



Noether symmetry approach in multiple scalar fields scenario

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ABSTRACT

In this Letter, we find suitable potentials in the multiple scalar fields scenario by using the Noether symmetry approach. We discussed three models with multiple scalar fields: N -quintessence with positive kinetic terms, N -phantom with negative kinetic terms and N -quintom with both positive and negative kinetic terms. In the N -quintessence case, the exponential potential which could be derived from several theoretic models is obtained from the Noether conditions. In the N -phantom case, the potential $\frac{V_0}{2}(1 - \cos(\sqrt{\frac{3N}{2}} \frac{\phi}{m_{pl}}))$, which could be derived from the Pseudo Nambu–Goldstone boson model, is chosen as the Noether conditions required. In the N -quintom case, we derive a relation $DV'_{\phi q} = -\tilde{D}V'_{\phi p}$ between the potential forms for the quintessence-like fields and the phantom-like fields by using the Noether symmetry.

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1. Introduction

Scalar field theory which is related to particle physics has become the generic playground for building cosmological models, both in the early and late accelerating periods of our universe [1,2]. Although the dynamics of these accelerations is likely to contain several scalar fields, it is normally assumed that only one of these fields remained dynamically significant for a long time. However, realistic theoretical models, embedded in grand unified or super symmetric theories, must necessarily be theories of multiple fields. The simplest multiple scalar fields scenario which we will consider is first originated from the assisted inflation scenario [3]. The essential point of this scenario is that inflation is not driven by any single field but a collection of N fields. These fields have the same

and worthy of putting other theoretical difficulties aside temporarily. Then, it is natural to ask why don't we discuss the multiple scalar fields with different kinetic terms. According to the classification of the scalar fields,¹ we can discuss three types of fields in the simplest multiple scalar fields scenario, which are the quintessence type of fields with positive kinetic terms, the phantom type of fields with negative kinetic terms, the quintom type of fields with both positive and negative kinetic terms. In this Letter, we call them N -quintessence, N -phantom, N -quintom for convenience.

But, as in the single scalar field case, we have to ask how to choose the potentials from the various models for those multiple scalar fields. In this Letter, we will deal with this problem of choice from a point of view of symmetry. The Noether symmetry has been brought to you by CORE, provided by Elsevier - Publisher Connector. lutions in cosmological models motivated at a fundamental level.

This Letter is organized as follows. In Section 2, we introduce the multiple scalar fields models. In Section 3, the Noether symmetry approach will be introduced and applied to both N -quintessence and N -phantom cases to get exact solutions. In Section 4, we discuss the application of Noether symmetry approach to N -quintom case in connection with its solution. In Section 5, we give out the evolution of our universe in N -quintessence and

Meanwhile, the observations suggest the equation of state (EoS) parameter of dark energy is in the range of $-1.21 \leq \omega \leq -0.89$ [5]. Since the quintessence type of matter could not give the possibility that $\omega < -1$, the extended paradigms (e.g. phantom and quintom) are proposed. Phantom type of matter with negative kinetic energy has well-known problems, but, nevertheless, was implicitly suggested in cosmological models and have also been widely studied as dark energy. It is phenomenologically significant

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¹ The quintessence with positive kinetic term was proposed in Ref. [6]; the phantom with negative kinetic term was suggested in Ref. [7]; and the quintom with both positive and negative kinetic terms was proposed in Ref. [8].

N -phantom cases. Finally, a short summary will be presented in Section 6.

2. N -quintessence, N -phantom and N -quintom scalar field model scenario

As stated in the introduction, usually, only one scalar field is enough to accelerate the universe, but a single field is not natural. The application of the multiple scalar fields in cosmology should be seriously considered. Here, we assume that the geometry of space–time is described by the flat FRW (Friedmann–Robertson–Walker) metric which seems to be consistent with today's cosmological observations

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2, \quad (1)$$

where a is the scale factor. After setting the number of the scalar fields as N , the action of the multiple scalar fields can be written as

$$S_\phi = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \sum_{i=1}^N \left(\epsilon \frac{\dot{\phi}_i^2}{2} - V(\phi_i) \right) \right], \quad (2)$$

where $\epsilon = 1$ denotes the quintessence fields with the positive kinetic terms, $\epsilon = -1$ denotes the phantom fields with the negative kinetic term. Meanwhile, as we consider both the vector fields and the matter in the system, the total action is

$$S_{tot} = S_\phi + S_m, \quad (3)$$

where S_m is the action for matter. The density of the matter can be expressed as $\rho_m = \rho_{m0}(a_0/a)^{3\gamma}$, where ρ_{m0} is an initial constant and $0 < \gamma \leq 2$. Here, we limit our analysis to $\gamma = 1$ which corresponds to the pressureless matter with $P_m = 0$.

We assume the vector fields are non-interacting, their influences on each other are through their effects on the expansion. Considering all the scalar fields have the same potentials and initial conditions, action (2) could be simplified as

$$S_{\phi 1} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + N \left(\epsilon \frac{\dot{\phi}^2}{2} - V(\phi) \right) \right]. \quad (4)$$

When $\epsilon = 1$, we call the related scenario N -quintessence. While $\epsilon = -1$, we call the related scenario N -phantom.

For the N -quintom case, we assume the fields with same kinetic terms have the same potentials and initial conditions, the action can be written as

$$S_{\phi 2} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + N_q \left(\frac{\dot{\phi}_q^2}{2} - V(\phi_q) \right) + N_p \left(-\frac{\dot{\phi}_p^2}{2} - V(\phi_p) \right) \right], \quad (5)$$

where ϕ_q is the scalar field with the positive kinetic terms, N_q is the number of the corresponding quintessence type fields; ϕ_p is the scalar field with the negative kinetic terms, N_p is the number of the corresponding phantom type fields. This paradigm has been proved of crossing $\omega_\phi = -1$ when $N_q = N_p = 1$ [8].

