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Strong unique continuation for general elliptic equations in 2D

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ABSTRACT

We prove that solutions to elliptic equations in two variables in divergence form, possibly non-self-adjoint and with lower order terms, satisfy the strong unique continuation property.

1. Introduction

Given a bounded connected open set $\Omega \subset \mathbb{R}^2$, we consider weak solutions $u \in W^{1,2}(\Omega)$ to the elliptic equation

$$Lu = 0, \quad \text{in } \Omega, \tag{1.1}$$

where L is defined as follows

$$Lu = -\operatorname{div}(A\nabla u + uB) + C \cdot \nabla u + du. \tag{1.2}$$

Here, the coefficients A, B, C, d are assumed to satisfy the hypotheses listed below. For simplicity, and with no loss of generality, they are assumed to be defined on all of $\mathbb{R}^2$.

$A = \{a_{ij}\}$ is a positive definite, *possibly non-symmetric*, 2×2 matrix with $L^\infty(\mathbb{R}^2)$ entries. We express its uniform ellipticity by the following bounds (see, for instance, [7] for their equivalence to other customary formulations of ellipticity). For a given $K \geqslant 1$

$$\begin{aligned} &A(x)\xi \cdot \xi \geqslant K^{-1}|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \mathbb{R}^2, \\ &A^{-1}(x)\xi \cdot \xi \geqslant K^{-1}|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \mathbb{R}^2. \end{aligned} \tag{1.3}$$

For a given $q > 2$, $B = \{b_i\}$, $C = \{c_i\}$ are vector-valued functions belonging to $L^q(\mathbb{R}^2;\mathbb{R}^2)$ and d is an $L^{q/2}(\mathbb{R}^2)$ function. Altogether, for a given $\kappa > 0$, they are assumed to satisfy

$$\|B\|_{L^q(\mathbb{R}^2)} + \|C\|_{L^q(\mathbb{R}^2)} + \|d\|_{L^{q/2}(\mathbb{R}^2)} \leqslant \kappa. \tag{1.4}$$

The aim of this note is to prove the following theorem.

Theorem 1.1. *The operator L has the strong unique continuation property.*

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As is well known, this means that if a solution u to (1.1) has a zero of infinite order at a point $x_0 \in \Omega$ then $u = 0$ identically in Ω.

In order to summarize the past history of results of unique continuation for elliptic equations in two variables, we may start with a remark due to Martio [14]:

> In the plane uniqueness results can be proved using the theory of quasiconformal mappings and special representation theorems.

In fact Martio is referring to Chapter 6 in the book by Bers, John and Schechter [10], which in turn is mainly based on the papers by Bers and Nirenberg [9] and by Bojarski [11]. In these items, a theory of first order elliptic systems in the plane (the so-called Beltrami systems) is developed, and in particular it is applied to the unique continuation of second order elliptic equations in *non-divergence* form, see for instance [10, §6.4]. Thus it seems that up to 1988, a proof of the unique continuation for equations in divergence form and L^∞ coefficients in the principal part, was not available. On the other hand, assuming some regularity on the coefficients in the principal part, results of unique continuation were known, one can refer to Carleman [12] and Hartman and Wintner [13].

In 1992, motivated by inverse boundary problems [3,5], the author noticed that the approach based on Beltrami equations and on the representation formulas of Bers–Nirenberg and Bojarski could be used also to prove unique continuation for pure divergence elliptic equations, that is equations like (1.1), (1.2) with $B = C = 0$ and $d = 0$, [1]. This result appeared in print soon afterwards in a joint work with Magnanini [6]. Next, in [2], the unique continuation was shown for operators of the form

$$Lu = -\operatorname{div}(A\nabla u) + du, \tag{1.5}$$

with A symmetric and d bounded. The idea there was that, on a sufficiently small disk, we can find a positive solution w to $Lw = 0$. Hence by a classical trick, see for instance Miranda [16, Ch. I, §5], we note that any solution u to $Lu = 0$ can be locally factored as $u = wv$ where v solves

