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Avoidance of partitions of a three-element set ☆

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Abstract

Klazar defined and studied a notion of pattern avoidance for set partitions, which is an analogue of pattern avoidance for permutations. Sagan considered partitions which avoid a single partition of three elements. We enumerate partitions which avoid any family of partitions of a 3-element set as was done by Simion and Schmidt for permutations. We also consider even and odd set partitions. We provide enumerative results for set partitions restricted by generalized partition patterns, which are an analogue of the generalized permutation patterns of Babson and Steingrímsson. Finally, in the spirit of work done by Babson and Steingrímsson, we will show how these generalized partition patterns can be used to describe set partition statistics.

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1. Introduction

Pattern avoidance in permutations was first introduced by Knuth in [18], and is currently an area of very active research. An approach to studying pattern avoidance and containment that deals with set partitions was introduced and studied by Klazar in [15–17] and continued by Sagan in [22]. The extensively studied non-crossing partitions defined by Kreweras [19] can be viewed as those which avoid a specific pattern with four elements. We will focus on the enumeration of all partitions of an n-element set which avoid a family of partitions of a 3-element set. To make

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these notions of pattern containment for permutations and set partitions precise and to see their connections we will need some definitions.

If $f : S \to T$ is a function from set S to set T, then f acts element-wise on objects constructed from S. For example, if $a_1a_2\dots a_n$ is a permutation of elements of S then $f(a_1a_2\dots a_n) = f(a_1)f(a_2)\dots f(a_n)$. Also, define $[n]$ to be the set $\{1, 2, \dots, n\}$ and $[k, n]$ to be the set $\{k, k + 1, \dots, n\}$.

Suppose that $S \subseteq \mathbb{Z}$ is a set with $\#S = n$, then the *standardization* map corresponding to S is the unique order preserving bijection $St_S : S \to [n]$. For example if $S = \{2, 5, 7, 10\}$ then $St_S(2) = 1$, $St_S(5) = 2$, $St_S(7) = 3$, and $St_S(10) = 4$. When it is clear from context what set the standardization map is acting on, we will omit the subscript S.

Let $p = a_1a_2\dots a_k \in S_k$ be a given permutation, called the *pattern*, where S_k is the symmetric group on k letters. A permutation $q = b_1b_2\dots b_n \in S_n$ *contains* the pattern p if there is a subsequence $q' = b_{i_1}b_{i_2}\dots b_{i_k}$ of q with $St(q') = p$. Otherwise q *avoids* p. For example the permutation $q = 32145$ contains 6 copies of the pattern 213, namely 324, 325, 314, 315, 214, and 215. On the other hand q avoids the pattern 132. For $R \subseteq S_k$, let

$$S_n(R) = \{q \in S_n\colon\ q \text{ avoids every pattern } p \in R\}.$$

The problem of enumerating $S_n(R)$ for $R \subseteq S_3$ was considered by Simion and Schmidt [26]. We will consider the analogous problem for patterns in partitions.

A *partition* π of set $S \subseteq \mathbb{Z}$, written $\pi \vdash S$, is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \dots, B_k$ of S called *blocks* such that $\bigcup_{i=1}^{k} B_i = S$. We write $\pi = B_1/B_2/\dots/B_k$ and define the *length* of π, written $\ell(\pi)$, to be the number of blocks. Since the order of the blocks does not matter, we will always write our partitions in the *canonical order* where

$$\min B_1 < \min B_2 < \dots < \min B_k.$$

We will also always write the elements of each block in increasing order. For example, $137/26/45 \vdash [7]$ has length 3.

Let

$$\Pi_n = \big\{\pi \vdash [n]\big\}$$

be the set of all partitions of $[n]$. Suppose σ is a set partition of length m and π is a partition of length ℓ. Then σ *contains* π, written $\pi \subseteq \sigma$, if there are ℓ different blocks of σ each containing a block of π. For example $\sigma = 137/26/45$ contains $\pi = 2/37/5$ but does not contain $\pi' = 2/37/6$ because 2 and 6 are in the same block of σ.

Let $\pi \in \Pi_k$ be a given set partition called the *pattern*. A partition $\sigma \in \Pi_n$ *contains the pattern* π if there is some $\sigma' \subseteq \sigma$ with $St(\sigma') = \pi$. Otherwise π *avoids* σ. For example $\sigma = 137/26/45$ contains six copies of the pattern $\pi = 14/2/3$, namely 17/2/4, 17/2/5, 17/4/6, 17/5/6, 26/3/4, and 26/3/5. It is important to note here that when looking for a copy of π in σ, the order of the blocks does not matter. On the other hand, consider the pattern $\pi' = 1/234$. To be contained in σ the copy of the block 234 of π' must be contained in a block of size three or larger. The only such block of σ is 137. It is impossible to find an element smaller than 1, so σ does not contain a copy of π'. For $R \subseteq \Pi_k$, let

$$\Pi_n(R) = \{\sigma \in \Pi_n\colon\ \sigma \text{ avoids every pattern } \pi \in R\}.$$

The set of non-crossing partitions mentioned above may be defined as the set $\Pi_n(13/24)$. It is known that $\#\Pi_n(13/24) = C_n$, where C_n is the nth Catalan number [19,27]. For a survey of results about non-crossing partitions see Simion's paper [25].

Sagan [22] has provided enumerative results for $\Pi_n(R)$ when $\#R = 1$. In the spirit of work done by Simion and Schmidt on permutation patterns [26], we will enumerate $\Pi_n(R)$ for $\#R \geqslant 2$. We then define the sign of a partition and enumerate the set of signed partitions of $[n]$ avoiding particular patterns. In Section 5, we define generalized patterns analogous to the generalized permutation patterns of Babson and Steingrímsson [1], and provide enumerative results for those. Finally, we will show how these generalized partition patterns can be used to describe set partition statistics.

2. Double restrictions

In this section we will consider the case of $\#\Pi_n(R)$ where $\#R = 2$. Given a set partition $\sigma = B_1/B_2/\dots/B_k \vdash [n]$, let $\sigma^c = B_1^c/B_2^c/\dots/B_k^c$ be the *complement* of σ where

$$B_i^c = \{n - a + 1\colon\ a \in B_i\}.$$

For example if $\sigma = 126/3/45$ then $\sigma^c = 156/23/4$. The following result is obvious, so we omit the proof.

Proposition 2.1 *(Sagan). For $n \geqslant 1$,*

$$\Pi_n\big(\sigma^c\big) = \big\{\pi^c\colon\ \pi \in \Pi_n(\sigma)\big\},$$
$$\#\Pi_n\big(\sigma^c\big) = \#\Pi_n(\sigma).$$

The following lemma is an immediate consequence of Proposition 2.1.

Lemma 2.2.

$$\#\Pi_n(12/3, 123) = \#\Pi_n(1/23, 123),$$
$$\#\Pi_n(1/2/3, 12/3) = \#\Pi_n(1/2/3, 1/23),$$
$$\#\Pi_n(12/3, 13/2) = \#\Pi_n(1/23, 13/2).$$

There are 10 different sets R with elements from Π_3 and $\#R = 2$, so by Lemma 2.2 there are seven different cases to consider. Note that $\#\Pi_0 = 1$ by letting the empty set partition itself. Since any partition in Π_1 or Π_2 cannot possibly contain a partition of [3], we have $\#\Pi_0(R) = 1$, $\#\Pi_1(R) = 1$ and $\#\Pi_2(R) = 2$ for all $R \subseteq \Pi_3$. The fact that $\#\Pi_3 = 5$ implies that $\#\Pi_3(R) = 3$ for any $R \subset \Pi_3$, with $\#R = 2$. Hence, it suffices to consider $n \geqslant 4$ in the following results.