3. The Noether symmetry approach in N -quintessence and N -phantom

In the case of N -quintessence and N -phantom, we take the scale factor a and the scalar field ϕ as independent dynamical variables in the system which the action (4) represents. Then the

configuration space could be chosen as $\mathcal{Q} = (a, \phi)$, while the related tangent space is $T\mathcal{Q} = (a, \phi, \dot{a}, \dot{\phi})$. To study the symmetries of the space under consideration, we need an effective point-like Lagrangian for the model whose variation with respect to its dynamical variables yields the correct equations of motion. However, based on action (4), it is proper to make the point-like Lagrangian as

$$\mathcal{L}_1 = \mathcal{L}_{\phi 1} + \mathcal{L}_m = 3a\dot{a}^2 - \frac{N}{m_{pl}^2} \left(\epsilon \frac{a^3 \dot{\phi}^2}{2} - a^3 V(\phi) \right) + \frac{\rho_{m0}}{m_{pl}^2}, \quad (6)$$

where the Planck mass is $m_{pl}^2 = (8\pi G)^{-1}$, and the term $\frac{\rho_{m0}}{m_{pl}^2}$ corresponds to the effects from matter.

Therefore, the total energy of the system $E_{\mathcal{L}_{\phi 1}}$, could be written in this way

$$E_{\mathcal{L}_1} = \frac{\partial \mathcal{L}_1}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L}_1 = a^3 \left(\frac{\epsilon N \dot{\phi}^2}{2} + NV(\phi) + \rho_{m0} a^{-3} - 3m_{pl}^2 H^2 \right). \quad (7)$$

If the above equation being considered as a constraint, with the vanishing of the “energy function”, it is just the Friedmann equation

$$H^2 = \frac{1}{3m_{pl}^2} \left[\frac{\epsilon N \dot{\phi}^2}{2} + NV(\phi) + \rho_{m0} a^{-3} \right]. \quad (8)$$

Furthermore, for a dynamical system, the Euler–Lagrangian equation is

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}_1}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}_1}{\partial q_i} = 0. \quad (9)$$

Based on the Lagrangian, in the N -quintessence and N -phantom case, the variable q_i is a and ϕ , respectively. When $q_i = a$, the Raychaudhuri equation could be gotten

$$\dot{H} = -\frac{m_{pl}^2}{2} (\rho_\phi + P_\phi + \rho_m) = -\frac{\epsilon N m_{pl}^2}{2} \dot{\phi}^2 - \frac{m_{pl}^2}{2} \rho_m, \quad (10)$$

where the energy density and the pressure of scalar fields are

$$\rho_\phi = \frac{\epsilon N}{2} \dot{\phi}^2 + NV(\phi), \quad (11)$$

$$P_\phi = \frac{\epsilon N}{2} \dot{\phi}^2 - NV(\phi). \quad (12)$$

What is more, the equation of state could also be obtained

$$\omega_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\epsilon \dot{\phi}^2/2 - V(\phi)}{\epsilon \dot{\phi}^2/2 + V(\phi)}. \quad (13)$$

Obviously, in the N -quintessence case where $\epsilon = 1$, $\omega_\phi > -1$; in the N -phantom case where $\epsilon = -1$, $\omega_\phi < -1$. Both of them could not cross $\omega_\phi = -1$, that is why we also consider N -quintom. In the case of $q_i = \phi$, the Euler–Lagrangian equation is the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + \epsilon V'_\phi = 0, \quad (14)$$

where the prime means $V'_\phi = dV/d\phi$. For the different value of ϵ , the quintessence makes the fields roll down the potential, while the phantom makes them roll up.

The above equations coincide with the results calculated from the Einstein equations, and prove that the point-like Lagrangian is consistent with the dynamical system.

As is well known in [10–12], Noether symmetry approach is a powerful tool in finding the solution for a given Lagrangian. From this method, it is possible to obtain a reduction, and possibly get a full integration of the system, whenever the cyclic variable of the system is found. The key point related to the Noether symmetry is a Lie algebra presented in the tangent space. Following [10–12], for the Lagrangian (6), firstly we define the Noether symmetry induced by a vector X on the tangent space $TQ = (a, \phi, \dot{a}, \dot{\phi})$ which is

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\phi}}, \quad (15)$$

where α and β are generic functions of a and ϕ . The Lagrangian is invariant under the transformation X if

$$L_X \mathcal{L}_1 = \alpha \frac{\partial \mathcal{L}_1}{\partial a} + \frac{d\alpha}{dt} \frac{\partial \mathcal{L}_1}{\partial \dot{a}} + \beta \frac{\partial \mathcal{L}_1}{\partial \phi} + \frac{d\beta}{dt} \frac{\partial \mathcal{L}_1}{\partial \dot{\phi}} = 0. \quad (16)$$

Given $L_X \mathcal{L}_1 = 0$ satisfied, there exists a Noether symmetry. Combined with the Lagrangian, this symmetry gives out

$$\alpha + 2a \frac{\partial \alpha}{\partial a} = 0, \quad (17)$$

$$6 \frac{\partial \alpha}{\partial \phi} - \epsilon N \frac{a^2}{m_{pl}^2} \frac{\partial \beta}{\partial a} = 0, \quad (18)$$

$$3\alpha + 2a \frac{\partial \beta}{\partial \phi} = 0, \quad (19)$$

$$3V(\phi)\alpha + aV'_\phi(\phi)\beta = 0, \quad (20)$$

which we call Noether conditions. The difference between the N -quintessence and N -phantom is in Eq. (18) as the parameter ϵ denotes.

