Existence of Multiple Solutions to the Equations of Monge–Ampère Type

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In this paper we prove the existence of nontrivial convex solutions to the Monge–Ampère equation det(D^2 u) = f(x, u, Du) with Dirichlet or Neumann boundary condition, where f(x, 0, 0) = 0. We also deal with the existence and multiplicity of convex solutions to the equation det(D^2 u) = λf(x, u, Du). Our results show that the existence of convex solutions of Monge–Ampère equations is analogous to that of positive solutions of semilinear elliptic equations.

1. Introduction

In recent years the Monge–Ampère equation
\[ \det(D^2 u) = f(x, u, Du) \quad \text{in } \Omega \] (1.1)
has been extensively studied, where Du and D^2 u are the gradient and the Hessian of the function u, respectively. Throughout this paper Ω is supposed to be a uniformly bounded convex domain in \( \mathbb{R}^n \) with \( C^4 \) boundary, \( n \geq 2 \). The existence and regularity of convex solutions of (1.1) have been derived by means of various methods (see [5, 7, 11, 15] and the references therein). But most of the authors considered the existence of one solution only. On the other hand, it is well known that there exist nontrivial solutions to the semilinear elliptic equation
\[ -\Delta u = f(x, u) \quad \text{in } \Omega \] (1.2)
under some assumptions on \( f(x, u) \). In particular many results on the existence of positive solutions of (1.2) have been obtained (see [9]). The multiplicity of solutions to the Dirichlet problem of (1.1) has also been studied in some special cases. For example, when \( \Omega = B(0, R) \), Kutev [8] and Tso [14] proved the existence of a nontrivial radial solution to (1.1) by integral methods and variational methods, respectively.
In this paper, we use the methods of degree theory and a priori estimation to prove, for general domain \( \Omega \), the existence of two or more convex solutions to the equations of Monge–Ampère type. Our results show that the existence of convex solutions of (1.1) is similar to that of positive solutions of the semilinear elliptic equation (1.2).

This paper is arranged as follows. In Section 2, we establish the a priori bounds for the solutions of (1.1). In Section 3, we prove the existence of a nonzero convex solution to (1.1) under zero Dirichlet boundary condition. In Section 4 we treat the Neumann problem and obtain the existence of a nontrivial solution. In Section 5, we discuss the existence and nonexistence of solutions to the eigenvalue problem

\[
\det(D^2u) = \lambda f(x, u, Du) \quad \text{in } \Omega.
\]  

Finally in Section 6, we extend the existence results to the equation of prescribed Gauss curvature

\[
\det(D^2u) = f(x, u)(1 + |Du|^2)^{(n+2)/2}.
\]  

For the convenience of later applications, we mention here some facts concerning the generalized solutions of Monge–Ampère equations. Let \( u \) be a convex function; we define the normal mapping \( P_u \) of \( u \) by

\[
P_u(x) = \{ p \in R^n, p \cdot (y - x) + u(x) \leq u(y) \forall y \in \Omega \},
\]

and \( P_u(E) = \bigcup_{x \in E} P_u(x) \) for any subset \( E \subseteq \Omega \). Then \( \mu(E) = \text{mes}(P_u(E)) \) is a completely additive measure [4]. If \( \mu(E) \) is an absolutely continuous measure and is represented by a measurable function \( f(x, u, Du) \), namely,

\[
\int_E f(x, u, Du) \, dx = \mu(E) \quad \text{for any Borel set } E \in \Omega,
\]

then \( u \) is said to be a generalized solution of \( \det(D^2u) = f \). The existence, comparison principle, and convergence of generalized solutions were proved (see [4, 12, 17]). Therefore one can use the method of supersolution and subsolution to solve (1.1) (see [1]). We describe the procedure briefly here. Suppose the continuous, convex functions \( \bar{u} \) and \( u \) are the supersolution and subsolution of

\[
\det(D^2u) = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

respectively, i.e.,

\[
\mu_{\bar{u}}(E) \leq (\mu_u(E)) \geq , \text{ resp. } \int_E f(x, u) \, dx \quad \forall \text{ Borel set } E \subseteq \Omega
\]

\( \bar{u} \geq 0 \) (\( u \leq 0 \), resp.) on \( \partial \Omega \).
Let \( u_0 = \tilde{u} \) (or \( u_0 = y \)), and let \( u_j \) be the solution of
\[
\det(D^2u) - Cu = f(x, u_{j-1}) - Cu_{j-1} \quad \text{in} \ \Omega,
\]
\[ u = 0 \quad \text{on} \ \partial\Omega. \]

Then \( \{u_j\} \) is monotone, and it converges to a generalized solution \( u \) of (1.1), where \( C > 0 \) is such a constant that \( f(x, z) - Cz \) decreases in \( z \) for \( z \in (\inf_{\Omega} u, \sup_{\Omega} \tilde{u}) \). If \( u_0, f \in C^2 \), and for any \( \delta > 0 \),
\[
\inf\{ f(x, z); x \in \Omega_{\delta}, z \in [\inf_{\Omega} u, \sup_{\Omega} \tilde{u}] \} \geq C(\delta) > 0
\]
then \( u \) is a classical solution of (1.1), where \( \Omega_{\delta} = \{ x \in \Omega; \text{dist}(x, \partial\Omega) > \delta \} \).

In this paper, only convex solutions are considered. We use \( C \) to denote various constants under control, and use \( \|u\|_{0,\Omega}, \|u\|_{k,\Omega}, \) and \( \|u\|_{k+1,\Omega} \) to represent \( \|u\|_{L^\infty(\Omega)}, \|u\|_{C^k(\Omega)}, \) and \( \|u\|_{C^{k+1}(\Omega)} \), respectively; the subscript \( \Omega \) is omitted when it is apparent, where \( k \) is any integer, \( \alpha \in (0,1] \).

Having completed this paper, the author was informed that Tso [19] had discussed the problem
\[
\det(D^2u) = f(x, u) \quad \text{in} \ \Omega,
\]
\[ u = 0 \quad \text{on} \ \partial\Omega, \quad (1.5) \]
and obtained the existence of nontrivial solutions to (1.5) by means of variational methods.

2. A Priori Bounds

We consider the problem
\[
\det(D^2u) = f(x, u, Du) \quad \text{in} \ \Omega,
\]
\[ \beta(x) D_{\gamma}u + \alpha(x) u = \psi(x) \quad \text{on} \ \partial\Omega, \quad (2.1) \]
where \( \gamma = \gamma(x) \) is the unit outward normal to \( \partial\Omega \), \( f \geq 0, \beta > 0, \alpha \) and \( \psi \) are continuous functions, and
\[ \alpha(x) \geq \alpha_0 > 0 \quad \forall x \in \Omega. \quad (2.2) \]
We suppose that
\[
f(x, z, p) \geq F(z) \forall x \in B(x_0, \delta), z < 0, p \in R^n \text{ for some ball } B(x_0, \delta) \subset \Omega \text{ and some nonincreasing function } F(z) \geq 0; \text{ and } \lim_{z \to -\infty} F(z)/|z|^n = +\infty. \quad (2.3) \]
Noting that \( D_{\gamma}u = \lim_{t \to 0} \frac{[u(x + t\gamma) - u(x)]}{t} \) is well defined if \( u \in C^{0,1}(\overline{\Omega}) \) is convex, we have
THEOREM 2.1. Suppose (2.2), (2.3) hold; if \( u \in C^{0,1}(\Omega) \) is a generalized solution of (2.1), then

\[
\| u \|_0 \leq M,
\]

where \( M \) depends only on \( \delta, \text{diam}(\Omega), \alpha_0, F(t), \) and \( \| \psi \|_0 \).