A partition $\sigma \vdash [n]$ is *layered* if σ is of the form $[1, i]/[i + 1, j]/[j + 1, k]/\dots/[\ell + 1, n]$. An example of a layered partition is $\sigma = 123/4/56/789$. A partition σ is a *matching* if $\#B \leqslant 2$ for every block B of σ.

We will use the following results of Sagan [22] repeatedly, so we state them now.

Proposition 2.3 *(Sagan).*

$$\Pi_n(1/2/3) = \{\sigma\colon l(\sigma) \leqslant 2\}, \tag{1}$$
$$\Pi_n(12/3) = \{\sigma = B_1/B_2/\ldots/B_k\colon \min B_i = i \textit{ for each } i, \textit{ and } [k+1,n] \subseteq B_i \textit{ for some } i\}, \tag{2}$$
$$\Pi_n(13/2) = \{\sigma\colon \sigma \textit{ is layered}\}, \tag{3}$$
$$\Pi_n(123) = \{\sigma\colon \sigma \textit{ is a matching}\}. \tag{4}$$

Proposition 2.4. *For all $n \geqslant 3$,*

$$\Pi_n(1/2/3, 12/3) = \{12\ldots n,\ 1/23\ldots n,\ 13\ldots n/2\},$$
$$\#\Pi_n(1/2/3, 12/3) = 3.$$

Proof. Let $\sigma \in \Pi_n(1/2/3, 12/3)$. By (1), σ may have at most two blocks. If $\ell(\sigma) = 1$ then $\sigma = 12\ldots n$. If $\ell(\sigma) = 2$ then by (2), we must have $[3,n] \subset B_i$ for $i = 1$ or 2. □

Proposition 2.5. *For all $n \geqslant 1$,*

$$\Pi_n(1/2/3, 13/2) = \{\sigma\colon \sigma = 12\ldots k/(k+1)(k+2)\ldots n \textit{ for some } k \in [n]\},$$
$$\#\Pi_n(1/2/3, 13/2) = n.$$

Proof. If $\sigma \in \Pi_n(1/2/3, 13/2)$ then σ is layered by (3), and $\ell(\sigma) \leqslant 2$ by (1). Hence σ is of the form described above. The enumeration follows immediately. □

Proposition 2.6.

$$\Pi_n(1/2/3, 123) = \begin{cases} \{12/34, 13/24, 14/23\}, & n = 4, \\ \emptyset, & n \geqslant 5, \end{cases}$$
$$\#\Pi_n(1/2/3, 123) = \begin{cases} 3, & n = 4, \\ 0, & n \geqslant 5. \end{cases}$$

Proof. If $n \geqslant 5$ and $\sigma \vdash [n]$, then $\ell(\sigma) \geqslant 3$ or σ has a block of size $\geqslant 3$ by the Pigeonhole Principle. Thus by (1) and (4), $\Pi_n(1/2/3, 123) = \emptyset$ for $n \geqslant 5$. The case $n = 4$ is easy to check. □

Proposition 2.7. *For all $n \geqslant 3$,*

$$\Pi_n(1/23, 12/3) = \{12\ldots n,\ 1/2/\ldots/n,\ 1n/2/3/\ldots/n-1\},$$
$$\#\Pi_n(1/23, 12/3) = 3.$$

Proof. Let $\sigma = B_1/B_2/\ldots/B_k$ avoid $12/3$. If $k = 1$ then $\sigma = 12\ldots n$, which avoids $1/23$. Similarly, when $k = n$, we have $\sigma = 1/2/\ldots/n$, which avoids $1/23$. If $k = n-1$ and $n \in B_i$ for $i \geqslant 2$ then B_1/B_i is a copy of $1/23$. Thus $n \in B_1$ and $\sigma = 1n/2/3/\ldots/n-1$. If $1 < k < n-1$ then, by (2), we must have $\{n-1, n\} \subseteq B_i$ for some i, and there is at least one more block. Hence σ contains a copy of $1/23$, and so this case can not occur. $\square$

Proposition 2.8. *For all $n \geqslant 1$,*

$$\Pi_n(12/3, 13/2) = \big\{\sigma = 1/2/\dots/k-1/k(k+1)\dots n,\ \textit{for some } k \in [n]\big\},$$
$$\#\Pi_n(12/3, 13/2) = n.$$

Proof. Suppose $\sigma = B_1/B_2/\dots/B_k \in \Pi_n(12/3, 13/2)$. Then by (2) we have $i \in B_i$ for each i and exactly one of the B_i contains $[k+1, n]$. From (3) we have that σ must be layered. So $[k+1, n] \in B_k$, and $B_k = [k, n]$. Thus there is exactly one $\sigma \in \Pi_n(12/3, 13/2)$ of length k for each $k \in [n]$. $\square$

Proposition 2.9. *For all $n \geqslant 1$,*

$$\Pi_n(12/3, 123) = \{\sigma = B_1/B_2/\dots/B_k\colon \min B_i = i,\ \textit{and } k = n-1 \textit{ or } n\},$$
$$\#\Pi_n(12/3, 123) = n.$$

Proof. Assume $\sigma = B_1/B_2/\dots/B_k \in \Pi_n(12/3, 123)$. Then by (2) and (4), $k = n-1$ or n. The result follows. $\square$

Let F_n be the nth Fibonacci number, initialized by $F_0 = 1$ and $F_1 = 1$. A *composition* of an integer n is an ordered collection of positive integers $n_1, n_2, \dots, n_k$ such that $n = n_1 + n_2 + \dots + n_k$. The n_i are called *parts*. It is easy to see that F_n counts the number of compositions of n with parts of size 1 or 2.

Proposition 2.10. *For all $n \geqslant 0$,*

$$\Pi_n(13/2, 123) = \{\sigma\colon \sigma \textit{ is a layered matching}\},$$
$$\#\Pi_n(13/2, 123) = F_n.$$

Proof. Any $\sigma \in \Pi_n(13/2, 123)$ must be layered by (3) and a matching by (4).

There is a bijection between the compositions of n with parts of size 1 or 2 and the partitions of $[n]$ that are layered matchings. If $\sigma \in \Pi_n(13/2, 123)$ and $\sigma = B_1/B_2/\dots/B_k$, then we map σ to the composition $n = n_1 + n_2 + \dots + n_k$ with $n_i = \#B_i$. $\square$

From the results above we know that

$$\#\Pi_n(1/2/3, 13/2) = \#\Pi_n(12/3, 13/2) = \#\Pi_n(12/3, 123) = n,$$

and we have a very nice description of the elements in each of these sets. It is interesting to note that one gets similar results when avoiding certain sets of permutations in S_3.

Proposition 2.11 *(Simion, Schmidt). For every $n \geqslant 1$,*

$$\#S_n(123,132,231) = \#S_n(123,213,312) = n,$$
$$\#S_n(132,231,321) = \#S_n(213,312,321) = n.$$

And:

$$\begin{aligned}
q \in S_n(123,132,231) \quad &\Leftrightarrow \quad q = (n, n-1, \ldots, k+1, k-1, k-2, \ldots, 2, 1, k),\\
q \in S_n(123,213,312) \quad &\Leftrightarrow \quad q = (n, n-1, \ldots, k+1, 1, 2, 3, \ldots, k),\\
q \in S_n(132,231,321) \quad &\Leftrightarrow \quad q = (n-1, n-2, \ldots, k+1, n, k, k-1, \ldots, 2, 1),\\
q \in S_n(213,312,321) \quad &\Leftrightarrow \quad q = (k-1, \ldots, 3, 2, 1, n, n-1, \ldots, k).
\end{aligned}$$

The Fibonacci numbers also occur when avoiding permutations.