What is more, the momentum potential can be defined as below

$$p_a = \frac{\partial \mathcal{L}_1}{\partial \dot{a}} = 6a\dot{a}, \quad (21)$$

$$p_\phi = \frac{\partial \mathcal{L}_1}{\partial \dot{\phi}} = -\frac{\epsilon N}{m_{pl}^2} a^3 \dot{\phi}. \quad (22)$$

Then we can express the constant of motion which is reproduced by the Noether symmetry

$$\alpha p_a + \beta p_\phi = Q = \mu_0, \quad (23)$$

where Q is called conserved charge and μ_0 is the related constant. The Noether constant of motion on shell gives a possibility of solving the system. More specifically, a symmetry exists if at least one of the functions α or β is different from zero. As a byproduct, the form of $V(\phi)$ is determined in correspondence with such a symmetry.

The cyclic variable can be regarded as a helpful tool of getting the exact description about the dynamical system. A point transformation $(a, \phi) \rightarrow (z, w)$ is effective to find the cyclic variable. It is

$$i_X z = \alpha \frac{\partial z}{\partial a} + \beta \frac{\partial z}{\partial \phi} = 1, \quad (24)$$

$$i_X w = \alpha \frac{\partial w}{\partial a} + \beta \frac{\partial w}{\partial \phi} = 0, \quad (25)$$

then the Lagrangian could be rewritten in term of the cyclic variables. After the transformation, the cyclic variable is z , and the constant of motion can be rewritten as $Q = p_z$. This will simplify our calculation effectively. A general discussion of this issue could be found in [10–12]. After introducing the Noether symmetry approach, we will discuss the solutions for the Noether conditions both in the N -quintessence and N -phantom in the following.

3.1. Exact solutions for N -quintessence

In the N -quintessence case where the sign of the kinetic terms takes the value $\epsilon = 1$, the Noether conditions are

$$\alpha + 2a \frac{\partial \alpha}{\partial a} = 0, \quad (26)$$

$$6 \frac{\partial \alpha}{\partial \phi} - N \frac{a^2}{m_{pl}^2} \frac{\partial \beta}{\partial a} = 0, \quad (27)$$

$$3\alpha + 2a \frac{\partial \beta}{\partial \phi} = 0, \quad (28)$$

$$3\alpha V(\phi) + a\beta V'_\phi(\phi) = 0. \quad (29)$$

When $N = 1$, the Noether conditions reduce to the single field case [11]. As indicated by Eq. (27), the effects of the multiple scalar fields are manifested by the number of the scalar fields N .

An obvious constant potential solution is

$$\alpha = 0, \quad \beta = \text{constant}, \quad V = \text{constant}. \quad (30)$$

In this solution, a is the cyclic variable. And the subsequent constant of motion gives out

$$\beta p_\phi = -\frac{\epsilon N}{m_{pl}^2} a^3 \dot{\phi} = Q = \mu_0. \quad (31)$$

The discussions could be divided into two cases simply. Firstly, when $\mu_0 = 0$, $\phi = \text{constant}$, this is a cosmological constant solution. Secondly, when $\mu_0 \neq 0$, the kinetic term $\dot{\phi} \propto a^3$. The scalar fields decay fast, even faster than the corresponding vector field solution [9]. These two cases are trivial respectively. In the following, we will concentrate our discussions on another solution which is

$$\alpha = \frac{\sigma_+}{\sqrt{a}}, \quad \beta = \frac{-3\lambda\sigma_-}{2a\sqrt{a}}, \quad (32)$$

$$V = V_0\sigma_-^2 = V_0(A^2 e^{2\lambda\phi} + B^2 e^{-2\lambda\phi} - 2AB), \quad (33)$$

where $\sigma_\pm = Ae^{\lambda\phi} \pm Be^{-\lambda\phi}$, $\lambda = \sqrt{3N/8m_{pl}^2}$, A and B are constants.

We can see that the potential is a combined exponential function. Indeed, there are some physical origins about this kind of potential. In higher-dimensional gravitational theories such as superstring and Kaluza–Klein theories [13], exponential potentials often appear from the curvature of internal spaces associated with the geometry of extra dimensions [14]. Moreover, it is known that exponential potential can arise in gaugino condensation as a non-perturbative effect and in the presence of supergravity corrections to global supersymmetric theories [15]. However, this kind of potential is picked up by Noether symmetry.

In particular, when $A = 0$, the Noether conditions show

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{3N}{2}} \frac{\phi}{m_{pl}}\right). \quad (34)$$

This kind of potential leads to a power-law expanding universe, with $a \propto t^{4/3}$, $\omega_\phi = -1/2$. The quintessence with an exponential potential was widely studied in cosmology, see, for example, Ref. [16]. It even has a scaling solution. In the following, based on the value of A , we will get the exact solutions from the point of view of Noether symmetry.