Proof. By the convexity it follows that \( u \) attains its maximum at some point \( x' \in \partial \Omega \) where \( D_x u \geq 0 \). From the boundary condition in (2.1) we have

\[
\sup_{\Omega} u = u(x') = \left[ \psi(x') - \beta(x') D_x u(x') \right]/\alpha(x')
\leq \| \psi \|_0/\alpha_0 =: K.
\]

Let \( M = - \inf_{\Omega} u \). We may suppose \( M \geq 0 \). By the convexity of \( u \) we have

\[
u(x) \leq \sup_{\Omega} u + \left[ \inf_{\Omega} u - \sup_{\Omega} u \right] \text{dist}(x, \partial \Omega)/\text{diam}(\Omega)
\leq K - \frac{\delta}{2d} M,
\]

\[
|D_x u(x)| \leq \left[ \sup_{\Omega} u - u(x) \right]/\text{dist}(x, \partial \Omega)
\leq \frac{2}{\delta} (K + M),
\]

where \( d = \text{diam}(\Omega) \). Set \( s = (2/\delta)(K + M) \); from (2.7) we have \( P_u(B(x_0, \frac{1}{2} s)) \subset B(0, s) = \{ x \in \mathbb{R}^n ; |x| < s \} \). Thus

\[
\text{mes}(B(0, s)) \geq \text{mes} \left( P_u \left( B \left( x_0, \frac{1}{2} \delta \right) \right) \right) = \mu_u \left( B \left( x_0, \frac{1}{2} \delta \right) \right)
\]

\[
= \int_{B(x_0, (1/2) \delta)} f(x, u, D_x u) \, dx \geq \int_{B(x_0, (1/2) \delta)} F(u) \, dx.
\]

From (2.6) and the monotonicity of \( F(t) \) we obtain

\[
\text{mes}(B(0, s)) \geq F \left( K - \frac{\delta}{2d} M \right) \text{mes} \left( B \left( x_0, \frac{1}{2} \delta \right) \right),
\]

namely,

\[
(K + M)^{-n} F \left( K - \frac{1}{2d} M \right) \leq C(n, \delta).
\]

Since \( K \) is a fixed constant, from (2.3) we obtain (2.4). \( \Box \)
Remark 2.1. If the Neumann boundary condition in (2.1) is replaced by an oblique derivative condition, Theorem 2.1 still holds.

In [10] Lions considered the eigenvalue problem

\[
\begin{align*}
\det(D^2u) &= |\lambda u|^n & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega
\end{align*}
\]  

(2.8)

and proved the existence of a unique \( \lambda_1 = \lambda_1(\Omega) \geq 0 \) such that (2.8) possesses a nonzero convex solution \( \phi \in C^\infty(\Omega) \cap C^{1,1}(\bar{\Omega}) \) for \( \lambda = \lambda_1 \). For any \( t > 0 \), \( t\phi \) is still a solution of (2.8). \( \lambda_1 \) is called the eigenvalue of the Monge–Ampère operator, and \( \phi \) its eigenfunction.

We now turn to the Dirichlet problem

\[
\begin{align*}
\det(D^2u) &= f(x, u, Du) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega.
\end{align*}
\]  

(2.9)

Suppose that there is a nonincreasing function \( F(t) \geq 0 \) such that

\[
f(x, z, p) \geq F(z) \quad \forall x \in \Omega, \quad z \leq 0, \quad p \in \mathbb{R}^n, \quad \text{and}
\]

\[
\lim_{z \to -\infty} F(z)/|z|^n > \lambda_1^n.
\]

(2.10)

We have

\[
\text{**Theorem 2.2.** Suppose (2.10) holds. If } u \in C^{0,1}(\bar{\Omega}) \text{ is a generalized solution of (2.9), then } \|u\|_0 \leq M, \text{ where } M \text{ depends only on } F, n, \Omega.
\]

**Proof.** For any fixed \( t > 0 \), denote \( \Omega' = \{tx; x \in \Omega\} \). By the uniqueness of eigenvalues of (2.8) we have \( \lambda_1(\Omega') = t^{-2}\lambda_1(\Omega) \). Without loss of generality we may suppose the origin 0 \( \in \Omega \), which implies \( \Omega' \subset \Omega \) for any \( t < 1 \).

If the conclusion of Theorem 2.2 is not true, then there exists a sequence of functions \( f_j(x, z, p) \) with \( f_j(x, z, p) \geq F(z) \) \( \forall x \in \Omega, \quad z \leq 0, \quad p \in \mathbb{R}^n \), so that the sequence of solutions \( u_j \) of

\[
\begin{align*}
\det(D^2u_j) &= f_j(x, u_j, Du_j) & \text{in } \Omega, \\
u_j &= 0 & \text{on } \partial\Omega
\end{align*}
\]

satisfies \( M_j = \|u_j\|_0 = -\inf_{\Omega_j} u_j \to \infty \) as \( j \to \infty \). From (2.10) there exist \( \tilde{\lambda} > \lambda_1 \) and \( t \in (0, 1) \) such that \( \lim_{z \to -\infty} F(z)/|z|^n \geq \tilde{\lambda}^n \) and \( \lambda_1(\Omega') = t^{-2}\lambda_1(\Omega) < \tilde{\lambda} \). Since \( u_j(x) \to -\infty \) uniformly in \( \Omega' \), we have

\[
\det(D^2u_j) \geq \tilde{\lambda}^n |u_j|^n \quad \text{in } \Omega'
\]

provided \( j \) is sufficiently large. Fix such a \( j \). Let \( \phi \), be the eigenfunction of the Monge–Ampère operator on \( \Omega' \). Replacing \( \phi \), by \( s\phi \), for some small \( s \)
we may suppose $\phi_j(x) > u_j(x)$ in $\Omega'$. Thus $\phi_j(x)$ and $u_j(x)$ are the super-solution and subsolution of the Dirichlet problem

$$\det(D^2 u) = |\lambda u|^{n^*} \quad \text{in } \Omega', \quad u = 0 \quad \text{on } \partial\Omega', \quad (2.11)$$

respectively. Therefore there is a solution $u$ of (2.11) which satisfies $\phi_j(x) \geq u(x) \geq u_j(x)$. This means both $\lambda$ and $\lambda^* \lambda_1$ are the eigenvalues of the Monge-Ampère operator on $\Omega'$, which contradicts the uniqueness of eigenvalues.

In comparison with the conditions for the a priori estimation of positive solutions of semilinear elliptic equation (1.2), the hypotheses in Theorems 2.1 and 2.2 are quite simple. From the above estimates we conclude the following nonexistence results.

**Corollary 2.3.** Suppose (2.2), (2.3) hold. Then there exists $\lambda_0 > 0$ such that the problem

$$\det(D^2 u) = f(x, u, Du) + \lambda \quad \text{in } \Omega, \quad \beta(x) D_\gamma u + \alpha(x) u = \psi(x) \quad \text{on } \partial\Omega. \quad (2.12)$$

admits no solution for $\lambda \geq \lambda_0$.

**Proof:** If $u_\lambda$ is a solution of (2.12), by Theorem 2.1 we have $\|u_\lambda\| \leq M$ for some $M > 0$ independent of $\lambda \geq 0$. Let $w(x)$ be the solution of

$$\det(D^2 w) = 1 \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega. \quad (2.13)$$

Then $\det(D^2(\lambda^{1/n}w + M)) = \lambda \leq \det(D^2 u_\lambda)$. By the comparison principle it follows that $\lambda^{1/n}w + M \geq u_\lambda \geq -M$, i.e., $\lambda \leq [2M/\inf_{\Omega} |w|]^n$.

Similarly we have

**Corollary 2.4.** Suppose $f(x, z, p)$ satisfies (2.10). Then there exists $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, there is no solution of

$$\det(D^2 u) = f(x, u, Du) + \lambda \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Moreover, we have

**Corollary 2.5.** Suppose $f \geq f_0 > 0$, (2.2) and (2.3) hold. Then there exists $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, there is no solution of

$$\det(D^2 u) = \lambda f(x, u, Du) \quad \text{in } \Omega, \quad \beta(x) D_\gamma u + \alpha(x) u = \psi(x) \quad \text{on } \partial\Omega. \quad (2.14)$$
Proof: If $\lambda \geq 1$ and $u_\lambda$ is a solution of (2.14), then $\|u_\lambda\|_0 \leq M$ for some $M > 0$ independent of $\lambda \geq 1$. Let $w(x)$ be the solution of (2.13). Then

$$
\text{det}(D^2((\lambda f_0)^{1/n} w)) = \lambda f_0 \leq \text{det}(D^2(u_\lambda - M)) \quad \text{in } \Omega,
$$

$$
w = 0 \geq u_\lambda - M \quad \text{on } \partial \Omega.
$$

By the comparison principle it follows that $(\lambda f_0)^{1/n} w \geq u_\lambda - M \geq -2M$, which implies $\lambda \leq [2M/f_0^{1/n} \inf_{\Omega} |w|]^{n}$. 