$$-\operatorname{div}\bigl(w^2 A\nabla v\bigr) = 0. \tag{1.6}$$

That is, loosely speaking, $w(-\operatorname{div}(A\nabla(w\cdot)) + dw\cdot) = -\operatorname{div}(w^2 A\nabla\cdot)$ and we have reduced ourselves again to a pure divergence form equation. Soon afterwards, Schulz [17] noticed that a slightly more involved, but similar trick, could be used to treat operators of the general form (1.2), with bounded lower order coefficients, but still, *symmetric* principal part!

Here we bypass this symmetry obstruction by using a reduction formula, Proposition 2.7, on the operator (1.2) which involves the use of two positive multipliers m, w instead of one. These multipliers are obtained as solutions in the small of two appropriately chosen elliptic equations. In fact, one of the two multipliers, w, is constructed, Definition 2.6, as a solution of an equation whose coefficients depend on the previously chosen function m, see Definition 2.5. This forces to take into consideration equations with unbounded lower order coefficients, and this is the main reason why, we have assumed since the beginning the integrability conditions (1.4).

In the following Section 2 we construct such multipliers and prove the reduction to a pure divergence equation, Proposition 2.7. In the final Section 3 we complete the proof of the main Theorem 1.1.

2. Preliminaries

In what follows, we denote by B_R a disk of radius R and arbitrary center.

Lemma 2.1. *There exists p, $2 < p < q$, only depending on K and q, and a number $R_0 > 0$, only depending on K, q and κ such that for every R, $0 < R \leqslant R_0$ and for every $F \in L^p(B_R; \mathbb{R}^2)$ there exists a unique weak solution $u \in W_0^{1,p}(B_R)$ to the equation*

$$Lu = -\operatorname{div} F \tag{2.1}$$

and it satisfies

$$\|\nabla u\|_{L^p(B_R)} \leqslant C\|F\|_{L^p(B_R)} \tag{2.2}$$

where $C > 0$ only depends on K and q.

Proof. This result is a minor, well-known, variation of a celebrated theorem of Meyers [15, Theorem 1]. In fact, in [15] it is proven that, considering the principal part L_0 of the operator L, that is

$$L_0 u = -\operatorname{div}(A\nabla u), \tag{2.3}$$

there exist p, $2 < p < q$, and $c_0 > 0$, only depending on K and q, such that for every $R > 0$ we have that $L_0 : W_0^{1,p}(B_R) \mapsto W^{-1,p}(B_R)$ is invertible and the following estimate holds

$$\|L_0 u\|_{W^{-1,p}(B_R)} \geqslant c_0\|\nabla u\|_{L^p(B_R)} \quad \text{for every } u \in W_0^{1,p}(B_R). \tag{2.4}$$

Let us denote by $M = L - L_0$ the remainder first order operator

$$Mu = -\operatorname{div}(uB) + C \cdot \nabla u + du. \tag{2.5}$$

Then Eq. (2.1) can be rewritten as

$$u + L_0^{-1} Mu = L_0^{-1}(-\operatorname{div} F), \tag{2.6}$$

and the thesis will follow provided we show that, for sufficiently small R the operator $L_0^{-1}M$ is a contraction on $W_0^{1,p}(B_R)$. By (1.4) and by a straightforward use of Sobolev inequalities, one obtains that, for every $R > 0$ and for every $u \in W_0^{1,p}(B_R)$, $v \in W_0^{1,p'}(B_R)$, we have

$$\begin{aligned}\bigl|\langle Mu, v\rangle\bigr| &= \left|\int_{B_R} uB \cdot \nabla v + vC \cdot \nabla u + duv\right| \\ &\leqslant C\kappa\bigl(R^{2(\frac{1}{2}-\frac{1}{q})} + R^{2(\frac{1}{p}-\frac{1}{q}+\frac{1}{2}-\frac{1}{q})}\bigr)\|\nabla v\|_{L^{p'}(B_R)}\|\nabla u\|_{L^p(B_{R_0})},\end{aligned} \tag{2.7}$$

here $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $W^{-1,p}(B_R)$ and $W_0^{1,p'}(B_R)$, and the constant $C > 0$ only depends on q and p, that is on q and K. Consequently, by (2.4), there exists $R_0 > 0$, only depending on q, K and κ such that $\|L_0^{-1}M\| \leqslant \frac{1}{2}$. And (2.2) follows with $C = \frac{2}{c_0}$. $\square$