3.1.1. When $A \neq 0$ and $B \neq 0$

If we put Eqs. (21), (22), (32) and (33) into Eq. (23), we find that the constant of motion is hard to obtain. Therefore, we search

the cyclic variable for help. By calculating Eqs. (24) and (25), we can get the following expressions for the new variables

$$z = \frac{a^{3/2}\sigma_+}{6AB}, \quad w = \frac{a^{3/2}\sigma_-}{6AB}, \quad (35)$$

where z is the cyclic variable. Correspondingly, ϕ and a could be expressed as

$$\phi = \frac{1}{2\lambda} \ln \frac{z+w}{z-w}, \quad a = [9AB(z^2 - w^2)]^{1/3}. \quad (36)$$

The resulting forms of potential and Lagrangian are

$$V(\phi) = V_0 \frac{4w^2}{z^2 - w^2}, \quad (37)$$

$$\mathcal{L}_{\phi 1} = 12AB \left[(\dot{z}^2 - \dot{w}^2) + \frac{3NV_0}{m_{pl}^2} w^2 \right]. \quad (38)$$

Using the Euler-Lagrangian equations, the above Lagrangian leads to the equations of motion for z and w ,

$$\ddot{z} = 0, \quad \ddot{w} = -\frac{3NV_0}{m_{pl}^2} w. \quad (39)$$

The solutions are

$$z = z_1 t + z_0, \quad (40)$$

$$w = w_1 \sin \left(\sqrt{\frac{3NV_0}{m_{pl}^2}} t + w_0 \right), \quad (41)$$

where z_0, z_1, w_0, w_1 are constants. Therefore, the exact evolution of the field and the scale factor could be given out as below

$$\phi = \frac{1}{2\lambda} \ln \frac{z_1 t + z_0 + w_1 \sin \left(\sqrt{\frac{3NV_0}{m_{pl}^2}} t + w_0 \right)}{z_1 t + z_0 - [w_1 \sin \left(\sqrt{\frac{3NV_0}{m_{pl}^2}} t + w_0 \right)]}, \quad (42)$$

$$a = \left[9AB \left((z_1 t + z_0)^2 - w_1^2 \sin^2 \left(\sqrt{\frac{3NV_0}{m_{pl}^2}} t + w_0 \right) \right) \right]^{1/3}. \quad (43)$$

If $z \ll w$, we could not get a physical value of ϕ , through the scale factor seems oscillate. And if $z \gg w$, ϕ is very small, but the universe will evolve as $a \propto t^{2/3}$. It is similar to the matter-dominated phase.

3.1.2. When $A = 0$ and $B \neq 0$

In this subsection, we continue to search the cyclic variables but for a different potential where $A = 0$ while $B \neq 0$. By calculating Eqs. (24) and (25), the expressions of the new variables are

$$z = \frac{a^{3/2}}{3\sigma_+}, \quad w = \frac{a^{3/2}}{3\sigma_-}, \quad (44)$$

where z is the cyclic variable. Then ϕ and a can be rewritten as

$$\phi = \frac{1}{2\lambda} \ln \left(\frac{B^2 z w}{9} \right), \quad a = \left(\frac{z}{w} \right)^{1/3}. \quad (45)$$

As a result, we get the potential and the Lagrangian in term of z and w

$$V(\phi) = \frac{V_0}{9zw}, \quad (46)$$

$$\mathcal{L}_{\phi 1} = \frac{-4}{3} \frac{\dot{z}\dot{w}}{w^2} + \frac{N}{m_{pl}^2} \frac{V_0}{9w^2}. \quad (47)$$

Apply the new Lagrangian to the Euler-Lagrangian equations, we obtain

$$\ddot{z} = \frac{3V_0}{2m_{pl}^2 w}, \quad \ddot{w} = \frac{2\dot{w}^2}{w}. \quad (48)$$

They lead to

$$z = - \left[\frac{V_0 w_2}{4m_{pl}^2} t^3 + \frac{3V_0 w_3}{4m_{pl}^2} t^2 + \frac{3V_0 w_4}{4m_{pl}^2} t + w_5 \right], \quad (49)$$

$$w = \frac{-1}{w_2 t + w_3}, \quad (50)$$

where w_2, w_3, w_4 are constants. Putting the above equations into Eq. (44), the evolutions of a and ϕ are

$$\phi = \sqrt{\frac{2}{3N}} m_{pl} \ln \left(B^2 \frac{\frac{V_0 w_2}{4m_{pl}^2} t^3 + \frac{3V_0 w_3}{4m_{pl}^2} t^2 + \frac{3V_0 w_4}{4m_{pl}^2} t + w_5}{w_2 t + w_3} \right), \quad (51)$$

$$a = \left[(w_2 t + w_3) \left(\frac{V_0 w_2}{4m_{pl}^2} t^3 + \frac{3V_0 w_3}{4m_{pl}^2} t^2 + \frac{3V_0 w_4}{4m_{pl}^2} t + w_5 \right) \right]^{1/3}. \quad (52)$$

When $z \propto t$, the scale factor evolves as $a \propto t^{2/3}$ which is similar to the matter-dominated phase. When $z \propto t^3$, the scale factor is $a \propto t^{4/3}$ which may accelerate the universe. This is an interesting solution that we need. We will discuss this solution in Section 5 in detail.

However, the N -quintessence scenario could be replaced by a single field paradigm with the similar evolutions $a \propto t^{4/3}$. We just need to change the corresponding parameter in the single field case as

$$V_{0s} = NV_0, \quad \lambda_s = \frac{\lambda}{\sqrt{N}}. \quad (53)$$

The reason for this behavior is that each field experiences the ‘downhill’ force from its own potential, it feels the friction from all the scalar fields via their contribution to the expansion rate.

The case $B = 0, A \neq 0$ is treated exactly in the same way and the results are the same, except for the substitution of A for B . In summary, it must be noted that our results include some already known models. The exponential potential not only make the acceleration last a long time, but also satisfy the Noether conditions.

As for the comparison with the observations, one field results have been derived by Ref. [10]. In the N -quintessence case, the range of parameter will be changed because of N . Considering our purpose is on the choice of the potential, we will not discuss this subject in detail.

3.2. Exact solutions for N -phantom

For the N -phantom case where $\epsilon = -1$, the Noether conditions are

$$\alpha + 2a \frac{\partial \alpha}{\partial a} = 0, \quad (54)$$

$$6 \frac{\partial \alpha}{\partial \phi} + N \frac{a^2}{m_{pl}^2} \frac{\partial \beta}{\partial a} = 0, \quad (55)$$

$$3\alpha + 2a \frac{\partial \beta}{\partial \phi} = 0, \quad (56)$$

$$3\alpha V(\phi) + a\beta V'_{\phi}(\phi) = 0. \quad (57)$$

Compared to the N -quintessence case, the difference arises in Eq. (55) by the sign of the kinetic terms.