Remark 2.2. Having established the $L^\infty$ bounds for the solutions of (2.1), we can obtain the $C^{3+\nu}$ estimation by virtue of the regularity results of Monge–Ampère equations. We state it as follows.

Suppose $u \in C^3(\Omega) \cap C^{0,1}(\bar{\Omega})$ is a convex solution to the equation

$$
\text{det}(D^2u) = f(x, u, Du) \quad \text{in } \Omega
$$

with the Dirichlet boundary condition $u = \psi$ on $\partial \Omega$. If $f \in C^2$, $\psi \in C^4$, $f(x, u, Du) \geq f_0 > 0$, and

$$
f(x, z, p) \leq \mu(z) |p|^{n+1} \quad \forall x \in \Omega, \quad |p| \geq 1
$$

for some continuous function $\mu(z)$, then

$$
\|u\|_{3+\nu} \leq M \quad \text{for some } \nu \in (0, 1).
$$

where $M$ depends only on $\|u\|_0$, $\|\psi\|_4$, $f_0$, and $f$ up to its second derivatives (see [7]). If $u \in C^3(\bar{\Omega})$ is a convex solution of (2.15) with the Neumann boundary condition $D\gamma u + \alpha(x) u = \psi$, and $f \in C^2$, $\alpha$, $\psi \in C^3$, $\alpha \geq \alpha_0 > 0$, $f(x, u, Du) \geq f_0 > 0$, then $u$ satisfies (2.17), where $M$ depends only on $\|u\|_0$, $\|\psi\|_3$, $\|\alpha\|_3$, $\alpha_0$, $f_0$, and $f$ up to its second derivatives (see [11]).

3. Existence of Nontrivial Solutions to the Dirichlet Problem

We consider the problem

$$
\text{det}(D^2u) = f(x, u, Du) \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{on } \partial \Omega.
$$

Suppose $f(x, z, p) \in C^2(\bar{\Omega} \times R \times R^n)$ satisfies

$$
f(x, z, p) > 0 \quad \forall z < 0, \quad x \in \Omega, \quad p \in R^n; \tag{3.2}
$$

$$
\lim_{|p| + |z| \to 0} f(x, z, p)/(|z|^n + |p|^n) = 0 \quad \text{uniformly for } x \in \Omega. \tag{3.3}
$$
that is, \( \forall \epsilon > 0, \exists C_\epsilon > 0 \) such that if \(|z| + |p| < 1\),

\[
f(x, z, p) \leq \epsilon(|z|^n + |p|^n) + C_\epsilon(|z|^n + |p|^n)^{3/2}.
\]

(3.3)'

In addition, we suppose there exists a continuous function \( \mu(t) \) such that

\[
f(x, z, p) \leq \mu(z) d_x^\beta |p|^\gamma \quad \text{for all } |p| > 1, x \in \Omega,
\]

(3.4)

where \( d_x = \text{dist}(x, \partial \Omega), \beta \geq \max(0, \gamma - n - 1) \). We first prove

**Lemma 3.1.** Let \( u \in C^3(\Omega) \cap C^{0,1}(\overline{\Omega}) \) be a solution of (3.1). If \( f \) satisfies (3.4), then

\[
\|u\|_{C^{0,1}(\Omega)} \leq M_0,
\]

(3.5)

where \( M_0 \) depends only on \( \Omega, \|u\|_0, \mu(t), \beta, \gamma \). Furthermore, if \( f \) satisfies (3.2), then for any compact subset \( \Omega'' \) of \( \Omega \), we have

\[
\|u\|_{C^\lambda(\Omega'')} \leq M_1,
\]

(3.6)

where \( M_1 \) depends only on \( \Omega'', \Omega, \|u\|_0, \|u\|_{C^{0,1}}, \) and \( f \).

**Proof.** Inequality (3.5) is a direct consequence of Theorem 17.21 in [7]. To show (3.6) we denote \( k = -\sup_{\Omega''} u, \delta = \text{dist}(\Omega'', \partial \Omega) \). Then from the convexity of \( u \) it follows that \( k \geq \delta \|u\|_0 / \text{diam}(\Omega) \). Let \( \Omega' = \{x \in \Omega; u < \frac{1}{2}k\} \); from (3.5) we have \( \text{dist}(\Omega', \partial \Omega) \geq k/2M_0, \text{dist}(\Omega'', \partial \Omega') \geq k/2M_0 \). Hence \( f(x, u, Du) \geq a \) in \( \Omega' \) for some \( a > 0 \). Noting that \( u = \frac{1}{2}k \) on \( \partial \Omega' \), we have (see [7])

\[
\|u\|_{C^\lambda(\Omega')} \leq M_2, \quad M_2 \text{ depends on } \Omega'.
\]

And therefore by Theorem 17.14 in [7] we obtain (3.6). \( \square \)

**Remark 3.1.** From the proof we see that the constant \( M_1 \) in (3.6) depends only on the value of \( f \) in \( \Omega' \), i.e., \( M_1 \) is irrelevant to the values of \( f(x, z, p) \) where \( z \geq -\frac{1}{2}k = \frac{1}{2} \sup_{\Omega''} u \).

**Theorem 3.1.** Suppose \( f \in C^2(\overline{\Omega} \times R \times R^n) \) satisfies (2.3), (3.2)--(3.4). Then problem (3.1) possesses a nontrivial convex solution \( u \in C^{3+\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega}), \alpha \in (0, 1) \).

**Proof.** First we consider the case \( \gamma \leq n + 1 \).

For each \( \delta \in (0, 1) \), choose a nonnegative function \( f_\delta(x, z, p) \in C^2(\overline{\Omega} \times R \times R^n) \) satisfying (3.3)' with the constant \( C_\epsilon \) independent of
$\delta \in (0, 1)$, and $f_\delta(x, z, p) = 0$ for $|z| < \delta$, $f_\delta(x, z, p) = f(x, z, p)$ for $|z| > 2\delta$. Let $T_\delta$ be the mapping defined by $u = T_\delta v$ which is the solution of

$$\begin{align*}
det(D^2u) &= f_\delta(x, v, Du) + t \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*} \quad (3.7)$$

Then for any $\varepsilon > 0$, $T_\delta$ is a completely continuous mapping from $(t, v) \in [\varepsilon, \infty) \times C^3_0(\overline{\Omega})$ to $C^3_0(\overline{\Omega})$, where $C^3_0(\overline{\Omega})$ is the set of functions in $C^3(\overline{\Omega})$ which vanish on $\partial\Omega$. Since $f_\delta(x, z, p) = 0$ for $|z| < 6$, we have

$$T_\delta(v) = T_\delta(0) = t^{1/n}T_1(0) = t^{1/n}T_1(v) \quad \text{for all } v \in B_\delta,$$

where $T_1 = T_{1,1} = 1$, $B_\delta = \{u \in C^3_0(\overline{\Omega}); \|u\|_3 < \delta\}$. Thus there exists $\theta(\delta) > 0$ such that $T_\delta(B_\delta) \subset B_\delta$ for $t \in (0, \theta(\delta))$, and

$$\deg(I - T_\delta, B_\delta, 0) = \deg(I, B_\delta, 0) = 1, \quad t \in (0, \theta(\delta)). \quad (3.8)$$

For any $\varepsilon > 0$, and $t \in [\varepsilon, \lambda_0]$, by Theorem 2.1 and Remark 2.2, the solution $u = u_t$ of the problem

$$\begin{align*}
det(D^2u) &= f_\delta(x, u, Du) + t \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*} \quad (3.9)$$

satisfies $\|u\|_3 + a < M_\varepsilon$ for some $M_\varepsilon > 0$ independent of $t$, where $\lambda_0 > 0$ is such a constant that (3.9) has no solution for $t > \lambda_0$ (see Corollary 2.3). Hence $\deg(I - T_\delta, B_{M_\varepsilon}, 0) = \gamma$ is well defined and independent of $t \in [\varepsilon, \lambda_0]$. Let $t = \lambda_0$. We get $\gamma = 0$. From (3.8) we therefore deduce that

$$\deg(I - T_\delta, B_{M_\varepsilon}, \overline{B_\delta}, 0) = -1, \quad t \in [\varepsilon, \lambda(\delta)]. \quad (3.10)$$

By the arbitrariness of $\varepsilon > 0$, problem (3.9) possesses a solution $u_{\delta, \varepsilon} \notin \overline{B_\delta}$ for $t < \theta(\delta)$.