Lemma 2.2. *Under the same assumptions of Lemma* 2.1, *and letting R_0, p be as introduced in the same lemma, given $F \in L^q(B_R;\mathbb{R}^2)$ and $f \in L^s(B_R)$ with $\frac{1}{s} \leqslant \frac{1}{2} + \frac{1}{q}$, for every $R \leqslant R_0$ there exists a unique weak solution $u \in W_0^{1,p}(B_R)$ to the equation*

$$Lu = -\operatorname{div} F + f \tag{2.8}$$

and it satisfies

$$\|\nabla u\|_{L^p(B_R)} \leqslant C\bigl(R^{2(\frac{1}{p}-\frac{1}{q})}\|F\|_{L^q(B_R)} + R^{2(\frac{1}{p}-\frac{1}{s})+1}\|f\|_{L^s(B_R)}\bigr), \tag{2.9}$$

and also

$$\|u\|_{L^\infty(B_R)} \leqslant CR^{1-\frac{2}{p}}\bigl(R^{2(\frac{1}{p}-\frac{1}{q})}\|F\|_{L^q(B_R)} + R^{2(\frac{1}{p}-\frac{1}{s})+1}\|f\|_{L^s(B_R)}\bigr), \tag{2.10}$$

where $C > 0$ only depends on K, q and s.

Proof. This is also a well-known consequence of Meyers' result [15, Theorem 1]. In fact we may easily construct $G \in L^q(B_R;\mathbb{R}^2)$ such that $-\operatorname{div} G = f$ and also

$$\|G\|_{L^q(B_R)} \leqslant CR^{2(\frac{1}{q}-\frac{1}{s})+1}\|f\|_{L^s(B_R)}, \tag{2.11}$$

where $C > 0$ only depends on q and s. Hence, applying Lemma 2.1 with F replaced with $F + G$, we obtain (2.9), (2.10) follows by a Sobolev inequality. $\square$

We record here another result, of the same flavor as the previous lemmas, which shall be used later on.

Lemma 2.3. *Let $u \in W^{1,2}(\Omega)$ be a weak solution to* (1.1) *in Ω, and let p, $2 < p < q$, be the exponent introduced in Lemma* 2.1. *For any two concentric balls $B_\rho \subset B_r \subset \Omega$, we have*

$$\|\nabla u\|_{L^p(B_\rho)} \leqslant Cr^{2(\frac{1}{p}-1)}\|u\|_{L^2(B_r)}, \tag{2.12}$$

where $C > 0$ only depends on K, κ, q and on the ratio $\frac{r}{\rho}$.

Proof. This is indeed Meyers' higher integrability theorem [15, Theorem 2]. It may also be obtained in a straightforward manner from Lemma 2.1 above with the aid of a smooth cutoff function. $\square$

The proposition below provides the main tool in the construction of the required multipliers.