Obviously, the simplest solution is

$$\alpha = 0, \quad \beta = \text{constant}, \quad V = \text{constant}. \quad (58)$$

This constant potential solution is similar to the corresponding solution in the N -quintessence case, we don't discuss this fast decaying case.

However, another interesting solution is

$$\alpha = \frac{2C \cos\left(\frac{1}{2}\sqrt{\frac{3N}{2}}\frac{\phi}{m_{pl}}\right)}{\sqrt{a}}, \quad (59)$$

$$\beta = \frac{-2\sqrt{6}C \sin\left(\sqrt{\frac{-3N}{8}}\frac{\phi}{m_{pl}}\right)}{a\sqrt{a}}, \quad (60)$$

$$V(\phi) = V_0 \sin^2\left(\frac{1}{2}\sqrt{\frac{3N}{2}}\phi\right) = \frac{V_0}{2} \left(1 - \cos\left(\sqrt{\frac{3N}{2}}\frac{\phi}{m_{pl}}\right)\right), \quad (61)$$

where C is a constant. When $N = 1$, there are some differences between the results in Ref. [17] and ours. The form of the potential could be called PNGB (Pseudo Nambu–Goldstone Bosons) potential resulting from explicit breaking of a shift symmetry [18].

To find the exact evolution of the universe, as the calculations in the N -quintessence case, we need the help of the cyclic variables. According to Eqs. (24) and (25), a transformation could be done from (a, ϕ) to (z, w) ,

$$\phi = \arctan \frac{w}{z}, \quad a = (3C)^{2/3} (z^2 + w^2)^{1/3}, \quad (62)$$

then we can rewrite the potential and the Lagrangian as

$$V = \frac{V_0 w^2}{z^2 + w^2}, \quad (63)$$

$$\mathcal{L}_{\phi 1} = 9C^2 \left[\frac{4}{3} (z^2 + w^2) + V_0 \frac{Nw^2}{m_{pl}^2} \right]. \quad (64)$$

The Lagrangian leads to the equations of motion for the new variables

$$\ddot{z} = 0, \quad \ddot{w} = \frac{3NV_0 w}{4m_{pl}^2}. \quad (65)$$

The solutions are

$$z = z_3 t + z_2, \quad (66)$$

$$w = w_6 \exp\left(\sqrt{\frac{3NV_0}{4m_{pl}^2}} t\right), \quad (67)$$

where z_2, z_3 and w_6 are constant. However, by using the cyclic variable z , we get the evolutions of the field and the scale factor,

$$\phi = \arctan \frac{w_6 \exp\left(\sqrt{\frac{3NV_0}{4m_{pl}^2}} t\right)}{z_3 t + z_2}, \quad (68)$$

$$a = (3C)^{2/3} \left[(z_3 t + z_2)^2 + w_6^2 \exp\left(\sqrt{\frac{3NV_0}{m_{pl}^2}} t\right) \right]^{1/3}. \quad (69)$$

If $z \gg w$, the values of fields are nearly zero, $a \propto t^{2/3}$, it is the matter-dominated solution. When $z \ll w$, the universe evolves as $a \propto \exp\left(\sqrt{\frac{3NV_0}{m_{pl}^2}} t\right)$, this is the de Sitter solution. We will discuss this solution in Section 5 in detail.

As for the comparison with the observations, one field results have been derived by Ref. [17]. In the N -phantom case, the range of parameter will be changed because of N . Considering our purpose is on the choice of the potential, we will not discuss this subject in detail.

4. Noether symmetry in N -quintom case

The quintom scenario is proposed to fit the observable data [5]. N -quintessence and N -phantom could not cross $\omega_\phi = -1$ as we see. However, N -quintom has an attractive feature that it may cross $\omega_\phi = -1$ which is a possibility implied by the data. After adding the Noether symmetry, this property should be rechecked. Though in the ‘‘cosmic triad’’ vector field case, Noether symmetry provides an interesting constraint on the potentials [9] for the quintom case with ω_ϕ crossing -1 . However, it is worthy of trying the Noether symmetry approach in the N -quintom case. According to the action (5), the point-like Lagrangian is

$$\begin{aligned} \mathcal{L}_2 = & 3aa^2 - \frac{N_q}{m_{pl}^2} \left(\frac{a^3 \dot{\phi}_q^2}{2} - a^3 V_q \right) \\ & - \frac{N_p}{m_{pl}^2} \left(\frac{-a^3 \dot{\phi}_p^2}{2} - a^3 V_p \right) + \rho_{m0}. \end{aligned} \quad (70)$$