Proposition 2.12 *(Simion, Schmidt). For every $n \geqslant 1$,*

$$\#S_n(123,132,213) = F_n.$$

There is a simple map $\Phi : \Pi_n \to S_n$, given by sending $\sigma = B_1/B_2/\ldots/B_k$ to $B_k B_{k-1} \ldots B_1$. For example, $\Phi(1/23/4/56) = 564231$.

Proposition 2.13. *The map Φ restricts to a bijection from the set $\Pi_n(13/2,123)$ to the set $S_n(123,132,213)$.*

Proof. We may describe $q \in S_n(123,132,213)$ recursively. To avoid the patterns 123 and 213, we must have $q^{-1}(n) \leqslant 2$. If $q^{-1}(n) = 1$ then the remaining positions form a permutation in $S_{n-1}(123,132,213)$. If $q^{-1}(n) = 2$ then $q^{-1}(n-1) = 1$, otherwise there will be a copy of 132 in q. The remaining positions form a permutation in $S_{n-2}(123,132,213)$.

Suppose $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(13/2,123)$, then $B_k = \{n\}$ or $\{n-1,n\}$. The permutation $\Phi(\sigma)$ thus begins with n or $n-1, n$. Inductively, one can see that this restriction of the map Φ is well defined.

To prove that the restricted Φ is a bijection we provide its inverse map. Let $q = q_1 q_2 \ldots q_n \in S_n(123,132,213)$ then we say that q_k is a *descent* if $q_k > q_{k+1}$. Let $D = \{q_{i_1}, q_{i_2}, \ldots, q_{i_\ell}\}$ be the set of descents of q, with $i_1 < i_2 < \cdots < i_\ell$. Then

$$\Phi^{-1}(q) = q_{i_\ell+1} q_{i_\ell+2} \ldots q_n / q_{i_{\ell-1}+1} \ldots q_{i_\ell} / \ldots / q_1 \ldots q_{i_1}.$$

For example $\Phi^{-1}(564231) = 1/23/4/56$ because its descent set is $D = \{3,4,6\}$.

We now show that Φ^{-1} is well defined. Every $q \in S_n(123,132,213)$ must have a descent in at least one of its first two positions. After this initial descent there may be no more than one position between any two descents. Thus the blocks of $\Phi^{-1}(q)$ will have size at most 2, and from the description of the elements of $S_n(123,132,213)$ above $\Phi^{-1}(q)$ will be layered.

The fact that Φ and Φ^{-1} are inverses follows easily from the descriptions of the maps. $\square$

3. Higher order restrictions

We begin, as with double restrictions, by reducing the number of cases. The following lemma is a consequence of Proposition 2.1.

Table 1
Enumeration of partitions restricted by 3 patterns

R	$\Pi_n(R)$	$\#\Pi_n(R)$
$\{1/2/3, 12/3, 13/2\}$	$\{12\ldots n, 1/23\ldots n\}$	2
$\{1/2/3, 12/3, 123\}$	$\emptyset$	0
$\{1/2/3, 13/2, 123\}$	$\{12/34\}$	1 if $n = 4$
	$\emptyset$	0 if $n \geqslant 5$
$\{1/2/3, 1/23, 12/3\}$	$\{12\ldots n\}$	1
$\{12/3, 13/2, 123\}$	$\{1/2/\ldots/n, 1/2/\ldots/n-2/(n-1)n\}$	2
$\{1/23, 12/3, 13/2\}$	$\{123\ldots n, 1/2/\ldots/n\}$	2
$\{1/23, 12/3, 123\}$	$\{1/2/\ldots/n, 1n/2/3/\ldots/n-1\}$	2

Lemma 3.1.

$$\#\Pi_n(1/2/3, 12/3, 123) = \#\Pi_n(1/2/3, 1/23, 123),$$

$$\#\Pi_n(1/2/3, 12/3, 13/2) = \#\Pi_n(1/2/3, 1/23, 13/2),$$

$$\#\Pi_n(12/3, 13/2, 123) = \#\Pi_n(1/23, 13/2, 123).$$

The results for $\#\Pi_n(R)$ where $\#R = 3$ are easy to prove. Table 1 describes these sets and gives their enumeration for $n \geqslant 4$. The following proposition describes $\#\Pi_n(R)$ for $\#R \geqslant 4$. We omit the simple proof.

Proposition 3.2. *For $R \subseteq \Pi_3$ with $\#R \geqslant 4$ and $n \geqslant 4$,*

$$\#\Pi_n(R) = \begin{cases} 0 & \text{if } \{1/2/3, 123\} \subseteq R, \\ 1 & \text{else}. \end{cases}$$

4. Even and odd set partitions

In this section we will consider the number of even and odd partitions of the set $[n]$, which avoid a single pattern of length three. A partition $\sigma \vdash [n]$ with $\ell(\sigma) = k$ has *sign*,

$$\operatorname{sgn}(\sigma) = (-1)^{n-k}.$$

Even partitions σ satisfy $\operatorname{sgn}(\sigma) = 1$, and *odd* partitions σ satisfy $\operatorname{sgn}(\sigma) = -1$. We will use the following notation:

$$E\Pi_n(\pi) = \bigl\{\sigma \vdash [n]\colon \operatorname{sgn}(\sigma) = 1\bigr\},$$

$$O\Pi_n(\pi) = \bigl\{\sigma \vdash [n]\colon \operatorname{sgn}(\sigma) = -1\bigr\}.$$

The following follows directly from the definitions.

Lemma 4.1. *The sign of σ is the same as the sign of σ^c. Thus* $\#E\Pi_n(12/3) = \#E\Pi_n(1/23)$ *and* $\#O\Pi_n(12/3) = \#O\Pi_n(1/23)$.

We will use the following result of Sagan [22] repeatedly, so we state it now. Define the *double factorial* by

$$(2i)!! = 1 \cdot 3 \cdot 5 \cdots (2i-1).$$

Proposition 4.2 *(Sagan).*

$$\#\Pi_n(1/2/3) = 2^{n-1}, \tag{5}$$

$$\#\Pi_n(12/3) = \binom{n}{2} + 1, \tag{6}$$

$$\#\Pi_n(13/2) = 2^{n-1}, \tag{7}$$

$$\#\Pi_n(123) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} (2i)!!. \tag{8}$$

We now consider single restrictions. By Lemma 4.1 there are only four cases.