Proposition 2.4. *Under the same assumptions as above, and letting R_0, p be as before, there exists R_1, $0 < R_1 \leqslant R_0$, only depending on K, κ and q, such that there exists $u \in W^{1,p}(B_{R_1})$ which is a weak solution to*

$$Lu = 0, \quad \text{in } B_{R_1}, \tag{2.13}$$

and it satisfies

$$\frac{1}{2} \leqslant u \leqslant 2, \tag{2.14}$$

$$\|\nabla u\|_{L^p(B_{R_1})} \leqslant 1. \tag{2.15}$$

Proof. Let $z \in W_0^{1,p}(B_R)$ be the solution to (2.1) obtained in Lemma 2.2 when F, f are replaced with $-B$, $-d$, respectively. We may choose $R = R_1$ small enough so that

$$|z| \leqslant \frac{1}{2}, \tag{2.16}$$

$$\|\nabla z\|_{L^p(B_{R_1})} \leqslant 1. \tag{2.17}$$

The thesis follows by picking $u = z + 1$. $\square$

In the next definitions we construct the multipliers m, w and two auxiliary elliptic operators.

Definition 2.5. We define the multiplier m as the solution obtained in the previous Proposition 2.4, when, in the operator L introduced in (1.2) the coefficient vector B is replaced with 0. That is, m is a weak solution to

$$-\operatorname{div}(A\nabla m) + C \cdot \nabla m + dm = 0 \quad \text{in } B_{R_1}, \tag{2.18}$$

and it satisfies the bounds (2.14), (2.15). Consequently, we define in B_{R_1} the following set of coefficients

$$\begin{aligned} \tilde{A} &= mA^T, \\ \tilde{B} &= mC - A\nabla m, \\ \tilde{C} &= mB, \end{aligned} \tag{2.19}$$

here the superscript $(\cdot)^T$ denotes the transpose. Accordingly, we set

$$\tilde{L}u = -\operatorname{div}(\tilde{A}\nabla u + u\tilde{B}) + \tilde{C} \cdot \nabla u. \tag{2.20}$$

Observe that the following bounds are easily obtained

$$\begin{aligned} &\tilde{A}(x)\xi \cdot \xi \geqslant \frac{1}{2K}|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in B_{R_1}, \\ &\tilde{A}^{-1}(x)\xi \cdot \xi \geqslant \frac{1}{2K}|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in B_{R_1}, \end{aligned} \tag{2.21}$$

$$\|\tilde{B}\|_{L^p(B_{R_1})} + \|\tilde{C}\|_{L^p(B_{R_1})} \leqslant 2\left(\pi R_1^2\right)^{\frac{1}{p} - \frac{1}{q}} \kappa + K. \tag{2.22}$$

Definition 2.6. Let R_0, R_1 and p be as before. By applying Proposition 2.4 to the operator $\tilde{L}$ we find that there exist R_2, $0 < R_2 \leqslant R_1$, and t, $2 < t < p$ only depending on K, κ and q, such that there exists $w \in W^{1,t}(B_{R_2})$ which is a weak solution to

$$\tilde{L}w = 0, \quad \text{in } B_{R_2}, \tag{2.23}$$

and it satisfies

$$\frac{1}{2} \leqslant w \leqslant 2, \tag{2.24}$$

$$\|\nabla w\|_{L^t(B_{R_2})} \leqslant 1. \tag{2.25}$$

Such a function w shall be our second multiplier. Let us fix any disk $B_R \subset \Omega$ with $R \leqslant R_2$. In B_R we define

$$\begin{aligned} \hat{A} &= mwA, \\ \hat{B} &= wA\nabla m + mwB - mA^T\nabla w - mwC, \end{aligned} \tag{2.26}$$

and consequently, we set

$$\hat{L}u = -\operatorname{div}(\hat{A}\nabla u + u\hat{B}). \tag{2.27}$$

Note that the following bounds are easily verified:

$$\hat{A}(x)\xi\cdot\xi \geqslant \frac{1}{4K}|\xi|^2, \quad \text{for every } \xi\in\mathbb{R}^2 \text{ and for a.e. } x\in B_R,$$

$$\hat{A}^{-1}(x)\xi\cdot\xi \geqslant \frac{1}{4K}|\xi|^2, \quad \text{for every } \xi\in\mathbb{R}^2 \text{ and for a.e. } x\in B_R, \tag{2.28}$$

$$\|\hat{B}\|_{L^t(B_R)} \leqslant 2K\big(1+\big(\pi R^2\big)^{\frac{1}{t}-\frac{1}{p}}\big) + 4\big(\pi R^2\big)^{\frac{1}{t}-\frac{1}{q}}\kappa. \tag{2.29}$$

Proposition 2.7. *For any $v\in W^{1,2}(B_R)$ we have*

$$\hat{L}v = wL(mv), \tag{2.30}$$

as elements of $W^{-1,2}(B_R)$.