Based on the above point-like Lagrangian, the total energy and the Euler–Lagrangian equation will give out the Friedmann equation, the Raychaudhuri equation and the equations of motion

$$H^2 = \frac{1}{3m_{pl}^2} \left[N_q \left(\frac{\dot{\phi}_q^2}{2} + V_q \right) + N_p \left(-\frac{\dot{\phi}_p^2}{2} + V_p \right) + \rho_m \right], \quad (71)$$

$$\begin{aligned} \dot{H} = & -\frac{m_{pl}^2}{2} (\rho_\phi + P_\phi + \rho_m) \\ = & -\frac{m_{pl}^2 N_q}{2} \dot{\phi}_q^2 + \frac{m_{pl}^2 N_p}{2} \dot{\phi}_p^2 - \frac{m_{pl}^2}{2} \rho_m, \end{aligned} \quad (72)$$

$$\ddot{\phi}_q + 3H\dot{\phi}_q + V'_q = 0, \quad (73)$$

$$\ddot{\phi}_p + 3H\dot{\phi}_p - V'_p = 0, \quad (74)$$

where the primes mean $V'_{\phi q} = dV_q/d\phi_q$ and $V'_{\phi p} = dV_p/d\phi_p$. The energy density and the pressure which could be derived from the action (5) are

$$\rho_\phi = N_q \left(\frac{\dot{\phi}_q^2}{2} + V_q \right) + N_p \left(-\frac{\dot{\phi}_p^2}{2} + V_p \right), \quad (75)$$

$$P_\phi = N_q \left(\frac{\dot{\phi}_q^2}{2} - V_q \right) - N_p \left(\frac{\dot{\phi}_p^2}{2} + V_p \right). \quad (76)$$

So the EoS parameter is

$$\omega_\phi = \frac{N_q \left(\frac{\dot{\phi}_q^2}{2} - V_q \right) - N_p \left(\frac{\dot{\phi}_p^2}{2} + V_p \right)}{N_q \left(\frac{\dot{\phi}_q^2}{2} + V_q \right) + N_p \left(-\frac{\dot{\phi}_p^2}{2} + V_p \right)}. \quad (77)$$

Now, we should choose a new configuration space $\mathcal{Q} = (a, \phi_q, \phi_p)$ with the corresponding tangent space $T\mathcal{Q} = (a, \dot{\phi}_q, \dot{\phi}_p, \dot{a}, \ddot{\phi}_q, \ddot{\phi}_p)$. And the vector generator which induce the Noether symmetry is changed to

$$\tilde{X} = \tilde{\alpha} \frac{\partial}{\partial a} + \tilde{\beta} \frac{\partial}{\partial \phi_q} + \gamma \frac{\partial}{\partial \phi_p} + \dot{\tilde{\alpha}} \frac{\partial}{\partial \dot{a}} + \dot{\tilde{\beta}} \frac{\partial}{\partial \dot{\phi}_q} + \dot{\gamma} \frac{\partial}{\partial \dot{\phi}_p}, \quad (78)$$

where $\tilde{\alpha}$, $\tilde{\beta}$ and γ are generic functions of the variables a , ϕ_q and ϕ_p . The Noether symmetry requires the Lie derivative

of the Lagrangian vanishes which means $L_{\tilde{\chi}}\mathcal{L}_2 = 0$. Following Refs. [10–12], the Noether conditions can be obtained

$$\tilde{\alpha} + 2a \frac{\partial \tilde{\alpha}}{\partial a} = 0, \quad (79)$$

$$6 \frac{\partial \tilde{\alpha}}{\partial \phi_q} - \frac{N_q a^2}{m_{pl}^2} \frac{\partial \tilde{\beta}}{\partial a} = 0, \quad (80)$$

$$6 \frac{\partial \tilde{\alpha}}{\partial \phi_p} + \frac{N_p a^2}{m_{pl}^2} \frac{\partial \gamma}{\partial a} = 0, \quad (81)$$

$$3\tilde{\alpha} + 2a \frac{\partial \tilde{\beta}}{\partial \phi_q} = 0, \quad (82)$$

$$3\tilde{\alpha} + 2a \frac{\partial \gamma}{\partial \phi_p} = 0, \quad (83)$$

$$3(V_q + V_p)\tilde{\alpha} + aV'_{\phi_q}\tilde{\beta} + aV'_{\phi_p}\gamma = 0. \quad (84)$$

There is an obvious solution that is

$$\tilde{\alpha} = 0, \quad \tilde{\beta} = D, \quad \gamma = \tilde{D}, \quad (85)$$

where D and \tilde{D} are integral constants. The symmetry exists, if and only if at least one of the parameter $\tilde{\alpha}$, $\tilde{\beta}$, γ is not zero. Based on the Noether conditions, we find a condition relating the potential forms of the quintessence-like fields and the phantom-like fields, that is

$$DV'_{\phi_q} = -\tilde{D}V'_{\phi_p}. \quad (86)$$

And the constant of motion corresponding to this solution is

$$-DN_q a^3 \dot{\phi}_q + \tilde{D}N_p a^3 \dot{\phi}_p = Q = \mu_0. \quad (87)$$

In the following discussion, based on the value of D , \tilde{D} and μ_0 , we try to discuss the solutions, especially for the value of EoS parameter.

4.1. When $D \neq 0$ and $\tilde{D} = 0$

If $D \neq 0$ and $\tilde{D} = 0$, we can get $V'_{\phi_q} = 0$, the quintessence-like matter has a constant potential. And from the constant of motion, we can get $\dot{\phi}_q^2 \propto \mu_0^2 a^{-6}$. However, based on the value of μ_0 , we divide the situation into two cases to discuss.

Case a), when $\mu_0 \neq 0$, the kinetic terms of the quintessence decay fast, while their potentials are constant, and no constraint on the phantom type of matter, which leads to $w < -1$ at last.

Case b), when $\mu_0 = 0$, the quintessence scalar field is a constant. This case is similar to a phantom model with cosmological constant. The interesting thing is that we could not give any constraint on the phantom-like matters.

The case $D = 0$ and $\tilde{D} \neq 0$ could be treated exactly in the same way. And the results are the same, except for the non-constrained field is changed to the quintessence-like type.

4.2. When $D \neq 0$ and $\tilde{D} \neq 0$

4.2.1. The $\mu_0 \neq 0$ case

In this case, the conserved charge is not zero. From Eq. (87), we get that $\dot{\phi}_q = D\dot{\phi}_p/\tilde{D} \propto a^{-3}$. It means that the kinetic terms of the scalar field decay fast. The equations of motion leads to $V'_{\phi_p} = V'_{\phi_q} = 0$, i.e., the potentials are constant. However, the EoS parameter evolves to $\omega_\phi = -1$ until the kinetic terms of the scalar fields vanish.