Proposition 4.3. *For all odd $n \geqslant 1$,*

$$\#E\Pi_n(1/2/3) = 1,$$
$$\#O\Pi_n(1/2/3) = 2^{n-1} - 1.$$

For all even $n \geqslant 2$,

$$\#E\Pi_n(1/2/3) = 2^{n-1} - 1,$$
$$\#O\Pi_n(1/2/3) = 1.$$

Proof. By (1), any $\sigma \in \Pi_n(1/2/3)$ must have $\ell(\sigma) \leqslant 2$. If n is odd, then a partition of length 1 will be even and a partition of length 2 will be odd. There is only one partition of length 1, and $\#O\Pi_n(\pi) + \#E\Pi_n(\pi) = \#\Pi_n(\pi)$ for any pattern π. Thus, the result holds for odd n by (5). The proof for even n is similar. $\square$

Proposition 4.4. *For all odd $n \geqslant 0$,*

$$\#E\Pi_n(12/3) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1,$$

$$\#O\Pi_n(12/3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. By (2) we have, for n odd,

$$\#E\Pi_n(12/3) = 1 + \sum_{k=0}^{\frac{n-3}{2}} (2k+1) = 1 + \frac{(n-1)^2}{4} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1,$$

and by (6)

$$\#O\Pi_n(12/3) = \binom{n}{2} + 1 - \frac{(n-1)^2}{4} - 1 = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

The proof for even n is similar. □

Proposition 4.5. *For all $n \geqslant 1$,*

$$\#O\Pi_n(13/2) = \#E\Pi_n(13/2) = 2^{n-2}.$$

Proof. By (7) it suffices to give a sign reversing involution $\psi : \Pi_n(13/2) \to \Pi_n(13/2)$. By (3), $\sigma \in \Pi_n(13/2)$ is layered, so it is of the form $\sigma = B_1/B_2/\ldots/B_k$, where either $B_k = \{n\}$ or $B_k \supset \{n\}$. Let

$$\psi(\sigma) = \begin{cases} B_1/B_2/\ldots/B_{k-1} \cup \{n\} & \text{if } B_k = \{n\}, \\ B_1/B_2/\ldots/B_k - \{n\}/n & \text{if } B_k \supset \{n\}. \end{cases}$$

Notice that $\psi(\sigma)$ is still layered for any $\sigma \in \Pi_n(13/2)$, so ψ is well defined. And, ψ is its own inverse because it either moves n into the block preceding it if $\{n\}$ is a block and into its own block otherwise. Also, ψ changes the sign of σ by either increasing or decreasing the length of σ by 1. □

Proposition 4.6. *For all $n \geqslant 1$,*

$$\#E\Pi_n(123) = \sum_{i=0}^{\lfloor \frac{n-2}{4} \rfloor} \binom{n}{4i+2} (4i+2)!!,$$

$$\#O\Pi_n(123) = \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor} \binom{n}{4i} (4i)!!.$$

Proof. Any $\sigma \in \Pi_n(123)$ is a matching. If i blocks of σ have 2 elements each and the remaining blocks are singletons then σ has $i + (n - 2i) = n - i$ blocks. Thus $\operatorname{sgn}(\sigma) = (-1)^{n-(n-i)} = (-1)^i$. So the even and odd counts are obtained by taking the appropriate terms from (8). □

Table 2 gives the results for $\#E\Pi_n(R)$ and $\#O\Pi_n(R)$ where $\#R \geqslant 2$ and $n \geqslant 4$. We prove the enumeration of $E\Pi_n(13/2, 123)$ and $O\Pi_n(13/2, 123)$ as an example and leave the rest to the reader.

Proposition 4.7.

$$\#E\Pi_n(13/2, 123) = \begin{cases} \lceil F_n/2 \rceil & \text{for } n \equiv 0, 1 \pmod 6, \\ F_n/2 & \text{for } n \equiv 2, 5 \pmod 6, \\ \lfloor F_n/2 \rfloor & \text{for } n \equiv 3, 4 \pmod 6; \end{cases}$$

$$\#O\Pi_n(13/2, 123) = \begin{cases} \lfloor F_n/2 \rfloor & \text{for } n \equiv 0, 1 \pmod 6, \\ F_n/2 & \text{for } n \equiv 2, 5 \pmod 6, \\ \lceil F_n/2 \rceil & \text{for } n \equiv 3, 4 \pmod 6. \end{cases}$$

Table 2
Enumeration of even and odd partitions restricted by at least 2 patterns

R	$\#E\Pi_n(R)$	$\#O\Pi_n(R)$
$\{1/2/3, 12/3\}$	1 for n odd 2 for n even	2 for n odd 1 for n even
$\{1/2/3, 13/2\}$	1 for n odd $n-1$ for n even	$n-1$ for n odd 1 for n even
$\{1/2/3, 123\}$	3 for $n=4$ 0 for $n \geqslant 5$	0
$\{1/23, 12/3\}$	2 for n odd 1 for n even	1 for n odd 2 for n even
$\{12/3, 13/2\}$	$\lceil n/2 \rceil$	$\lfloor n/2 \rfloor$
$\{12/3, 123\}$	1	$n-1$
$\{13/2, 123\}$	$\lceil F_n/2 \rceil$ for $n \equiv 0, 1 \pmod 6$ $F_n/2$ for $n \equiv 2, 5 \pmod 6$ $\lfloor F_n/2 \rfloor$ for $n \equiv 3, 4 \pmod 6$	$\lfloor F_n/2 \rfloor$ for $n \equiv 0, 1 \pmod 6$ $F_n/2$ for $n \equiv 2, 5 \pmod 6$ $\lceil F_n/2 \rceil$ for $n \equiv 3, 4 \pmod 6$
$\{1/2/3, 1/23, 12/3\}$	1 for n odd 0 for n even	0 for n odd 1 for n even
$\{1/2/3, 12/3, 13/2\}$	1	1
$\{1/2/3, 12/3, 123\}$	0	0
$\{1/2/3, 13/2, 123\}$	1	0
$\{1/23, 12/3, 13/2\}$	2 for n odd 1 for n even	0 for n odd 1 for n even
$\{1/23, 12/3, 123\}$	1	1
$\{12/3, 13/2, 123\}$	1	1
$\{1/2/3, 1/23, 12/3, 13/2\}$	1 for n even 0 for n odd	0 for n odd 1 for n even
$\{1/2/3, 1/23, 12/3, 123\}$	0	0
$\{1/2/3, 12/3, 13/2, 123\}$	0	0
$\{1/23, 12/3, 13/2, 123\}$	1	0
$\{1/2/3, 1/23, 12/3, 13/2, 123\}$	0	0

Proof. Let $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(13/2, 123)$. Then $B_k = \{n\}$ or $\{n-1, n\}$. If $B_k = \{n\}$ then $B_1/B_2/\ldots/B_{k-1}$ is a layered matching of $[n-1]$ and $\operatorname{sgn}(B_1/B_2/\ldots/B_{k-1}) = \operatorname{sgn}(\sigma)$. If $B_k = \{n-1, n\}$ then $B_1/B_2/\ldots/B_{k-1}$ is a layered matching of $[n-2]$ and $\operatorname{sgn}(B_1/B_2/\ldots/B_{k-1}) = -\operatorname{sgn}(\sigma)$. Thus we have that

$$\#E\Pi_n(13/2, 123) = \#E\Pi_{n-1}(13/2, 123) + \#O\Pi_{n-2}(13/2, 123).$$

Similarly,

$$\#O\Pi_n(13/2, 123) = \#O\Pi_{n-1}(13/2, 123) + \#E\Pi_{n-2}(13/2, 123).$$

Now induct on n. To show that the proposition is true when $0 \leqslant n \leqslant 5$ is easy. This leaves us with twelve cases to check for the inductive step. We will show one of them. It is easy to see that F_n is odd unless $n \equiv 2, 5 \pmod 6$.