Let $t(\delta) = 2^{-n}\delta^ne; \varepsilon > 0$ small will be determined below. We claim that $u_{\delta, \varepsilon} = u_{\delta, t(\delta)} \to u_0 \neq 0$ as $\delta \to 0$ (extract a subsequence if necessary), and $u_0 \in C^{3^*+n}(\Omega) \cap C^{0,1}(\overline{\Omega})$ is the solution of (3.1).

Indeed, denote $s = -\inf_\Omega u_{\delta, \varepsilon}$. Then $s \geq \delta$ (otherwise $\theta u_{\delta, \varepsilon}$ is a fixed point of $T_{\theta t(\delta)}$, which contradicts $T_{\theta t(\delta)}(\overline{B_\delta}) \subset B_\delta$, where $\theta = \delta/\|u_{\delta, \varepsilon}\|_3 < (0, 1)$). We verify further that $s \geq C_0$ for some $C_0 > 0$ independent of $t \in (0, t(\delta))$. Indeed, if $s \leq 1$, by the convexity of $u_{\delta, \varepsilon}$ we have (suppose $\text{diam}(\Omega) = 1$)

$$|Du_{\delta, \varepsilon}(x)| \geq s/\text{diam}(\Omega) \geq s \quad \text{on } \partial\Omega, \quad (3.11)$$

and

$$|Du_{\delta, \varepsilon}(x)| \geq (u_{\delta, \varepsilon}(x) - \inf u)/\text{diam}(\Omega) \geq s/2 \quad \text{if } u_{\delta, \varepsilon} > -s/2.$$
Thus

\[ |u_{\delta,t}|^n + |Du_{\delta,t}|^n \geq 2^{-n}s^n. \]

From (3.3)', we obtain

\[
f_{\delta}(x, u, Du) + t \leq t + \varepsilon_1 (|u|^n + |Du|^n) + C_{\varepsilon_1} (|u|^n + |Du|^n)^2 \leq (\varepsilon + \varepsilon_1)(|u|^n + |Du|^n) + C_{\varepsilon_1} (|u|^n + |Du|^n)^2 \]

for \( t \leq t(\delta) \) and \(|u| + |Du| < 1\), where \( u = u_{\delta,t} \). Denote \( \Omega' = \{ x \in \Omega; |Du| < s \} \). From (3.11) we have \( \partial \Omega \cap \Omega' = \emptyset \), and \(|u|^n + |Du|^n < 2s^n \leq 2\) for \( x \in \Omega' \). Hence

\[
\int_{\Omega'} \frac{\det(D^2u)}{|u|^n + |Du|^n} \, dx = \int_{\Omega'} \frac{f_{\delta}(x, u, Du) + t}{|u|^n + |Du|^n} \, dx \\
\leq \int_{\Omega'} [\varepsilon + \varepsilon_1 + C_{\varepsilon_1} (|u|^n + |Du|^n)] \, dx \\
\leq [\varepsilon + \varepsilon_1 + 2s^nC_{\varepsilon_1}] \text{mes}(\Omega').
\]

On the other hand,

\[
\int_{\Omega'} \frac{\det(D^2u)}{|u|^n + |Du|^n} \, dx = \int_{B(0,s)} \frac{dp}{|u|^n + |p|^n} \\
\geq \int_{B(0,s)} \frac{dp}{s^n + |p|^n} \geq \int_{B(0,1)} \frac{dp}{1 + |p|^n} = C(n),
\]

where \( u = u_{\delta,t} \). Combining the above inequalities we obtain

\[ \varepsilon + \varepsilon_1 + 2s^nC_{\varepsilon_1} \geq C(n, \Omega). \]

Let \( \varepsilon_1 = \frac{1}{4}C(n, \Omega) \), \( \varepsilon = \min \{ \varepsilon_1, 2^n\delta^{-n}\theta(\delta) \} \). Then \( t(\delta) = 2^{-n}\delta^n\varepsilon < \theta(\delta) \), and

\[ s = \|u_{\delta,t(\delta)}\|_0 \geq C_0 > 0. \]  \hspace{1cm} (3.12)

Denote \( u_0 = u_{\delta,t(\delta)} \). From Lemma 3.1, there is a subsequence of \((u_0)\), still denoted by \((u_0)\), which converges to \( u_0 \neq 0 \) in \( C^0(\overline{\Omega}) \), and \( u_0 \in C^{0,1}(\overline{\Omega}) \) is a generalized solution of (3.5).

We claim \( u_0 \) is a classical solution. Indeed, for any fixed \( \Omega' \subset \Omega \), from (3.12) and by the convexity of \( u_\delta \), we have \( u_\delta < -2\theta \) in \( \Omega' \) for some \( \theta > 0 \) sufficiently small. This means \( f_{\delta}(x, u_\delta, Du_\delta) = f(x, u_\delta, Du_\delta) \) in \( \Omega' \) for \( \delta \in (0, \theta) \). By (3.6) and Remark 3.1 it follows that \( \|u_\delta\|_{C^0(\Omega')} < C \) for some \( C \) independent of \( \delta \). Let \( \delta \to 0 \). We obtain \( u_0 \in C^{2+\alpha}(\Omega') \). By the arbitrariness of \( \Omega' \), and the elliptic Schauder theory it follows that \( u_0 \in C^{3+\alpha}(\Omega), \alpha \in (0, 1) \).
Finally we treat the case $\gamma > n + 1$. We choose $f_m \in C^2(\Omega \times R \times R^n)$ satisfying (3.2), (3.3)' with the constant $C_\epsilon$ independent of $m$, and $f_m = f$ when $|p| < m$, $f_m(x, z, p) \leq \mu_m(z) |p|^n$ when $|p| > m$ (if we choose suitable $h_m(p) \in C^2(R^n, R^n)$, then $f_m = f(x, z, h_m(p))$ is the desired function). Let $u_m \in C^{3+\epsilon}(\Omega) \cap C^{0,1}(\Omega)$ be the solution of

$$\det(D^2u) = f_m(x, u, Du) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$ 

From Theorem 2.1 and Lemma 3.1 we have $\sup |Du_m| \leq M$ for some $M$ independent of $m$. Therefore $u_m$ is a solution of (3.1) for any $m > M$.

**Remark 3.2.** Theorem 3.1 still holds if condition (2.3) is replaced by (2.10), since from Theorem 2.2 we still have the boundedness of solutions of (3.1).

We now consider the simpler case

$$\det(D^2u) = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \quad \text{(3.13)}$$

Suppose $f(x, u)$ satisfies

$$\lim_{u \to 0} f(x, u)/|u|^n < \lambda_1^n, \quad \text{(3.14)}$$

$$\lim_{u \to \infty} f(x, u)/|u|^n > \lambda_1^n \quad \text{(3.15)}$$

uniformly for $x \in \Omega$, where $\lambda_1$ is the eigenvalue of the Monge–Ampère operator.

**Theorem 3.2.** If $f(x, u) \in C^2(\Omega \times R)$ satisfies (3.14), (3.15), and

$$f(x, u) > 0 \quad \forall x \in \Omega, u < 0, \quad \text{(3.16)}$$

then there exists a nontrivial solution $u \in C^3(\Omega) \cap C^{0,1}(\Omega)$ of (2.1).