4.2.2. The $\mu_0 = 0$ case

In this case, the conserved charge vanishes, so $DN_q \dot{\phi}_q = \tilde{D}N_p \dot{\phi}_p$, combined with Eq. (86) and the equations of motion, $N_q = N_p$ is obtained. We put these results into Eq. (77), and get

$$\omega_\phi = \frac{\frac{(1-D^2/\tilde{D}^2)\dot{\phi}_q^2}{2} - V_q - V_p}{\frac{(1-D^2/\tilde{D}^2)\dot{\phi}_q^2}{2} + V_q + V_p}. \quad (88)$$

If $D/\tilde{D} < 1$, $\dot{\phi}_p^2 < \dot{\phi}_q^2$, $\omega_\phi > -1$. The physical meaning is that if the quintessence type fields slowly vary compared with the phantom type fields, the quintessence will take the dominating role, and make $\omega_\phi > -1$. And we can discuss the $D/\tilde{D} > 1$ case in the same way, where the phantom type fields will take the dominating role and $\omega_\phi < -1$. However, this solution is new. And if it cross $\omega_\phi = -1$, the ratio D/\tilde{D} should be variable. However, as Noether symmetry approach required, D/\tilde{D} is constant. It means in N -quintom case, after adding Noether symmetry, we could not make this scenario cross $\omega_\phi = -1$.

In a short summary, even the Noether symmetry does not give an explicit potential in N -quintom case, it gives a constraint on the forms of the scalar field potentials. If we try to connect this model to the observations such as SNIa data, we must choose a proper potential. Unfortunately, the observations will give constraints to the potential parameter not the parameter related to Noether symmetry which we are interested here. And this symmetry restricts the EoS parameter of crossing $\omega_\phi = -1$.

5. From deceleration to acceleration

Based on the exact potential forms given by Noether symmetry in N -quintessence and N -phantom models, the evolution of our universe could be analyzed. Firstly, two new variables $y = \phi/m_{pl}$, $u = \ln(a/a_0)$ are needed. Then we can define the fractional energy density of dust matter as $\Omega_m = \rho_m/3H^2 m_{pl}^2 = \Omega_{m0}(H_0/H)^2 \exp(-3u)$, and the fractional energy density of scalar fields $\Omega_\phi = \rho_\phi/3H^2 m_{pl}^2$ which depends on the exact potential form.

In N -quintessence model, we discuss the possible accelerating solution which is presented in Eq. (33) with $A = 0$, $B \neq 0$ and $\lambda = \sqrt{3}/2$. Ω_ϕ can be written down as

$$\Omega_\phi = \frac{y'^2}{6} + \Omega_V \exp(-\lambda y), \quad (89)$$

where $\Omega_V = V_0 B^2/3H^2 m_{pl}^2 = \Omega_{V0}(H_0/H)^2$. Then, we can simplify Eq. (8) and (14) as

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{m0} \exp(-3u) + \Omega_{V0} \exp(-\lambda y)}{1 - y'^2/6}, \quad (90)$$

$$y'' = 3\lambda \Omega_V \exp(-\lambda y) - \left[\frac{3}{2}\Omega_m + 3\Omega_V \exp(-\lambda y)\right]y', \quad (91)$$

where a prime denotes the derivative with respect to u . Following the numerical calculation method used in Refs. [17,19], the evolution of the fractional energy densities can be plotted. We choose $\Omega_m = 1$ in the matter dominated epoch around $a/a_0 \approx 1/12$ or $u = -2.5$ as initial condition. Fig. 1 shows today's fractional density Ω_{m0} is nearly 0.6 which is contradictable with the widest observational results $\Omega_{m0} = 0.3 \pm 0.1$ [20].

Furthermore, setting $N = 1$ and using the potential in Eq. (61), Ω_ϕ in N -phantom case reads

$$\Omega_\phi = \frac{y'^2}{6} + \Omega_V \left(1 - \cos\left(\sqrt{\frac{3N}{2}}y\right)\right), \quad (92)$$

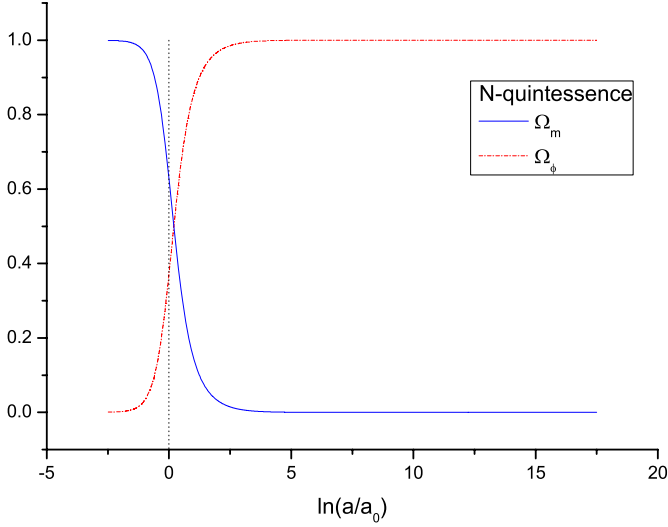


Fig. 1. The evolutions of fractional energy densities Ω_ϕ and Ω_m in N -quintessence model.