Suppose that $n \equiv 4 \pmod 6$. Then we have

$$\begin{aligned}\#E\Pi_n(13/2, 123) &= \#E\Pi_{n-1}(13/2, 123) + \#O\Pi_{n-2}(13/2, 123)\\ &= \lfloor F_{n-1}/2 \rfloor + F_{n-2}/2\\ &= \frac{F_{n-1} - 1 + F_{n-2}}{2}\\ &= \lfloor F_n/2 \rfloor. \qquad \square\end{aligned}$$

5. Generalized partition patterns

Babson and Steingrímsson [1] defined generalized patterns for permutations. These were patterns in which certain elements were required to be consecutive. Generalized permutation patterns were used to describe permutation statistics and classify Mahonian statistics. In this section we will define a similar notion for set partition patterns and consider the avoidance case. In the next section we will show that generalized partition patterns can be used to describe set partition statistics.

Recall that if $\sigma = B_1/B_2/\ldots/B_k$ is a partition then the blocks are written in such a way that $\min B_1 < \min B_2 < \cdots < \min B_k$. This gives us a well-defined notion of adjacency of blocks, where we consider B_i as being adjacent to both B_{i-1} and B_{i+1}. Consider the partition $\sigma = 147/25/36$ and the pattern $\pi = 13/2$. Suppose now that a copy of π must appear in adjacent blocks. Then $17/2$ is still a copy, but $17/3$ is not. We may also have the blocks in the restricted copy of $13/2$ in the opposite order making $25/4$ a copy of π in σ. We will denote π with the adjacency restriction by the *generalized pattern* $\rho = 13|2$. In general, we will denote block adjacency using a vertical bar.

Recall that the elements of a block are put in order by size, which gives us a way to consider adjacent elements. Now, suppose we want to find a copy of $13/2$ in $\sigma = 147/25/36$, but we require that the elements that represent 1 and 3 in this copy are adjacent. In this case $14/3$ is a copy of $13/2$, but $17/6$ is not, since 1 and 7 are not adjacent in their block. We will denote this by the *generalized pattern* $\rho = \widehat{13}/2$. In general, we will denote element adjacency by placing an arc over the elements, which must be adjacent.

If ρ is a generalized pattern, then the notation $\Pi_n(\rho)$ denotes the set of partitions of $[n]$, which avoid ρ. Similarly, if R is any set of generalized patterns then $\Pi_n(R)$ is the set of partitions of $[n]$, which avoid all generalized patterns in R.

We are interested in enumerating the $\Pi_n(R)$ where R is a set of partitions of [3] at least one of which contains an adjacency restriction. It turns out that the adjacency restrictions do not actually restrict most of the original patterns. This is summed up in the next lemma.

Lemma 5.1. *The following are true for generalized patterns*:

$$\Pi_n(1/2/3) = \Pi_n(1|2/3) = \Pi_n(1/2|3) = \Pi_n(1|2|3),$$

$$\Pi_n(1/23) = \Pi_n(1|23) = \Pi_n(1/\widehat{23}) = \Pi_n(1|\widehat{23}),$$

$$\Pi_n(13/2) = \Pi_n(\widehat{13}/2) = \Pi_n(13|2) = \Pi_n(\widehat{13}|2),$$

$$\Pi_n(123) = \Pi_n(\widehat{12}3) = \Pi_n(1\widehat{23}) = \Pi_n(\widehat{12}\widehat{23}),$$

$$\Pi_n(12/3) = \Pi_n(\widehat{12}/3),$$
$$\Pi_n(12|3) = \Pi_n(\widehat{12}|3).$$

Proof. We will only prove the second line as the others are very similar. First we show that $\Pi_n(1/23) = \Pi_n(1|23)$. It is obvious that if a partition $\sigma \vdash [n]$ contains a copy of $1|23$ then it contains a copy of $1/23$. So it will suffice to show the other containment holds. Let $\sigma = B_1/B_2/\ldots/B_k \vdash [n]$ contain a copy a/bc of $1/23$. Suppose $a \in B_s$ and $b, c \in B_t$. If $s < t$ then the block B_{t-1} exists and $\min B_{t-1} < \min B_t \leqslant b < c$. Letting $d = \min B_{t-1}$ gives a copy d/bc of $1|23$ in σ. If $s > t$ then B_{t+1} exists and $\min B_{t+1} \leqslant a < b < c$. Letting $e = \min B_{t+1}$ gives a copy e/bc of $1|23$ in σ. We remind the reader that the adjacent blocks of the copy of $1|23$ may appear in either order in σ.

Now we will show that $\Pi_n(1/23) = \Pi_n(1/\widehat{23})$. Again, it suffices to show that if $\sigma \vdash [n]$ contains a copy of $1/23$ then it contains a copy of $1/\widehat{23}$. Given a copy a/bc of $1/23$ in σ, if b and c are not adjacent in their block B then let d be the minimum of all of the elements of B which are larger than b. Thus a/bd is a copy of $1/\widehat{23}$ in σ. These two observations can be used to prove the remaining equality. □

Let R be a set of generalized patterns, and let S be the same set with adjacency restrictions dropped. That is if, for example, $1|\widehat{23} \in R$ then $1/23 \in S$, and S only contains patterns without adjacency restrictions. Lemma 5.1 says that unless $12|3$ or $\widehat{12}|3 \in R$, we have that $\Pi_n(R) = \Pi_n(S)$. However, since we have $\Pi_n(12|3) = \Pi_n(\widehat{12}|3)$, we only need to consider cases when $12|3 \in R$. The sets $\Pi_n(S)$ were enumerated in Sections 2 and 3, so we need only enumerate the sets $\Pi_n(S \cup \{12|3\})$ where $S \subseteq \Pi_3 - \{12/3\}$.

Proposition 5.2. *Let $S \subseteq \Pi_3 - \{12/3\}$ then $\Pi_n(S \cup \{12|3\}) = \Pi_n(S \cup \{12/3\})$ unless $S = \emptyset$ or $\{123\}$.*

Proof. The cases where $\#S \geqslant 2$ follow automatically from those with $\#S = 1$ and Lemma 5.1. The three cases with $\#S = 1$ are very similar, so we will only prove the statement for $S = \{13/2\}$. Let $\sigma \in \Pi_n(13/2, 12|3)$, then σ must be layered. Thus any copy of $12/3$ in σ easily reduces to a copy of $12|3$ as in the proof of Lemma 5.1. □

The following lemma describes the elements of $\Pi_n(12|3)$.

Lemma 5.3. *We have $\sigma \in \Pi_n(12|3)$ if and only if whenever a block B_t of σ satisfies $\#B_t \geqslant 2$, then*

$$\#B_{t-1} = 1 \quad \text{and} \quad \#B_{t+1} = 1.$$

Furthermore, if $B_{t+1} = \{a\}$ then $a < b$ for every $b \in B_t - \{\min B_t\}$.

Proof. First we show that $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(12|3)$ can be described as above. Let $\#B_t \geqslant 2$ and suppose that B_{t-1} contains at least 2 elements and let $a < b$ be the two smallest elements of B_{t-1}. Let $c < d$ be the two smallest elements of B_t. By the definition of canonical order, $a < c$. If $b < d$, then ab/d is a copy of $12|3$. If $b > d$, then cd/b is a copy of $12|3$ another contradiction. The proof that $\#B_{t+1} = 1$ is similar. The single element in B_{t+1} must be larger than c by definition. If it is larger than any other element of B_t we will again have an unwanted copy of 12|3.

Now, suppose that $\sigma \in \Pi_n$ has the structure described above. Then it is straightforward to show that σ cannot contain a copy of 12|3. $\square$

First we will consider the case where $S = \emptyset$ in Proposition 5.2. Let $a_n = \#\Pi_n(12|3)$ and let

$$f(x) = \sum_{n \geqslant 0} a_n \frac{x^n}{n!}$$

be the corresponding exponential generating function.

