigma, m, T)$, *to assume that*

$$\Psi_n(\sigma, m, T) = 0, \qquad \forall m, \sigma \text{ and } \forall n \leq M = 5. \tag{18}$$

Then

$$T = \{\Psi(0)|_{n \leq M}\}^{-1}. \tag{19}$$

Harmonic analysis by means of the time localization principle formula (9), which states that $E_m^n = 0$, $\forall m \& n \in N$, when $f_m(t)$ remains identical to $f_{m-1}(t)$, can be realized practically by an electronic circuit like the one sketched in Figure 2. The circuit for this on-line regularized differentiation invoking tonometer (ORDIT) contains two analogue integrators ($\int$), one differentiator ($\mathcal{D}$), two summing units ($\sum$), a dividing unit ($\div$), one DC amplifier (2) with a gain 2, two inverters (-1), six delay units (T), eight multipliers (X), a sampling unit (S), one adjustable amplifier (AA), and a sinusoidal oscillator (SO).

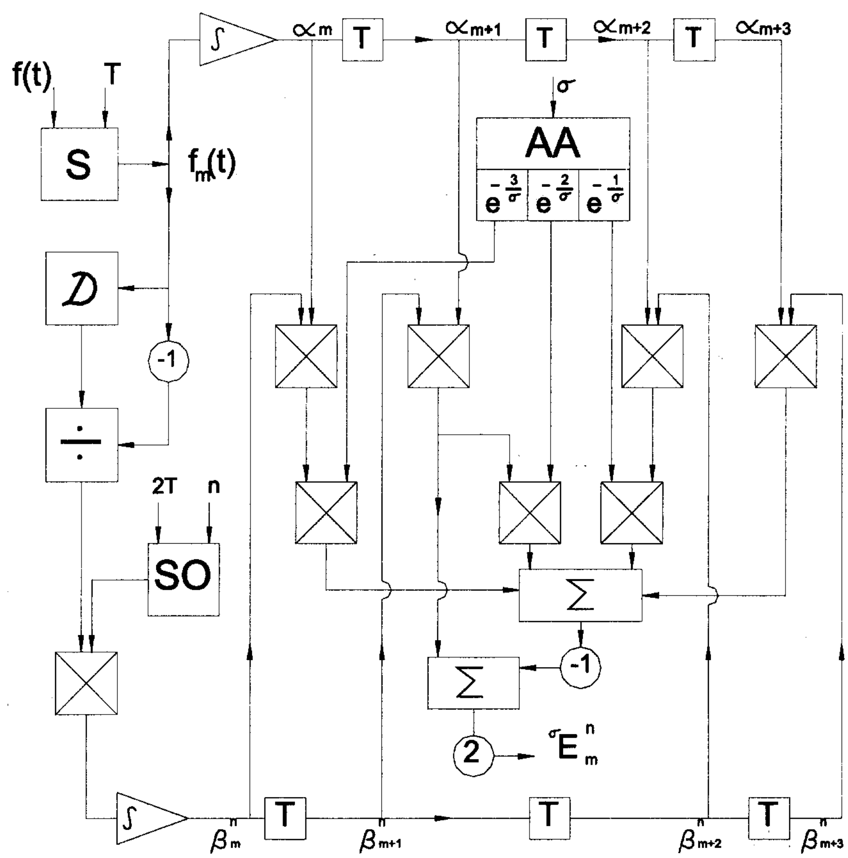

Figure 2. Diagram for the on-line regularized derivative invoking tonometer (ORDIT).

There is, however, the expected usual practical difficulty of the operational amplifier overloading when using a differentiation unit in any circuit. Equally well, one should moreover not overlook the basic theoretical restriction on the utility of Theorem 1 for E_n^m. This being its limited applicability to the rather wide a.t. space for $f(t)$, excludes in particular [12]

(i) any sharply varying $f_m(t)$ with a $\ln f_m(t)$ having a piecewise discontinuous $[\ln f_m(t)]''$, or having an unbounded $[\ln f_m(t)]'$ over Q_m, and
(ii) any particular $f_m(t)$ leading to a divergent associated β_m^n integral over Q_m.

We note finally that a similar formula to (12) could possibly be derived based on a time-frequency localization approach, but that lies outside the scope of this work.

3. DIFFERENTIATION-FREE ANALYSIS

Motivated by the current wide availability of microprocessors and microcomputers for process analysis or patient monitoring [13] and to bypass the shortcomings of operating the ORDIT, we shall propose here a differentiation-free amplitude spectral tonometer (AST).