Proof. Let $u = mv$ and note that $u\in W^{1,2}(B_R)$. Let $F\in L^2(B_R;\mathbb{R}^2)$ be such that $-\operatorname{div} F = Lu$ in the sense of $W^{-1,2}(B_R)$. Let $\psi\in C_0^\infty(B_R)$ be an arbitrary test function. Denote $\varphi = w\psi$ and observe that $\varphi\in W_0^{1,2}(B_R)$. Consequently

$$\int A\nabla u\cdot\nabla\varphi + uB\cdot\nabla\varphi + \varphi C\cdot\nabla u + du\varphi = \int F\cdot\nabla\varphi, \tag{2.31}$$

here, and in the rest of this proof, integrals are intended over B_R. Using the chain rule on the products $u = mv$, $\varphi = w\psi$, we obtain

$$\begin{aligned}
&\int (mwA\nabla v\cdot\nabla\psi + m\psi A\nabla v\cdot\nabla w\\
&\quad + vwA\nabla m\cdot\nabla\psi + v\psi A\nabla m\cdot\nabla w\\
&\quad + mvwB\cdot\nabla\psi + mv\psi B\cdot\nabla w\\
&\quad + mw\psi C\cdot\nabla v + vw\psi C\cdot\nabla m\\
&\quad + dmvw\psi) = \int F\cdot\nabla(w\psi).
\end{aligned} \tag{2.32}$$

By the identity $\psi\nabla v = \nabla(\psi v) - v\nabla\psi$, we may use the following substitutions

$$m\psi A\nabla v\cdot\nabla w = mA\nabla(\psi v)\cdot\nabla w - mvA\nabla\psi\cdot\nabla w, \tag{2.33}$$

$$mw\psi C\cdot\nabla v = mwC\cdot\nabla(\psi v) - mwvC\cdot\nabla\psi. \tag{2.34}$$

Therefore

$$\begin{aligned}
&\int (mwA\nabla v\cdot\nabla\psi + vwA\nabla m\cdot\nabla\psi + mvwB\cdot\nabla\psi - mvA\nabla\psi\cdot\nabla w - mwvC\cdot\nabla\psi)\\
&\quad + \int \big(mA\nabla(\psi v)\cdot\nabla w + mwC\cdot\nabla(\psi v)\big)\\
&\quad + \int (v\psi A\nabla m\cdot\nabla w + mv\psi B\cdot\nabla w + vw\psi C\cdot\nabla m + dmvw\psi) = \int F\cdot\nabla(w\psi).
\end{aligned} \tag{2.35}$$

Again by the chain rule, we may substitute

$$v\psi A\nabla m\cdot\nabla w = A\nabla m\cdot\nabla(v\psi w) - wA\nabla m\cdot\nabla(v\psi), \tag{2.36}$$

and obtain

$$\begin{aligned}
&\int (mwA\nabla v\cdot\nabla\psi + vwA\nabla m\cdot\nabla\psi + mvwB\cdot\nabla\psi - mvA\nabla\psi\cdot\nabla w - mwvC\cdot\nabla\psi)\\
&\quad + \int \big(mA\nabla(\psi v)\cdot\nabla w + mwC\cdot\nabla(\psi v)\big)\\
&\quad + \int \big(mv\psi B\cdot\nabla w - wA\nabla m\cdot\nabla(v\psi)\big)\\
&\quad + \int \big(A\nabla m\cdot\nabla(v\psi w) + vw\psi C\cdot\nabla m + dmvw\psi\big) = \int F\cdot\nabla(w\psi).
\end{aligned} \tag{2.37}$$