Proposition 5.4. *For $n \geqslant 2$,*

$$a_n = a_{n-1} + 1 + \sum_{k=1}^{n-2} \binom{n-2}{k} a_{n-k-2}$$

with the initial conditions $a_0 = 1$ and $a_1 = 1$, and $f(x)$ satisfies the differential equation

$$y'' = y' + y\bigl(e^x - 1\bigr) + e^x.$$

Proof. That $\#\Pi_0(12|3) = \#\Pi_1(12|3) = 1$ is obvious. Let $\sigma = B_1/B_2/\dots/B_k \in \Pi_n(12|3)$. Either $\#B_1 = 1$ or $\#B_1 \geqslant 2$. If $\#B_1 = 1$ then, by the definition of canonical order, $B_1 = \{1\}$. Clearly any 12|3 avoiding partition of the set $[2, n]$ will still avoid 12|3 if we prepend the block $\{1\}$. This gives the first term of the recursion.

Now suppose that $\#B_1 \geqslant 2$, then either $\sigma = 12\dots n$ or not. The case where $\sigma = 12\dots n$ is counted by the 1 in the recursion. If $\sigma \neq 12\dots n$ then, by Lemma 5.3, we must have $B_2 = \{2\}$. If k of the elements from $[3, n]$ are in B_1, then the remaining $n - k - 2$ elements must form a 12|3 avoiding partition. This establishes the recursion.

Using the recursion to produce the differential equation satisfied by $f(x)$ is routine and is left the reader. $\square$

The substitution $y = ue^{x/2}$ simplifies the equation to

$$u'' = u\left(e^x - \frac{3}{4}\right) + e^{x/2}.$$

Using Maple, we obtain the solution

$$\begin{aligned} u = {} & C_1 \cdot I_{\sqrt{-3}}\bigl(2e^{x/2}\bigr) + C_2 \cdot K_{\sqrt{-3}}\bigl(e^{x/2}\bigr) \\ & + 2I_{\sqrt{-3}}\bigl(2e^{x/2}\bigr) \int K_{\sqrt{-3}}\bigl(e^{x/2}e^{x/2}\bigr)\, dx \\ & - 2K_{\sqrt{-3}}\bigl(e^{x/2}\bigr) \int I_{\sqrt{-3}}\bigl(2e^{x/2}\bigr)e^{x/2}\, dx, \end{aligned}$$

for certain constants C_1 and C_2, where $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of the first and second kinds, respectively. There are known combinatorial interpretations for certain Bessel functions. See, for example, [2] and [11]. It is unlikely, however, that there is a combinatorial interpretation for the Bessel functions appearing in the exponential generating function $f(x) = ue^{-x/2}$, since $K_{\sqrt{-3}}(e^{x/2})$ is not well defined as a formal power series.

Now, we turn our focus to $\Pi_n(123, 12|3)$. Let $b_n = \#\Pi_n(123, 12|3)$ and

$$g(x) = \sum_{n \geqslant 0} b_n \frac{x^n}{n!}$$

be the corresponding exponential generating function.

The proof of the following proposition is very similar to the proof of Proposition 5.4 and is omitted.

Proposition 5.5. *For $n \geqslant 3$,*

$$b_n = b_{n-1} + (n-2)b_{n-3}$$

with the initial conditions $b_0 = 1$, $b_1 = 1$, and $b_2 = 2$. Also, $g(x)$ satisfies the differential equation

$$y''' = y'' + xy' + y.$$

Using Maple, we obtain the solution

$$\begin{aligned} y = {} & D_1 e^{x/2} Ai(1/4 + x) + D_2 e^{x/2} Bi(1/4 + x) \\ & + D_3 e^{x/2} \biggl(Ai(1/4 + x) \int Bi(1/4 + x) e^{-x/2}\, dx \\ & - \int Ai(1/4 + x) e^{-x/2}\, dx\, Bi(1/4 + x) \biggr), \end{aligned}$$

for constants D_1, D_2, and D_3, where Ai and Bi are Airy functions.

It is not terribly surprising that Airy functions appear, since these functions are closely related to Bessel functions and $\Pi_n(123, 12|3)$ is a subset of the set $\Pi_n(12|3)$. There do not seem to be any existing combinatorial interpretations of Airy functions. There is also unlikely to be a combinatorial interpretation of this generating function due to the fact that $Ai(1/4 + x)$ is not well defined as a formal power series.

For completeness we will consider the cases where odd and even set partitions avoid generalized set partitions. As before only the cases $O\Pi_n(R)$ and $E\Pi_n(R)$ where $R = \{12|3\}$ or $\{123, 12|3\}$ are new.

Let $oa_n = \#O\Pi_n(12|3)$ and $ea_n = \#E\Pi_n(12|3)$. Let $ob_n = \#O\Pi_n(123, 12|3)$ and $eb_n = \#E\Pi_n(123, 12|3)$. The following propositions easily follow from the recursions above. We let χ be the truth function, where χ of a statement is 1 if the statement is true and 0 if the statement is false.

Proposition 5.6. *For $n \geqslant 2$,*

$$oa_n = oa_{n-1} + \chi(n \text{ is even}) + \sum_{l=2,\ l \text{ even}}^{n-2} \binom{n-2}{l} oa_{n-2-l} + \sum_{l=1,\ l \text{ odd}}^{n-2} \binom{n-2}{l} ea_{n-2-l},$$

and

$$ea_n = ea_{n-1} + \chi(n \text{ is odd}) + \sum_{l=2,\ l \text{ even}}^{n-2} \binom{n-2}{l} ea_{n-2-l} + \sum_{l=1,\ l \text{ odd}}^{n-2} \binom{n-2}{l} oa_{n-2-l}.$$

Proposition 5.7. *For $n \geqslant 3$*

$$ob_n = ob_{n-1} + (n-2)eb_{n-3},$$

and

$$eb_n = eb_{n-1} + (n-2)ob_{n-3}.$$

6. Set partition statistics

Carlitz [5,6] and Gould [12] were the first to give versions of the q-Stirling numbers of the second kind. In [20], Milne introduces an inversion and dual inversion statistic on set partitions, whose distributions over partitions of $[n]$ with k blocks produce these two q-Stirling numbers of the second kind. Later, Sagan [23] introduced the major index and dual major index of a set partition, whose distributions produced the same two q-Stirling numbers of the second kind. At around the same time, Wachs and White [28] investigated four natural statistics, which they called *lb*, *ls*, *rb*, and *rs*, again producing the same two q-Stirling numbers of the second kind. Other statistics of interest are the number of crossings, nestings and alignments of a partition, see for example [4,9], or [14]. In this section we will show that all of these statistics can be described in the language of generalized partition patterns.

We will need some more notation. Consider the pattern $\pi = 1/23$. If we are looking for a copy of π in $\sigma = 137/26/45$, but we want the element representing 1 in the copy to be the minimum of its block then $1/45$ is a copy, but $3/45$ is not. We will represent this generalized pattern by $\overset{\frown}{}1/23$. And in general, we will denote such a generalized pattern by putting an arc over the first element of the block, in which we want the minimum to occur. In the same fashion, if we want the element representing 1 in a copy of $1/23$ to be the maximum in its block, then we denote the pattern by $1\overset{\frown}{}/23$. If we want the element representing 1 in a copy of $1/23$ to be both the minimum and the maximum of its block, then we denote the pattern by $\overset{\frown}{}1\overset{\frown}{}/23$.