**Proof.** For any given $\delta \in (0, 1)$, let $f_\delta(x, u)$ be a sequence of $C^2$ functions which satisfies (3.3), $f_\delta \leq f$, $f_\delta(x, u) = f(x, u)$ for $u < -\delta$, and $\lim_{u \to 0} f_\delta(x, u)/|u|^n = 0$. By Theorem 3.1, there exists a solution $u_\delta \in C^{3+\epsilon}(\Omega) \cap C^{0,1}(\Omega)$ of

$$\det(D^2u) = f_\delta(x, u) + \delta |u|^{2n} \quad \text{in } \Omega, u = 0 \quad \text{on } \partial \Omega.$$ 

From (3.14) there exist $\bar{\lambda} \in (0, \lambda_1)$ and $C_0 > 0$ such that for any $\delta \in (0, 1)$,

$$f_\delta(x, u) + \delta |u|^{2n} \leq \bar{\lambda} |u|^n \quad \forall (x, u) \in \Omega \times [- C_0, 0]. \quad \text{(3.17)}$$
We claim that

$$\| u_\delta \| = -\inf_{\Omega} u_\delta \geq C_0.$$  (3.18)

Indeed, if (3.18) does not hold, then from (3.17), $u_\delta$ is a supersolution of

$$\det(D^2 u) = |\dd u|^n \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega.$$  (3.19)

Let $\phi$ be the eigenfunction of the Monge–Ampère operator. Replacing $\phi$ by $t\phi$ for some large $t$ we may suppose $\phi < u_\delta$ in $\Omega$. Hence $u_\delta$ and $\phi$ are the supersolution and subsolution of (3.19), respectively. Therefore, (3.19) has a solution $u$ satisfying $u_\delta \geq u \geq \phi$ in $\Omega$. This means both $\lambda$ and $\lambda_1$ are the eigenvalues of the Monge–Ampère operator; this is a contradiction.

Consequently from Theorem 2.2 and Lemma 3.1, similar to the proof of Theorem 3.1 we see that there exists a subsequence of $u_\delta$ which converges to $u_{\delta_0}$ as $\delta \to 0$ and $u_{\delta_0} \in C^{3+\gamma}(Q) \cap C^{0,1}(\overline{Q})$ is a solution of (3.13). [1]

Remark 3.3. Theorem 3.2 still holds, if condition (3.15) is replaced by (2.3).

Remark 3.4. Observing that the $C^{1,1}$ bounds do not need the strict positivity of the right-hand side of the Monge–Ampère equation (see [18]), we see that the solution $u$ of (3.13) belongs to $C^{1,1}(\overline{Q})$, even if condition (3.16) is removed.

We have treated the case when $f^{1/n}(x, z, p)$ is superlinear in $z$ as $z \to \infty$. The other case is that $f^{1/n}(x, z, p)$ is sublinear as $z \to \infty$, that is,

$$\lim_{z \to -\infty} f(x, z, p)/|z|^n = 0 \quad \text{uniformly for} \quad x \in \Omega, \quad p \in \mathbb{R}^n.$$  (3.20)

If $f$ satisfies

$$\lim_{z \to 0} f(x, z, p)/|z|^n > \lambda_1^n \quad \text{uniformly for} \quad x \in \Omega, \quad p \in \mathbb{R}^n,$$  (3.21)

$$f(x, 0, 0) = 0,$$  (3.22)

then as in Section 2 we can establish the a priori bounds for the solutions of (3.5), and using degree theory as above we are able to derive the existence of a nontrivial solution. But in this case, $t\phi$ and $T\phi$ are the supersolution and subsolution of (3.5), respectively, provided $t$ is small enough and $T$ large enough, where $\phi$ is the eigenfunction of (2.8). Therefore we obtain a nontrivial solution of (3.5) immediately.
4. **Existence of Nontrivial Solutions to the Neumann Problem**

In this section we are concerned with the problem of the existence of a function \( u \) satisfying

\[
\begin{align*}
\det(D^2u) - f(x, u, Du) & \quad \text{in } \Omega, \\
D_{\gamma}u + \alpha(x)u & = 0 \quad \text{on } \partial\Omega,
\end{align*}
\]

where \( \gamma \) is the unit outward normal to \( \partial\Omega \), \( \alpha(x) \in C^3(\overline{\Omega}) \), \( f \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \).

We suppose

\[
\alpha(x) > \alpha_0 > 0; \quad (4.2)
\]

\[
\lim_{|z| + |p| \to 0} f(x, z, p)(|z|^n + |p|^n) = 0 \quad \text{uniformly for } x \in \Omega; \quad (4.3)
\]

\[
f(x, z, p) > 0, \quad z < 0, \quad x \in \overline{\Omega}, \quad p \in \mathbb{R}^n. \quad (4.4)
\]

We begin by proving the following lemma.

**Lemma 4.1.** Suppose \( u \in C^{0,1}(\overline{\Omega}) \) is a convex function and satisfies the Neumann condition in (4.1). Then \( u \leq 0 \) in \( \Omega \), and

\[
C_0 \leq \sup_{\Omega} u / \inf_{\Omega} u \leq 1,
\]

where \( C_0 \) depends only on \( \Omega \), \( \| \alpha(x) \|_0 \).

**Proof.** The lemma is trivial if \( u = 0 \). So we suppose \( u \neq 0 \). From the convexity it follows that \( u \) attains its maximum at some point \( x_0 \in \partial\Omega \) where \( D_{\gamma}u > 0 \). Hence \( u(x) \leq u(x_0) = -D_{\gamma}u(x_0)/\alpha(x_0) < 0. \)

Denote \( M = -\inf_{\partial\Omega} u, \ m = -\sup_{\partial\Omega} u. \) Then \( M \geq m > 0. \) Let \( x \in \Omega \) be such a point that \( |x - x_0| = \text{dist}(x, \partial\Omega) \), and \( (x_0 - x)/|x_0 - x| = \gamma(x_0) \). From the convexity of \( u \) we have

\[
\begin{align*}
\frac{\delta}{d} (m - M) - m, \\
D_{\gamma}u(x_0) \geq (u(x_0) - u(x))/|x - x_0| = -\frac{1}{\delta} (m + u(x)) \geq (M - m)/d,
\end{align*}
\]

where \( d = \text{diam}(\Omega), \ \delta = |x - x_0|. \) Consequently the Neumann condition in (4.1) gives

\[
\frac{1}{d} (M - m) + \alpha(x_0) u(x_0) \leq 0.
\]

Since \( u(x_0) = -m \), we obtain \( M < (d\alpha(x_0) + 1) m. \)
The main result of this section is

**Theorem 4.1.** Suppose $f \in C^2(\overline{\Omega} \times R \times R^n)$, $\alpha(x) \in C^3(\overline{\Omega})$, and (2.3), (4.2)–(4.4) hold. Then problem (4.1) admits a nontrivial solution $u \in C^{3+\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$.

**Proof.** Denote $B = \{u \in C^3(\overline{\Omega}); D_u + \alpha(x) u = 0 \text{ on } \partial\Omega\}$. Then $B$ is a closed linear subspace of $C^3(\overline{\Omega})$. From (4.3), $\forall \varepsilon > 0$, we have

$$f(x, z, p) \leq \varepsilon (|z|^n + |p|^n) + C_\varepsilon (|z|^n + |p|^n)^2 \quad \forall |z| + |p| < 1. \quad (4.5)$$

For any given $\delta \in (0, 1)$, let $f_\delta(x, z, p) \in C^2(\overline{\Omega} \times R \times R^n)$ be a function satisfying (4.5) with the constant $C_\varepsilon$ independent of $\delta \in (0, 1)$, and $f_\delta(x, z, p) = 0$ for $|z| < \delta$, $f_\delta(x, z, p) = f(x, z, p)$ for $|z| > 2\delta$. Let $T_\delta$ be the mapping defined by $u = T_\delta v$ which is the solution of

$$\det(D^2u) = f_\delta(x, u, Du) + t \quad \text{in } \Omega$$
$$D_u + \alpha(x) u = 0 \quad \text{on } \partial\Omega. \quad (4.6)$$