As before, any finite length record $f_m(t)$ of $f(t)$ when stored in the memory of a microcomputer may be displayed later as $f_m(\tau)$ on a τ display axis with $\tau = t - MT$, and will have the even-periodic extension

$$f_m(\tau) = \frac{1}{2}A_m^0 + \sum_{n=1}^{\infty} A_m^n \cos n\frac{\pi}{T}\tau. \tag{20}$$

Moreover, during the first half-period interval $[0, T]$, the above extension will overlap with the even-periodic extensions of any of the pairs $f_{m\pm2}(\tau)$, $m = 2, 3, 4, \dots, \infty$ of finite length records defined, namely

$$f_{m\pm2}(\tau) = \frac{1}{2}A_{m\pm2}^0 + \sum_{n=1}^{\infty} A_{m\pm2}^n \cos n\frac{\pi}{T}\tau.$$

We define then the n^{th} harmonic amplitude total variation over the Q_m interval, relative to the preceding Q_{m-2} interval, as

$$G_m^n = G_n\left(Q_m \setminus Q_{m-2}\right), \qquad m = 2, 3, 4, \dots, \infty, \tag{21}$$

satisfying

$$G_m^n = A_m^n - A_{m-2}^n = \frac{2}{T}\int_{Q^m} [f_m(\tau) - f_{m-2}(\tau)] \cos n\frac{\pi}{T}\tau \, d\tau. \tag{22}$$

Obviously, if the $f_m(\tau)$ finite time record is identical to the $f_{m-2}(\tau)$ record the $\{G_m^n\}$ set will contain only zero elements; we are not comparing here with $f_{m-1}(\tau)$ for technical reasons pertaining to the functioning of on-line microprocessors. The evaluation of the $\{A_m^n\}$ and $\{A_{m-2}^n\}$ sets is, however, always accompanied with errors of practical and computational nature, and the inverse problem of recovering $f_m(\tau)$ from a given $\{A_m^n\}$ set may not have a unique solution. To illustrate this fact let σ_m be the total difference between the Fourier coefficients of the Q_m and Q_{m-2} intervals in the l^2-norm

$$\frac{1}{2}\left(A_m^0 - A_{m-2}^0\right)^2 + \sum_{n=1}^{\infty}\left(A_m^n - A_{m-2}^n\right)^2 \le \sigma_m^2. \tag{23}$$

A challenging problem in the harmonic analysis of two different finite time records, $f_m(\tau)$ and its approximant $f_m^\delta(\tau)$, would naturally be: given the set $\{A_m^n\}$ of Fourier coefficients, generate at a given point $\tau \in Q_m$, the record $f_m(\tau)$ with error $\eta(\sigma_m) \ge \|f_m - f_m^\delta\|_{L^2}$, tending to zero as $\sigma_m \to 0$. So, let

$$\theta = \frac{2}{\pi}\left[\sum_{n=1}^{\infty} n^{-2}\right],$$

to assume that the error in assigning the Fourier coefficients satisfies in particular

$$A_m^0 - A_{m-2}^0 = \gamma_m,$$
$$A_m^n - A_{m-2}^n = G_m^n = \frac{T\sigma_m}{\theta\pi n}, \qquad n = 2, 3, 4, \dots, \infty.$$

Clearly then, replacement of $f_m(\tau)$ with $f_{m-2}(\tau)$ brings about the error

$$\frac{1}{2}\left(A_m^0 - A_{m-2}^0\right) + \sum_{n=1}^{\infty}\left(A_m^n - A_{m-2}^n\right) \cos n\frac{\pi}{T}\tau.$$

Note interestingly, here, that at the point $\tau = 0$, for example, the above error becomes unbounded,

$$\frac{1}{2}\left(A_m^0 - A_{m-2}^0\right) + \sum_{n=1}^{\infty}\left(A_m^n - A_{m-2}^n\right) = \frac{1}{2}\gamma_m + \left[\frac{T\sigma_m}{\theta\pi}\right]\sum_{n=1}^{\infty} n^{-1} \to \infty,$$

for all small but finite σ_m and may remain finite even when $\sigma_m = 0$. Therefore, convergence of the set $\{A_{m-2}^n\}$ to $\{A_m^n\}$ in the l^2-norm does not guarantee pointwise convergence of $f_{m-2}(\tau)$ to $f_m(\tau)$ over Q_m.