In the sequel, if we say ρ is a pattern then ρ may or may not have adjacency restrictions. Let ρ be a pattern and $\sigma \in \Pi_n$. Then ρ will be treated as a function from Π_n to the nonnegative integers by letting $\rho(\sigma)$ be the number of copies of ρ in σ. If we have patterns $\rho_1, \rho_2, \ldots, \rho_\ell$ then

$$(\rho_1 + \rho_2 + \cdots + \rho_\ell)(\sigma) = \rho_1(\sigma) + \rho_2(\sigma) + \cdots + \rho_\ell(\sigma).$$

We begin with the inversion statistic. Let $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n$ and $b \in B_i$. We will say that (b, B_j) is an *inversion* if $b > \min B_j$ and $i < j$. Define the *inversion number* of σ, written $\mathrm{inv}(\sigma)$, to be the number of inversions in σ.

We may calculate $\mathrm{inv}(\sigma)$ by summing, over all elements $b \in [n]$, the number of inversions of the form (b, B_j). This observation leads to the next proposition.

Proposition 6.1. *For any $\sigma \in \Pi_n$,*

$$\mathrm{inv}(\sigma) = (\overset{\frown}{1}3/\overset{\frown}{2})(\sigma).$$

Proof. We will show that there is a one to one correspondence between inversions and copies of $\overset{\frown}{1}3/\overset{\frown}{2}$. Let $\sigma = B_1/B_2/\ldots/B_k$. Let $b \in B_i$ and (b, B_j) be an inversion. If $a = \min B_i$ and $c = \min B_j$ then (b, B_j) corresponds to the copy ab/c of $\overset{\frown}{1}3/\overset{\frown}{2}$. Conversely, if ab/c is a copy of $\overset{\frown}{1}3/\overset{\frown}{2}$, then $a = \min B_i$ and $c = \min B_j$ where $i < j$ since $a < c$. Also, $b > c = \min B_j$. Thus, the copy ab/c yields the inversion (b, B_j). $\square$

Let $\sigma = B_1/B_2/\ldots/B_k$ be a partition. We will say that (b, B_{i+1}) is a *descent* of σ if $b \in B_i$ and $b > \min B_{i+1}$. Let d_i be the number of descents of σ in block B_i. Then the major index of σ is

$$\mathrm{maj}(\sigma) = \sum_{i=1}^{k-1} i d_i = d_1 + 2d_2 + \cdots + (k-1)d_{k-1}.$$

Notice that each descent (b, B_{i+1}) contributes i to the major index.

Proposition 6.2. *For any $\sigma \in \Pi_n$,*

$$\mathrm{maj}(\sigma) = (\overset{\frown}{1}3|\overset{\frown}{2} + \overset{\frown}{1}/\overset{\frown}{2}4|\overset{\frown}{3})(\sigma).$$

Proof. Let $\sigma = B_1/B_2/\ldots/B_k$ and $b \in B_i$. Let $\rho_1 = \overset{\frown}{1}3|\overset{\frown}{2}$ and $\rho_2 = \overset{\frown}{1}/\overset{\frown}{2}4|\overset{\frown}{3}$. We will first show that (b, B_{i+1}) is a descent if and only if b represents the 3 in a copy of ρ_1, or, for $i \geqslant 2$, the 4 in a copy of ρ_2. Then we will show that each descent (b, B_{i+1}) contributes i to the right-hand side.

Let (b, B_{i+1}) be a descent. If $a = \min B_i$ and $c = \min B_{i+1}$ then ab/c is a copy of ρ_1 where b represents the 3. If additionally $i \geqslant 2$ and we let $d = \min B_j$ where $j < i$ then $d/ab/c$ is a copy of ρ_2, in which b represents the 4. For the converse, let ab/c be a copy of ρ_1, then $c = \min B_{i+1}$ for some i, and (b, B_{i+1}) is a descent. Similarly, a copy $d/ab/c$ of ρ_2 with $c = \min B_{i+1}$ for some $i \geqslant 2$ produces the descent (b, B_{i+1}).

If (b, B_{i+1}) is a descent, then there is exactly one copy of ρ_2 with b representing 3, since the 1 in ρ_1 must be represented by $a = \min B_i$, and the 2 must be represented by $c = \min B_{i+1}$. Now, if b represents the 4 in a copy of ρ_2 then the 2 must be represented by $a = \min B_i$, and the 3 must be represented by $c = \min B_{i+1}$. But now the 1 may be represented by the minimum of any block appearing before B_i. So the total contribution of the two patterns is $1 + (i-1) = i$. $\square$

Let $\sigma = B_1/B_2/\dots/B_k$ and $b \in B_i$. The dual of a descent is an *ascent*, which is a pair (b, B_{i-1}) with $b > \min B_{i-1}$. Note that this is true that each $b \in B_i$ forms an ascent because of the canonical ordering. So, we define the dual major index to be

$$\widehat{\mathrm{maj}}(\sigma) = \sum_{i=2}^{k} (i-1)(\#B_i).$$

The *dual inversion number* of σ, written $\widehat{\mathrm{inv}}(\sigma)$, is the number of pairs (b, B_j) such that $b \in B_i$, $b > \min B_j$, and $i > j$. We will call these pairs *dual inversions*. Clearly, $\widehat{\mathrm{inv}}(\sigma) = \widehat{\mathrm{maj}}(\sigma)$ for any $\sigma \in \Pi_n$, since every ascent causes $i-1$ dual inversions.

Proposition 6.3. *For any $\sigma \in \Pi_n$,*

$$\widehat{\mathrm{inv}}(\sigma) = \widehat{\mathrm{maj}}(\sigma) = ({}^{\frown}1/{}^{\frown}2 + {}^{\frown}1/{}^{\frown}23)(\sigma).$$

Proof. Let $\sigma = B_1/B_2/\dots/B_k$. The proof that $\widehat{\mathrm{inv}}(\sigma) = ({}^{\frown}1/{}^{\frown}2 + {}^{\frown}1/{}^{\frown}23)(\sigma)$ is similar to the proof of Proposition 6.1. The only difference here is that the minimum of a block can represent the b in a dual inversion (b, B_j). This is taken care of by the first pattern. $\square$

Wachs and White [28] define four natural statistics on partitions by encoding the partitions as restricted growth functions. Their statistics are *lb*, *ls*, *rb*, and *rs*, which stand for left bigger, left smaller, right bigger and right smaller. For consistency, we will define these statistics without introducing restricted growth functions, and hence the names of the statistics may seem a little unusual.

Let $\sigma = B_1/B_2/\dots/B_k$. If $b \in B_i$, then we will say that (b, B_j) is:

- a *left bigger pair* of σ if $i < j$, and $b > \min B_j$,
- a *left smaller pair* of σ if $i > j$ and $b > \min B_j$,
- a *right bigger pair* of σ if $i < j$ and $b < \max B_j$,
- a *right smaller pair* of σ if $i > j$ and $b < \max B_j$.

Let $lb(\sigma)$, $ls(\sigma)$, $rb(\sigma)$, and $rs(\sigma)$ be, respectively, the number of left bigger pairs, the number of left smaller pairs, the number of right bigger pairs, and the number of right smaller pairs in σ.

Notice that (b, B_j) is a left bigger pair if and only if it is an inversion of σ, and (b, B_j) is a left smaller pair if and only if (b, B_j) is a dual inversion of σ. Thus we have from Propositions 6.2 and 6.3 that

$$lb(\sigma) = ({}^{\frown}13/{}^{\frown}2)(\sigma),$$
$$ls(\sigma) = ({}^{\frown}1/{}^{\frown}2 + {}^{\frown}1/{}^{\frown}23)(\sigma).$$

We will now consider the other two statistics.