Then $T_\delta$ is completely continuous from $[\varepsilon, \infty) \times B$ to $B$ for any $\varepsilon > 0$. Observing that $f_\delta(x, z, p) = 0$ for $|z| < \delta$, we have

$$T_\delta(v) = T_\delta(0) = t^{1/n}T_1(0) = t^{1/n}T_1(v) \quad \text{for any } v \in B_{\delta},$$

where $B_{\delta} = \{u \in \Omega; \|u\|_3 < \delta\}$. Hence there exists $\theta(\delta) > 0$ such that $T_\delta(B_{\delta}) \subset B_{\delta}$ for $t \in (0, \theta(\delta))$, and

$$\deg(I - T_\delta, B_{\delta}, 0) = \deg(I - t^{1/n}T_1, B_{\delta}, 0) = \deg(I, B_{\delta}, 0) = 1 \quad \text{for } t \in (0, \theta(\delta)). \quad (4.7)$$

On the other hand, $\forall \varepsilon > 0$, $\lambda_0 > 0$, from Theorem 2.1 and Remark 2.2, the solution $u = u_{1, \lambda}$, of

$$\det(D^2u) = f_\delta(x, u, Du) + t \quad \text{in } \Omega$$
$$D_u + \alpha(x) u = 0 \quad \text{on } \partial\Omega \quad (4.8)$$

satisfies

$$\|u\|_{3+\alpha} \leq M(\varepsilon, \lambda_0) \quad \text{for } t \in [\varepsilon, \lambda_0].$$

From Corollary 2.4 we may choose $\lambda_0$ so large that (4.8) has no solution for $t \geq \lambda_0$. Let $M_0 > M(\varepsilon, \lambda_0)$. Then the topological degree $\deg(I - T_\delta, B_{M_0}, 0) = \gamma$ is well defined and independent of $t \in [\varepsilon, \lambda_0]$. Let $t = \lambda_0$; we get $\gamma = 0$. From (4.7) we thus obtain $\deg(I - T_\delta, B_{M_0} \setminus B_{\delta}, 0) = -1$. Hence (4.8) admits a solution $u_{\delta, \lambda} \notin B_\delta$ for $t \in (0, \theta(\delta))$. 

We claim that there exists $t(\delta) > 0$ such that for any $t \in (0, t(\delta))$,
\[
\|u_{\delta, t}\|_0 = -\inf_{\Omega} u_{\delta, t} \geq C^*	ag{4.9}
\]
for some $C^*$ independent of $\delta \in (0, 1)$. Indeed, denote $s = -\inf_{\Omega} u_{\delta, t}$. Then $s > \delta$ (otherwise $\theta u_{\delta, t}$ is a fixed point of $T_{\theta t}$; this contradicts $T_{\theta t}(\bar{B}_\delta) \subset B_\delta$, where $\theta = \delta/\|u_{\delta}\|_3 \in (0, 1)$). Formula (4.9) is trivial if $s > 1$, so we may suppose $s \leq 1$. By the convexity of $u_{\delta, t}, \forall x \in \partial \Omega$, we have
\[
|Du_{\delta, t}(x)| \geq \frac{1}{d}(u_{\delta, t}(x) - \inf_{\Omega} u_{\delta, t})
\]
where $d = \text{diam}(\Omega)$. Hence
\[
|Du_{\delta, t}(x)| \geq C_1 s \quad \text{on } \partial \Omega, \quad C_1 = \alpha_0/(1 + dx_0). \tag{4.10}
\]
Let $t(\delta) = C_0^\alpha \delta^n \varepsilon_1$; $\varepsilon_1$ small will be determined below, $C_0$ is the constant in Lemma 4.1. From Lemma 4.1, we have $u_{\delta, t} \leq -C_0 s \leq -C_0 \delta$. Thus
\[
f_{\delta}(x, u, Du) + t \leq t + \varepsilon_1 (|u|^{n} + |Du|^{n}) + C_{\varepsilon_1} (|u|^{n} + |Du|^{n})^2
\]
\[
\leq (\varepsilon_1 + \varepsilon_1)(|u|^{n} + |Du|^{n}) + C_{\varepsilon_1} (|u|^{n} + |Du|^{n})^2
\]
for $t \in (0, t(\delta))$ and $|u| + |Du| < 1$, where $u = u_{\delta, t}$. Set $\Omega' = \{x \in \Omega; |Du_{\delta, t}| < C_1 s\}$. From (4.10) we have $\partial \Omega \cap \Omega' = \emptyset$ and $|u_{\delta, t}| \leq (1 + C^n_0) s^n \leq 1 + C_1$ on $\Omega'$. Thus
\[
\int_{\Omega'} \frac{\det(D^2 u)}{|u|^{n} + |Du|^{n}} \, dx \leq \int_{\Omega'} [\varepsilon_1 + C_{\varepsilon_1} (|u|^{n} + |Du|^{n})^2] \, dx
\]
\[
\leq [\varepsilon_1 + s^n(1 + C^n_0) C_{\varepsilon_1}] \text{mes}(\Omega').
\]
On the other hand,
\[
\int_{\Omega'} \frac{\det(D^2 u)}{|u|^{n} + |Du|^{n}} \, dx = \int_{B(0, C_1 s)} \frac{dp}{|u|^{n} + |p|^{n}}
\]
\[
\geq \int_{B(0, C_1 s)} \frac{dp}{s^n + |p|^{n}} \geq \int_{B(0, C_1)} \frac{dp}{1 + |p|^{n}} = C_2,
\]
where $u = u_{\delta, t}$. Combining the above inequalities we obtain
\[
\varepsilon_1 + s^n(1 + C^n_0) C_{\varepsilon_1} \geq C_3.
\]
Let $\varepsilon = \frac{1}{4}C_3, \varepsilon = \min(\frac{1}{4}C_3, (C_0\delta)^{-2} \theta(\delta))$. Then $t(\delta) = C_0\delta \varepsilon < \theta(\delta)$, and $s > C^*$ for some $C^* > 0$ independent of $\delta \in (0, 1)$. Hence (4.9) follows.

Consequently from Lemma 4.1 it follows that $u_{\delta, t} \leq -C_0C^*$ for $t < t(\delta)$. Let $\delta < \frac{1}{2}C_0C^*$ be fixed. Since $f_\delta(x, z, p) = f(x, z, p)$ for $|z| \geq 2\delta$, $u_{\delta, t}$ is a solution of

$$\begin{align*}
\det(D^2u) &= f(x, u, Du) + t \quad \text{in } \Omega, \\
D_\nu u + \alpha(x) u &= 0 \quad \text{on } \partial\Omega.
\end{align*}$$

Therefore from (4.4) and Theorem 2.1 we have

$$C_4^{-1} \geq f(x, u_{\delta, t}, Du_{\delta, t}) \geq C_4 > 0,$$

for some $C_4$ independent of $t \in (0, t(\delta))$. By the regularity results in [11] it follows that $\|u_{\delta, t}\|_3 \leq M$ for some $M > 0$ independent of $t \in (0, t(\delta))$. Extract a subsequence, still denoted by $(u_{\delta, t})$ so that $u_{\delta, t} \to u_0$ in $C^{2+\alpha}(\bar{\Omega})$ as $t \to 0$. Then $u_0 \neq 0$ is a solution of (4.1). From the elliptic Schauder theory it follows that $u_0 \in C^{3+\alpha}(\bar{\Omega})$. □

5. Existence of Multiple Solutions

In this section, we discuss the existence, nonexistence, and multiplicity of solutions to the eigenvalue problems of Monge–Ampère equations. First we consider the Dirichlet problem

$$\begin{align*}
\det(D^2u) &= \lambda f(x, u, Du) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*}$$

(5.1)

Suppose that

$$f(x, z, p) \geq f_0 > 0 \quad \forall (x, z, p) \in (\bar{\Omega} \times R \times R^n),$$

(5.2)

$$f(x, z, p) \leq \mu(z) |p|^{n+1} \quad \text{for } |p| > 1,$$

(5.3)

where $\mu(t)$ is a positive continuous function. We have

**Theorem 5.1.** Suppose $f(x, z, p) \in C^2(\bar{\Omega} \times R \times R^n)$ satisfies (5.2), (5.3), and (2.3). Then there exists $\lambda^* > 0$ such that problem (5.1) possesses at least two solutions for each $\lambda \in (0, \lambda^*)$, admits at least one solution for $\lambda = \lambda^*$, and has no solution for $\lambda > \lambda^*$.