Generally speaking, the inverse problem of recovering $f_m(\tau)$ from a given $\{A_m^n\}$ set (a version of which is represented by (12)), or the summation problem of Fourier series, constitutes an ill-posed problem whose resolution is an additional complication in the comparative practical harmonic analysis of two different finite time records. In this respect, according to a theorem by Tikhonov [9], the summation problem of a Fourier series $f_m(\tau)$ may be regularized by expanding $f_m(\tau)$ in the series

$$f_m(\tau) = \frac{1}{2}A_0^n + \sum_{n=1}^{\infty}\left[\frac{A_m^n}{1+\rho_m\pi^2 n^2/T^2}\right]\cos n\frac{\pi}{T}\tau, \tag{24}$$

in which each Fourier coefficient is multiplied by a regularizing factor $(1 + \rho_m\pi^2n^2/T^2)^{-1}$ containing a regularization parameter ρ_m that is of the same order of significance as σ_m. Taking (24) into account allows for replacing (22) by

$$G_m^0 = \frac{2}{T}\int_{Q^m}\left[\left(1+\frac{\rho_m\pi^2n^2}{T^2}\right)f_m(\tau) - \left(1+\frac{\rho_{m-2}\pi^2n^2}{T^2}\right)f_{m-2}(\tau)\right]\cos n\frac{\pi}{T}\tau\, d\tau. \tag{25}$$

One possible approach to evaluating the regularization parameters could be based on matching $f_m(\tau)$ to its nearest acceptable finite length record (which is $f_{m-2}(\tau)$) according to

$$\begin{aligned} f_m(\tau) &= \frac{1}{2}A_m^0 + \sum_{n=1}^{\infty}\left[\frac{A_m^n}{1+\rho_m\pi^2n^2/T^2}\right]\cos n\frac{\pi}{T}\tau \\ &\approx \frac{1}{2}\left(A_{m-2}^0 + \gamma_m\right) + \sum_{n=1}^{\infty} A_{m-2}^n \cos n\frac{\pi}{T}\tau. \end{aligned}$$

A good deal of theoretical and experimental work is needed in harmonic analysis on the problem of choosing a regularization parameter from the data. In our case, the data will be the $\{G_m^n\}$ set. Typically, cross validation, trial and error, or some other empirical procedure is used to determine the optimal value for ρ_m. In this respect, by further identification of coefficients of the same cosinusoidal functions on both sides of the previous relation, it follows that

$$\rho_m \approx \left[\frac{A_m^n}{A_{m-2}^n} - 1\right]\frac{T^2}{\pi^2 n^2}.$$

Now, to be able to obtain an estimate for this ρ_m we consider its asymptotic limit

$$\rho_m^* = \lim_{n\to\infty}\left[\frac{A_m^n}{A_{m-2}^n} - 1\right]\frac{T^2}{\pi^2n^2}, \tag{26}$$

which becomes independent of n only if it happens that $A_m^n/A_{m-2}^n = Kn^2 + 1$, where K is some arbitrary real constant.

Consider then, ρ as some empirical value of ρ_m^* that is valid for all m to introduce the notion of an extended uniformly regularized derivative

$$f_m(\rho,\tau) = \frac{1}{2T}\left[\left(1+\frac{\rho\pi^2n^2}{T^2}\right)f_m(\tau) - \left(1+\frac{\rho\pi^2n^2}{T^2}\right)f_{m-2}(\tau)\right], \tag{27}$$

which is effectively a coarse "backward difference" expression, in (25) and consider in it (26) to arrive at its regularized form

$$G^n_{m,\rho} = 4 \int_{Q^m} f_m(\rho, \tau) \cos n \frac{\pi}{T} \tau \, d\tau. \tag{28}$$

A zero-order regularization formula for $G^n_{m,0}$ corresponds clearly to the situation when $f_m(\rho, \tau) = f_m(0, \tau)$ in which $\rho = 0$.

The uniformly regularized differentiation-free tonometer URDFT based on (28) will certainly be free of most of the drawbacks of the ORDIT and can be practically realized for zero-order regularization by the electronic circuit sketched in Figure 3. This circuit is distinguishable from the circuit of Figure 2 by the replacement of its SO unit with a cosinusoidal oscillator (CO). Also, the absence of the differentiator from it is penalized by the incorporation of an on-line computer (OLC). Moreover, n of interest can be practically decided upon by means of a suitable frequency detector (FD) which will process simultaneously the incoming $f(t)$ train. The averaging time in this URDFT is dictated, however, by how fast the numerical operations are executed by the hardware used.