Proposition 6.4. *For any $\sigma \in \Pi_n$,*

$$rb(\sigma) = ({}^{\frown}1/{}^{\frown}2\,3^{\frown} + {}^{\frown}13/{}^{\frown}2\,4^{\frown} + {}^{\frown}1/{}^{\frown}2^{\frown} + {}^{\frown}12/{}^{\frown}3^{\frown})(\sigma).$$

Proof. Let $\sigma = B_1/B_2/\ldots/B_k$. The pattern $\frown 1/\frown 2\,3\frown$ counts right bigger pairs (b, B_j) where $b = \min B_i$ and $\#B_j \geqslant 2$. The pattern $\frown 13/\frown 2\,4\frown$ counts those pairs where $b \neq \min B_i$ and $\#B_j \geqslant 2$. The other two patterns correspond to the same two cases when $\#B_j = 1$. $\square$

The proof of the following proposition is similar to the proof of Proposition 6.4 and is omitted.

Proposition 6.5. *For any $\sigma \in \Pi_n$,*

$$rs(\sigma) = (\frown 13\frown/\frown 2 + \frown 14\frown/\frown 23)(\sigma).$$

There has long been interest in non-crossing partitions. Recall that the non-crossing partitions are those in the set $\Pi_n(13/24)$ for some n. Non-nesting partitions may be described as those in the set $\Pi_n(\widehat{14}/\widehat{23})$. Note that this definition of a non-nesting partition is not the only one. Klazar [15] defines non-nesting partitions as those in the set $\Pi_n(14/23)$.

Recently, however, there has been increasing interest in counting the number of crossings or nestings of a partition. In [9], Chen et al. show that the crossing number and nesting number are symmetrically distributed over Π_n by giving a bijection between partitions and vacillating tableaux. In [14], Kasraoui and Zeng give an involution of Π_n, which exchanges the crossing number and the nesting number while keeping another statistic, the number of alignments of two edges, fixed.

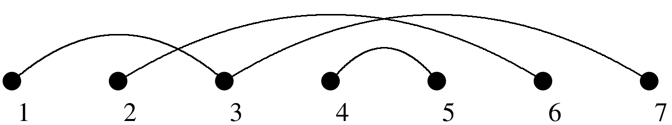
We will describe each of these statistics and show that they too may be translated into the language of patterns.

Let $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n$. We may rewrite σ as a set $P \subseteq [n] \times [n]$ in the following way. If $a, b \in B_i$ and there is no $c \in B_i$ such that $a < c < b$ then $(a, b) \in P$. If $B_i = \{d\}$ then $(d, d) \in P$. It is easy to see that P uniquely represents σ. We will call P the *standard representation* of σ.

Let A be a family $\{(i_1, j_1), (i_2, j_2)\} \subseteq P$. We will say that A is:

- a *crossing* if $i_1 < i_2 < j_1 < j_2$,
- a *nesting* if $i_1 < i_2 < j_2 < j_1$,
- an *alignment* if $i_1 < j_1 \leqslant i_2 < j_2$.

For example, the following diagram represents $\sigma = 137/26/45$, where an edge connects elements if they are adjacent in a block.



Notice that the pair $\{(1, 3), (2, 6)\}$ forms a crossing, the pair $\{(2, 6), (4, 5)\}$ forms a nesting and the pairs $\{(1, 3), (4, 5)\}$ and $\{(1, 3), (3, 7)\}$ each form an alignment of two edges.

Let $cr(\sigma)$ be the number of crossings in σ, $ne(\sigma)$ the number of nestings, and $al(\sigma)$ the number of alignments.

The following proposition is an easy consequence of the previous definitions.

Proposition 6.6. *For any $\sigma \in \Pi_n$,*

$$\begin{aligned} cr(\sigma) &= (\widehat{13}/\widehat{24})(\sigma), \\ ne(\sigma) &= (\widehat{14}/\widehat{23})(\sigma), \\ al(\sigma) &= (\widehat{12}/\widehat{34} + \widehat{12}\widehat{34} + \widehat{12}\widehat{3})(\sigma). \end{aligned}$$

Let $\sigma \in \Pi_n$ and P be the standard representation of σ. Consider the family $A = \{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\} \subseteq P$. Then A is a *k-crossing* if $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. We say A is a *k-nesting* if $i_1 < i_2 < \cdots < i_k < j_k < j_{k-1} < \cdots < j_1$. Let $cr_k(\sigma)$ be the number of k-crossings of σ and $ne_k(\sigma)$ be the number of k-nestings of σ. Notice that $cr = cr_2$ and $ne = ne_2$. The following proposition describes these two statistics as patterns.

Proposition 6.7. *For any $\sigma \in \Pi_n$,*

$$\begin{aligned} cr_k(\sigma) &= \big(\widehat{1(k+1)}/\widehat{2(k+2)}/\dots/\widehat{k(2k)}\big)(\sigma), \\ ne_k(\sigma) &= \big(\widehat{1(2k)}/\widehat{2(2k-1)}/\dots/\widehat{k(k+1)}\big)(\sigma). \end{aligned}$$

7. Future work

There has been an explosion in interest in permutation patterns recently, and this paper will hopefully help to generate interest in similar work with set partitions. Sections 2–4 focus mainly on the question of avoidance of a partition of a three element set, and there is more that can be done. Klazar [15–17], for example, has done work on avoidance of certain partitions of a four element set. The problem of avoiding more than one pattern in Π_4 is yet to be considered. Also, of interest is the problem of avoiding a family of patterns, which include patterns from both Π_4 and Π_3. Sagan [22] has provided enumerative results for four different infinite families of patterns.

This is just the tip of iceberg. We may also consider problems of containment. For example, what is the smallest n such that we can find a partition in Π_n, which contains all the patterns in Π_k? Also, for $\pi \in \Pi_k$, which $\sigma \in \Pi_n$ contain the maximal number of copies of π? The second question is similar to work initiated for packing of permutations by Price in [21].

In [1], Babson and Steingrímsson use generalized permutation patterns to classify Mahonian statistics. It is known that the distribution of the statistics of Milne, Sagan, and of Wachs and White on the set of partitions of $[n]$ with k blocks give nice q-analogues of the Stirling numbers of the second kind. Is there any way to use the generalized patterns for set partitions to classify the statistics which produce these nice q-analogues?

Another question which arises is: what distributions do we get if we examine these statistics on sets $\Pi_n(R)$ for some $R \subseteq \Pi_k$? The author is working with Sagan on a project [13] that answers this question for the restricted sets $\Pi_n(13/2)$ and $\Pi_n(13/2, 123)$. The distribution of the statistics ls and rb on $\Pi_n(13/2, 123)$ produce q-analogues of the Fibonacci numbers, which are closely related to q-Fibonacci numbers studied by Carlitz [7,8] and Cigler [10]. It is also interesting to note that these q-analogues arising from restricted set partitions are related to integer partitions. Such q-analogues can also be viewed as arising from statistics on compositions.

One partition being contained in another partition as a pattern produces a natural partial ordering on the family of all set partitions. This poset is likely to be quite beautiful and have nice structure. It is, of course, an analogue of the poset of permutations ordered by containment. For more information on this poset of permutations see [29]. The author is currently investigating properties of various posets of compositions related to a composition poset studied by Sagan and Vatter [24] and Björner and Sagan [3].

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