Now we note that $v\psi w \in W_0^{1,2}(B_R)$, therefore by (2.18) the fourth integral on the left-hand side vanishes. Then we can rearrange the terms as follows

$$\begin{aligned}&\int (mwA\nabla v\cdot\nabla\psi + vwA\nabla m\cdot\nabla\psi + mvwB\cdot\nabla\psi - mvA\nabla\psi\cdot\nabla w - mwvC\cdot\nabla\psi)\\&\quad+\int \big(mA^T\nabla w\cdot\nabla(\psi v) + mwC\cdot\nabla(\psi v) + mv\psi B\cdot\nabla w - wA\nabla m\cdot\nabla(v\psi)\big)\\&=\int F\cdot\nabla(w\psi).\end{aligned}\tag{2.38}$$

Again, we note that $v\psi \in W_0^{1,2}(B_R)$ and by (2.23), the second integral on the left-hand side is also vanishing. Finally, recalling the notation introduced in Definition 2.6, we arrive at

$$\int (\hat{A}\nabla v\cdot\nabla\psi + v\hat{B}\cdot\nabla\psi) = \int F\cdot\nabla(w\psi),\tag{2.39}$$

for every $\psi \in C_0^\infty(B_R)$, and hence by density, for every $\psi \in W_0^{1,2}(B_R)$. Note, in conclusion, that the functional $-w\operatorname{div}F$ given by $\langle -w\operatorname{div}F,\psi\rangle = \int F\cdot\nabla(w\psi)$ does indeed belong to $W^{-1,2}(B_R)$. $\square$

3. Proof of the main theorem

From now on, let u be a weak solution to (1.1), and let us fix any disk $B_R \subset \Omega$ with $R < R_2$. We denote

$$v = \frac{u}{m},\tag{3.1}$$

where m is the function introduced in Definition 2.5. Note that, by Proposition 2.4 and by Lemma 2.3, $v \in W^{1,t}(B_R)$ and by Proposition 2.7

$$\hat{L}v = 0,\tag{3.2}$$

in the weak sense.

The advantage is that from a pure divergence elliptic equation we can easily pass to a first order elliptic system of Beltrami type. The procedure is well known [1,6,17,7]. Denote

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\tag{3.3}$$

then, by (3.2), $J(\hat{A}\nabla v + v\hat{B})$ is weakly curl-free in B_R and therefore there exists a function $\tilde{v} \in W^{1,t}(B_R)$, unique up to an additive constant, such that

$$\nabla\tilde{v} = J(\hat{A}\nabla v + v\hat{B}),\tag{3.4}$$

and, since $t > 2$, $\tilde{v}$ is also Hölder continuous, thus we can normalize it by setting $\tilde{v}(x_0) = 0$, where x_0 denotes the center of B_R. Setting

$$f = v + i\tilde{v}\tag{3.5}$$

one has $f \in W^{1,t}(B_R;\mathbb{C})$ and, according to Bers and Nirenberg [9], one can rewrite (3.4), in terms of the complex coordinate $z = x_1 + ix_2$, as follows

$$f_{\bar{z}} = \mu f_z + \nu\overline{f_z} + \alpha f + \beta\overline{f}, \quad \text{in } B_R,\tag{3.6}$$

where the so-called complex dilatations μ, ν only depend (and can be explicitly expressed [7]) on $\hat{A}$, and the lower order coefficients α, β only depend on $\hat{A}$, $\hat{B}$. Moreover the following bounds are easily proven

$$|\mu| + |\nu| \leqslant k < 1, \quad \text{a.e. in } B_R,\tag{3.7}$$

where, in view of the ellipticity condition (2.28), the constant k only depends on K, see [7, Proposition 1.8] for a sharp bound. For the lower order coefficients, recalling (2.29), one can obtain

$$\|\alpha\|_{L^t(B_R)} + \|\beta\|_{L^t(B_R)} \leqslant C,\tag{3.8}$$

where $C > 0$ only depends on K, κ and q.