**Proof.** Denote $\lambda^* = \sup \{\lambda; (5.1) \text{ admits a solution}\}$. From Corollary 2.5 it follows that $\lambda^*$ is finite, and (5.1) has no solution for $\lambda > \lambda^*$. 
First we prove \( \lambda^* > 0 \). We define the mapping \( T \) of \( C^3(\overline{\Omega}) \) to \( C^3(\overline{\Omega}) \) by
\[
u = T(v)
\]
which is the solution of
\[
det(D^2u) = f(x, v, Dv) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.
\] (5.4)

Then the solutions of (5.1) correspond to the fixed points of \( \lambda^{1/n}T \). By the elliptic regularity we see that \( T \) is completely continuous. Hence there is \( \lambda > 0 \) such that \( \lambda^{1/n}T(\overline{B}_1) \subset B_1 \) for \( \lambda \in [0, \lambda^*] \), where \( B_1 \) is the unit ball in \( C^3(\overline{\Omega}) \). Thus \( \deg(I - \lambda^{1/n}T, B_1, 0) = \gamma \) is well defined and independent of \( \lambda \in (0, \lambda^*) \). Let \( \lambda = 0 \); we get \( \gamma = 1 \), which implies \( \lambda^* > \lambda > 0 \).

Next we prove (5.1) admits at least one solution for \( \lambda = \lambda^* \). Let \( \lambda_m \to \lambda^* \) be a sequence such that (5.1) admits a solution \( u_m \) for \( \lambda = \lambda_m \). Since \( \lambda^* > 0 \), we may suppose \( \lambda_m \geq \frac{1}{2}\lambda^* \) for all \( m \geq 1 \). From Theorem 2.1 and Remark 2.2 it follows that \( \|u_m\|_{3+\alpha} < M \) for some \( \alpha \in (0, 1) \) and \( M > 0 \). Therefore there is a subsequence of \( (u_m) \) which converges to \( u^* \in C^3(\overline{\Omega}) \); \( u^* \) is a solution of (5.1) for \( \lambda = \lambda^* \).

To complete the proof we have to prove that (5.1) possesses at least two solutions for \( \lambda \in (0, \lambda^*) \). Let \( \lambda_0 \in (0, \lambda^*) \) be any given. From Theorem 2.1 and Remark 2.2 we see that \( \|u\|_{3+\alpha} < M_0 \) for any solution \( u \) of (5.1) with \( M_0 \) independent of \( \lambda \in [\lambda_0, \lambda^* + 1] \). Thus \( \deg(I - \lambda^{1/n}T, B_{M_0}, 0) = \gamma \) is well defined for \( \lambda \in [\lambda_0, \lambda^* + 1] \). Since there is no solution of (5.1) when \( \lambda = \lambda^* + 1 \), it follows that
\[
\deg(I - \lambda^{1/n}T, B_{M}, 0) = 0, \quad \forall M > M_0, \quad \lambda \in [\lambda_0, \lambda^* + 1],
\] (5.5)

where \( B_M = \{u \in C^3_0(\overline{\Omega}); \|u\|_3 < M\} \).

Let \( \Phi = \{u \in C^3_0(\overline{\Omega}); u > u^* \quad \text{in} \quad \Omega, \quad \partial u/\partial y < \partial u^*/\partial y \quad \text{on} \quad \partial\Omega\} \), and \( \Phi_R = \Phi \cap B_R \). For \( R \) large we will find a solution in \( \Phi_R \) of (5.1) for \( \lambda = \lambda_0 \).

For all \( v \in C^3_0(\overline{\Omega}) \), let \( S_r(v) \in C^3_0(\overline{\Omega}) \) be the solution of
\[
det(D^2u) = t\lambda_0 f(x, v, Du) + (1 - t) \lambda_0 f(x, u^*, Du^*) \quad \text{in} \quad \Omega,
\quad u = 0\quad \text{on} \quad \partial\Omega.
\] (5.6)

Then \( S_r \) is completely continuous from \( (t, v) \in [0, 1] \times C^3_0(\overline{\Omega}) \) to \( C^3_0(\overline{\Omega}) \). The fixed points of \( S_r \) are equivalent to the solutions of (5.1) for \( \lambda = \lambda_0 \). If \( u \in \Phi_R \) is a solution of \( S_r(u) = u \), then \( 0 \geq u \geq u^* \). Thus \( \|u\|_3 \leq M_1 \), and \( S_r \) has no fixed points on \( \partial\Phi_R \cap \{\|u\|_3 = R\} \) for \( R > M_1 \). We claim that there is no fixed point of \( S_r \) on the remaining part of \( \partial\Phi_R \). For if \( u \) is such a fixed point, then either \( u = u^* \) on some point \( x' \in \Omega \), or \( \partial u/\partial y = \partial u^*/\partial y \) at some point \( x'' \) on \( \partial\Omega \). In the first case we have \( (D^2u) \geq (D^2u^*) \), \( Du = Du^* \), \( u = u^* \) at \( x' \). Therefore,
\[
det(D^2u) - det(D^2u^*) = (\lambda_0 - \lambda^*) f(x, u^*, Du^*) < 0;
\] (5.7)
this contradicts \((D^2u) \geq (D^2u^*)\). In the other case since \(u = u^* = 0\) on \(\partial \Omega\), we still have (5.7). Thus in a neighborhood \(\omega \in \Omega\) of \(x^*\),

\[\det(D^2u) < \det(D^2u^*).\]

Since \(u(x^*) = u^*(x^*)\), \(u \geq u^*\), by the maximum principle it follows that \(\partial u/\partial \gamma < \partial u^*/\partial \gamma\), also a contradiction. Consequently the topological degree

\[\text{deg}(I - S_1, \Phi_R, 0) - \gamma\]

is well defined and independent of \(t \in [0, 1]\). Let \(t = 0\). From (5.6) we have \(S_0(c) - (\lambda_0/\lambda^*)^{1/n} u^* \in \Phi_R\) for any \(c \in C^1(\Omega)\); hence \(\gamma = 1\). Therefore \(S_1\) has a fixed point \(u \in \Phi_R\) which is a solution of (5.1) for \(\lambda = \lambda_0\).

Noting that \(S_1 = \lambda_0^{1/n} T\), choosing \(M > R\) in (5.5), we obtain

\[\text{deg}(I - \lambda_0^{1/n} T, B_M \setminus \Phi_R, 0) = -1.\]

Therefore for \(\lambda \in (0, \lambda^*)\), (5.1) possesses at least two solutions

\[u_1 \in \Phi_R \quad \text{and} \quad u_2 \in B_M \setminus \Phi_R,\]

(5.8)

respectively. This completes the proof of Theorem 5.1.

Remark 5.1. It is not hard to see that condition (5.3) in Theorem 5.1 can be replaced by (3.4). Indeed, we may deal with this case as in the proof of Theorem 3.1, and obtain the desired results.

Theorem 5.1 states the existence of two solutions of (5.1). If \(f\) satisfies some other conditions, we are able to prove the existence of three or more solutions of (5.1). For example, we have

**Theorem 5.2.** Suppose \(f(x, z, p) \in C^2(\Omega \times R \times R^n)\) satisfies (2.3), (5.3). Suppose also that \(f(x, 0, 0) = 0\), and \(f(x, z, p) > g(|z| + |p|)\), where \(g(t) \in C(R)\) is a nondecreasing function satisfying

\[\lim_{t \to 0} g(t)/|t|^n = +\infty.\]

Then there exists \(\lambda^* > 0\) such that (5.1) admits at least three solutions for \(\lambda \in (0, \lambda^*)\), at least two solutions for \(\lambda = \lambda^*\), and has the unique solution \(u = 0\) for \(\lambda > \lambda^*\).