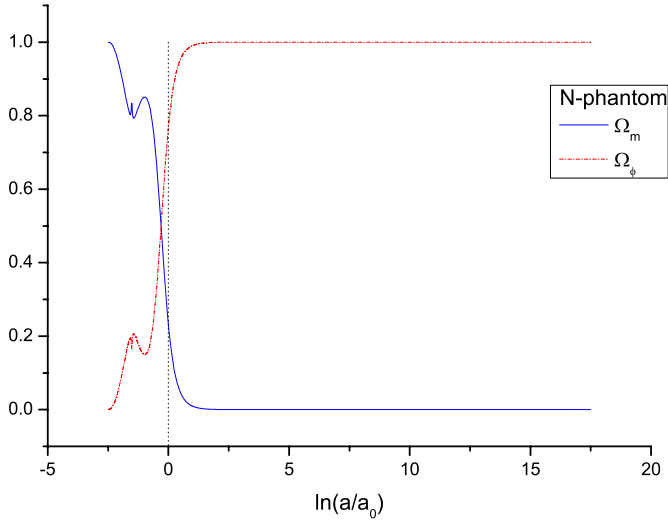


Fig. 2. The evolutions of fractional energy densities Ω_ϕ and Ω_m in N -phantom model.

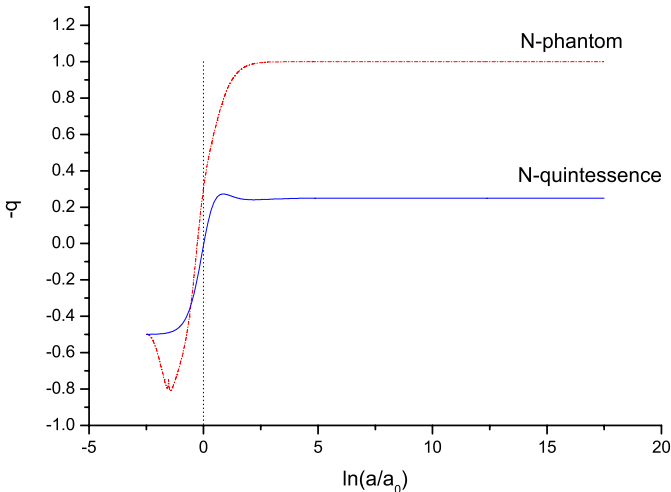


Fig. 3. The evolutions of the minus of the deceleration factor $-q$ in N -quintessence and N -phantom cases.

where $\Omega_V = V_0/3H^2 m_{pl}^2 = \Omega_{V0}(H_0/H)^2$. Then, the evolutions of scale factor and scalar field in N -phantom case are

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{m0} \exp(-3u) + \Omega_{V0}(1 - \cos(\sqrt{\frac{3N}{2}}y))}{1 - y'^2/6}, \quad (93)$$

$$y'' = -\frac{3}{2}\lambda\Omega_V \sin\left(\sqrt{\frac{3N}{2}}y\right) - \left[\frac{3}{2}\Omega_m + \frac{3\Omega_V}{2}\left(1 - \cos\left(\sqrt{\frac{3N}{2}}y\right)\right)\right]y'. \quad (94)$$

We can also start from the matter dominated epoch around $a/a_0 \approx 1/12$ or $u = -2.5$, and give out the evolutions of the fractional energy densities. Fig. 2 shows today's fractional density of dust matter Ω_{m0} is nearly 0.23 which is consistent with the observational results $\Omega_{m0} = 0.3 \pm 0.1$.

Specifically speaking, we can write down the acceleration (the minus of the deceleration factor)

$$\frac{\ddot{a}}{aH^2} = -q = \Omega_\phi - \frac{1}{3}y'^2 - \frac{1}{2}\Omega_m, \quad (95)$$

and plot its evolutions in N -quintessence and N -phantom cases. Fig. 3 shows N -quintessence with exponential potential chosen by Noether symmetry cannot make our universe accelerate ($-q \leq 0$), while N -phantom with the cosine potential can make our universe accelerate ($-q > 0$). These results coincide with the evolutions of the fractional energy densities in Figs. 1 and 2. In conclusion, today's acceleration heavily depends on the choice of potentials. This is the reason why we don't discuss the evolution of our universe in the N -quintom case where the Noether symmetry doesn't give out the exact form of potentials.

6. Conclusion

There is no immediate physical justification for the choice of $V(\phi)$ in multiple scalar fields. In this Letter, to choose proper potentials for multiple scalar fields scenario, and to be consistent with the observations which indicates the EoS parameter in the range of $-1.21 \leq \omega_\phi \leq -0.89$, we have studied the N -quintessence, N -phantom, N -quintom scalar fields models by the Noether symmetry approach. The existence of Noether symmetry implies that with respect to the infinitesimal generator of the desired symmetry, the Lie derivative of the related Lagrangian vanishes. As we have considered a flat FRW metric, the phase space in the N -quintessence and N -phantom was then constructed by taking the scale factor a and the scalar field ϕ as independent dynamical variables. In the N -quintom case, we have to expand the configuration space to $\mathcal{Q} = (a, \phi_q, \phi_p)$.

Specifically speaking, on the one hand, the Noether conditions depend on the cosmological dynamics which is determined by the potentials. On the other hand, the main consequence by adding the Noether symmetry is that we have selected the class of potentials and indicated the most reasonable, specific ones directly from the physical interpretation. In the N -quintessence case, we find the exponential potentials from the Noether conditions which could be derived from several theoretic models. In the N -phantom case, the suitable potential required by the Noether conditions is $\frac{V_0}{2}(1 - \cos(\sqrt{\frac{3N}{2}}\frac{\phi}{m_{pl}}))$ which is related to pseudo Nambu-Goldstone boson. The case of the N -quintom is very interesting. Although it does not give an explicit potential, it gives a constraint on the forms of the scalar field potentials.

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