We can now invoke the well-known representation theorem for solutions of equations of the form (3.6).

Theorem 3.1. *There exist a k-quasiconformal mapping χ from $\mathbb{C}$ onto itself, a holomorphic function F on $\chi(B_R)$ and a complex-valued Hölder continuous function s on B_R such that*

$$f = e^s F(\chi). \tag{3.9}$$

Moreover we have that the function χ and its inverse χ^{-1} satisfy the following Hölder continuity properties

$$\left|\chi(z) - \chi(\zeta)\right| \leqslant C|z-\zeta|^\eta, \quad \text{for any } z, \zeta \in B_R, \tag{3.10}$$

$$\left|\chi^{-1}(z) - \chi^{-1}(\zeta)\right| \leqslant C|z-\zeta|^\eta, \quad \text{for any } z, \zeta \in \chi(B_R), \tag{3.11}$$

and

$$\left|s(z) - s(\zeta)\right| \leqslant C|z-\zeta|^\eta, \quad \text{for any } z, \zeta \in B_R, \tag{3.12}$$

where C and η, $0 < \eta < 1$, only depend on K, κ and q.

Proof. This is a celebrated theorem of Bers and Nirenberg [9, p. 116], see also Bojarski [11, Theorem 4.3] and the book [10, Section 6.3]. □

It is now evident that if f is nontrivial then it may vanish only up to finite order, in fact in (3.9) the exponential e^s never vanishes, and $F(\chi)$ may have only isolated zeroes of finite order in view of (3.11). Only one small step remains in order to show the strong unique continuation property for $u = m\Re(e^s F(\chi))$.

Lemma 3.2. *Let $u \in W^{1,2}(\Omega)$ be a weak solution to* (1.1) *in Ω, and let p, $2 < p < q$, be the exponent introduced in Lemma* 2.1. *Let v, $\tilde{v}$ be as introduced above. For any two balls B_ρ, B_r concentric to B_R, $\rho < r < R$, we have*

$$\|\tilde{v}\|_{L^\infty(B_\rho)} \leqslant C\|v\|_{L^\infty(B_r)}, \tag{3.13}$$

where $C > 0$ only depends on K, κ, q and on the ratio $\frac{r}{\rho}$.

Proof. These bounds are straightforward consequences of (3.4), by the use of Lemma 2.3 applied to the operator $\hat{L}$ and by Sobolev inequalities. Note that use is made of the normalization $\tilde{v}(x_0) = 0$, where x_0 is the center of B_R. □

Proof of Theorem 1.1. Assume that a solution u to $Lu = 0$ has a zero of infinite order at a point $x_0 \in \Omega$, let $R < R_2$ such that the disk B_R, centered at x_0, is contained in Ω. By (3.1) also v has a zero of infinite order at x_0, and by (3.13), the same occurs to $\tilde{v}$. Hence also f, given by (3.5), does the same. By Theorem 3.1 we obtain that f, and hence u are identically zero in B_R. Then a standard continuity argument yields that u is identically zero in Ω. □

Concluding Remark. In previous studies, [4] by Escauriaza and the author and [8] by Rondi, Rosset, Vessella and the author, it has been ascertained that, when lower order terms are absent, or when the operator L is in the self-adjoint form $Lu = -\operatorname{div}(A\nabla u) + du$, with A symmetric, the representation Theorem 3.1 enables also to obtain quantitative estimates of unique continuation, such as doubling inequalities [4, Proposition 2], three-spheres inequalities [4, Proposition 1] and [8, Theorem 1.10], estimates of propagation of smallness [8, Theorems 5.1, 5.3] and stability estimates for Cauchy problems [8, Theorems 1.9, 7.1]. In view of the reduction to pure divergence form obtained in Proposition 2.7, all such types of results can be extended to equations of the form (1.1), (1.2) treated here. We refrain from details for the sake of brevity.

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