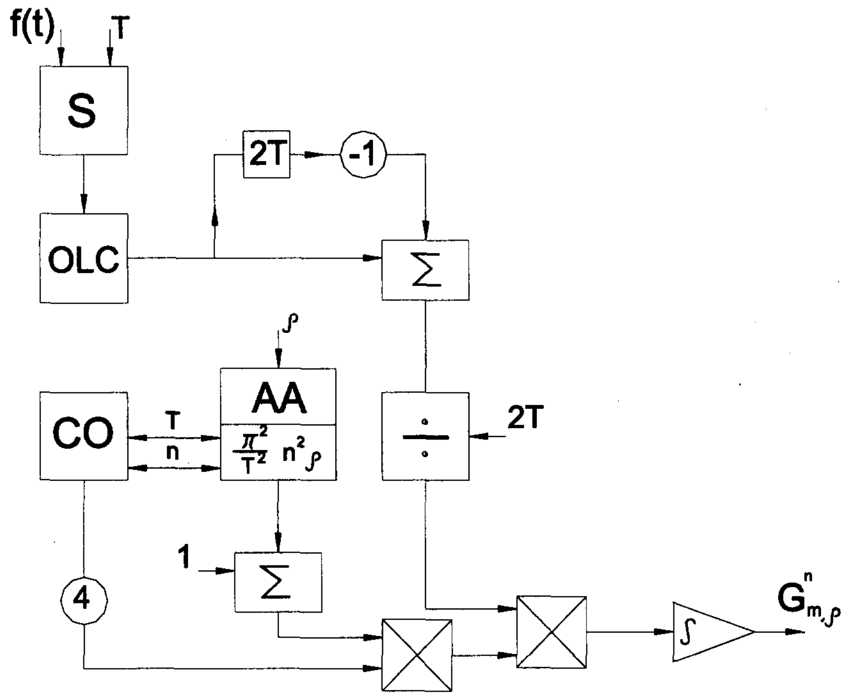

Figure 3. Block diagram for the uniformly regularized differentiation-free tonometer (URDFT).

A microcomputer may perhaps be better suited for most applications. Clearly, then if $\bar{T}$ is the computing time for a typical integral like $G^n_{m,\rho}$ by the used hardware, then the operation of a URDFT to be based on (28) can be feasible only when the averaging period satisfies

$$\bar{T} < T. \tag{29}$$

4. AMPLITUDE SPECTRAL TONOMETER

For an input data train and for any fixed harmonic n, the ORDIT will produce an output time sequence $\{E^n_m\}$, $m \in N$ exceeding a certain labeled preset level, a subset of which, $\{E^n_i\}$, $i \in I$, $n \in J \subset N$, is expected to be of overload signals. Alternatively, the URDFT will almost always produce a complementary $\{G^n_{m,\rho}\}^{\infty}_{m=2}$ regularized sequence of well-behaved outputs that exceed the preset labeled level.

If, in certain applications, one is interested only in the occurrence of the n-harmonic overloads, $n \in J \subset N$, that exceed the labeled level output of the $(I \times J)$ set only, then it is possible to combine the two analyzed circuits together as an amplitude spectral tonometer (AST) through a linear gate (LG), as illustrated in Figure 4, where each overload of the $\{E^n_i\}$, $i \in I$, $n \in J \subset N$ set serves as the gate trigger. Clearly, the AST operates on the symmetric difference of the spaces of the output signals from the ORDIT and the URDFT.

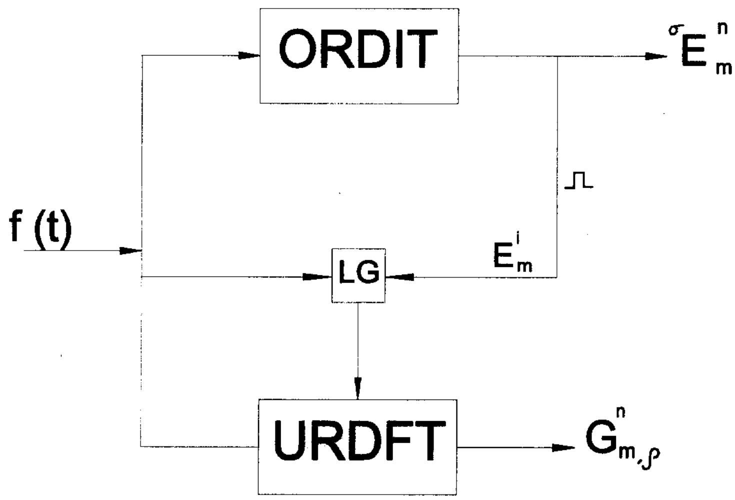

Figure 4. Block diagram for the amplitude spectral tonometer (AST).

Despite its apparent simplicity, one should nevertheless expect some practical difficulties in a robust functioning of the circuits of Figures 2 and 3. In addition to the differentiator $\mathcal{D}$, and even more than $\mathcal{D}$, the divider $(\div)$, which operates by the principle of subtracting logarithms, succeeded by exponentiation, has a restricted range (window) for operation. Circumventing this problem can only be achieved via the design of the trigger for the gate to be used in the eventual AST. The trigger can simply be operated by overload signals, not only from the $\mathcal{D}$, but also from the $(\div)$ unit.

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