**Proof.** Since \(u = 0\) is a solution of (5.1), it suffices to prove that there exists \(\lambda^* > 0\) such that (5.1) has at least two nonzero solutions for \(\lambda \in (0, \lambda^*)\), one nonzero solution for \(\lambda = \lambda^*\), and has no nonzero solution for \(\lambda > \lambda^*\).

Suppose \(u = u_\lambda\) is a nonzero solution of (5.1). Let \(M = \|u\|_0 = -\inf_{\Omega} u\). Then \(|Du| < M/\delta\) on \(\partial \Omega_\delta\), \(u \leq -\delta M/d\) in \(\Omega_\delta\), where \(d = \text{diam}(\Omega)\), \(\Omega_\delta = \{x; \text{dist}(x, \partial \Omega) > \delta\}\). Thus
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\[(M/\delta)^n \text{mes}(B(0, 1)) \geq \int_{Du(\Omega_\delta)} dp\]

\[= \int_{\Omega_\delta} \det(D^2 u) \, dx = \lambda \int_{\Omega_\delta} f(x, u, Du) \, dx \geq \int_{\Omega_\delta} g(|u|) \, dx \geq \lambda g \left( \frac{\delta}{d} M \right) \text{mes}(\Omega_\delta).\]

Choosing \(\delta > 0\) suitably we obtain

\[M^n \geq \lambda C_1 g(C_2 M), \tag{5.10}\]

where \(C_1, C_2\) depend only on \(\Omega\). Consequently,

\[M = \|u\|_0 \geq C_3(\lambda) > 0 \tag{5.11}\]

for some continuous function \(C_3(\lambda)\).

Let \(\lambda^* = \sup \{\lambda; (5.1)\text{ has nonzero solutions}\}\). We claim that \(\lambda^* < \infty\). Indeed, if there is a sequence \((\lambda_m)\) which tends to infinity such that (5.1) has a nontrivial solution \(u_m\) for \(\lambda = \lambda_m\). Without loss of generality we may suppose \(\lambda_m \geq 1\). From Theorem 2.1 we have \(M_m = \|u_m\|_0 \leq M\) for some \(M\) independent of \(m\). If \(M_m\) has a subsequence which converges to a positive constant, then \(\lambda_m g(C_2 M_m) \rightarrow +\infty\). This contradicts (5.10). If \(M_m \rightarrow 0\) then (5.9) contradicts (5.10). Hence \(\lambda^* < \infty\).

The results (5.10) and (5.11) still hold for the solution \(u = u_{\lambda,t}\) of

\[\det(D^2 u) = \lambda(f(x, u, Du) + t) \quad \text{in } \Omega,\]

\[u = 0 \quad \text{on } \partial \Omega \tag{5.12}\]

for any \(t \geq 0\). Applying Theorem 5.1 to (5.12), we obtain \(\lambda_{t^*} > 0\) such that (5.12) admits at least two solutions for \(\lambda \in (0, \lambda_{t^*})\), at least one solution \(u_{t^*}\) for \(\lambda = \lambda_{t^*}\), and has no solution for \(\lambda > \lambda_{t^*}\). Since for any \(t_2 > t_1 > 0\), \(u_{t^*}\) is a subsolution of (5.12) for \(t = t_1\), by Theorem 7.1 in [53] there is a solution of (5.12) for \(t = t_1\). Thus \(\lambda_{t^*} \geq \lambda_{t^*}\), and \(\lambda_{t^*} \wedge \lambda^* \in (0, \infty)\) as \(t \rightarrow 0\). Let \(t \rightarrow 0\). Similar to the proof of Theorem 5.1 we have \(u_{t^*} \rightarrow u^* \neq 0\); \(u^*\) is a solution of (5.1) for \(\lambda = \lambda^*\).

We now show that (5.1) has at least two nonzero solutions for \(\lambda \in (0, \lambda^*)\). Let \(\lambda_0 \in (0, \lambda^*)\) be any given, and let \(u\) be a solution of (5.1) with \(\lambda \in (\lambda_0, \lambda^*)\). By the convexity of \(u\) we have \(|Du(x)| \geq (u(x) - \inf_{\Omega} u)/d \geq (1/2d) C_3(\lambda)\) for \(x \in \Omega' = \{x \in \Omega; u(x) \geq -\frac{1}{2} C_3(\lambda)\}\), where \(d = \text{diam}(\Omega)\). Hence \(|u| + |Du| \geq (\frac{1}{2} + (1/2d)) C_3(\lambda) > 0\) in \(\Omega\), and \(f(x, u, Du) \geq g(|u| + |Du|) \geq a\) for some \(a > 0\) independent of \(\lambda \in (\lambda_0, \lambda^*)\). Therefore by Theorem 2.1 and Remark 2.2 we have \(\|u\|_3 \leq M\) for some \(M\) independent of \(\lambda \in (\lambda_0, \lambda^*)\). On the other hand, from (5.11) we see that, for
\( \lambda \in [\lambda_0, \lambda^* + 1] \), (5.1) has no solution on \( \partial B_\delta \) provided \( \delta \) is sufficiently small, where \( B_\delta = \{ u \in C^3_0(\bar{\Omega}) ; \| u \|_\delta < \delta \} \). Define the mapping \( S_r \) and \( T \) as in the proof of Theorem 5.1. Similar to (5.5) and (5.8) we have

\[
\deg(I - \lambda^{2/n}T, B_M \setminus \bar{B}_\delta, 0) = 0 \quad \forall \lambda \in [\lambda_0, \lambda^* + 1],
\]

\[
\deg(I - S_r, \Phi_R \setminus \bar{B}_\delta, 0) = \gamma = 1 \quad \text{for} \ t \in [0, 1].
\]

Since \( S_I = \lambda_0^{2/n}T \), let \( M > R \). We thus obtain two solutions of (5.1) which lie in \( \Phi_R \setminus \bar{B}_\delta \) and \( B_M \setminus \bar{B}_\delta \), respectively. By the arbitrariness of \( \lambda_0 \in (0, \lambda^*) \), we complete the proof of Theorem 5.2.

We have treated the Dirichlet problem and proved the existence of two or more solutions. These results still hold for the Neumann problem, even for the nonhomogeneous boundary problem. Here we consider the following problem

\[
\det(D^2u) = \lambda f(x, u, Du) \quad \text{in} \ \Omega,
\]

\[
D_\gamma u + \alpha(x) u = \psi(x) \quad \text{on} \ \partial\Omega.
\]

**Theorem 5.3.** Suppose \( f(x, z, p) \in C^2(\bar{\Omega} \times R \times R^n) \) satisfies (2.3), \( f(x, u, p) \geq f_0 > 0, \alpha(x), \psi(x) \in C^3(\partial\Omega), \alpha(x) \geq \alpha_0 \) for some \( \alpha_0 > 0 \). Then there exists \( \lambda^* > 0 \) such that problem (5.13) possesses at least two solutions for \( \lambda \in (0, \lambda^*) \), at least one solution for \( \lambda = \lambda^* \), and has no solution when \( \lambda > \lambda^* \).

**Proof.** Denote \( \lambda^* = \sup \{ \lambda ; (5.13) \text{ has a solution} \} \). From Corollary 2.5 it follows that \( \lambda^* \) is finite, and (5.13) has no solution when \( \lambda > \lambda^* \).

First we prove that \( \lambda^* > 0 \). Let \( u^* \in C^3(\bar{\Omega}) \) be a convex function satisfying, for \( \lambda > 0 \) sufficiently small,

\[
\det(D^2u^*) > \lambda f(x, u^*, Du^*) \quad \text{in} \ \Omega,
\]

\[
D_\gamma u^* + \alpha(x) u^* = \psi \quad \text{on} \ \partial\Omega.
\]

Denote \( B = \{ u \in C^3(\partial\Omega) ; D_\gamma u + \alpha(x) u = 0 \text{ on} \ \partial\Omega \} \), \( \Phi = \{ v \in B ; v > 0 \text{ on} \ \bar{\Omega} \} \), and \( \Phi_R = \{ u \in \Phi ; \| u \|_\delta < R \} \). For \( R \) large and \( \lambda \) small, we will find a convex solution of (5.13) with the form

\[
u = u^* + v, \quad v \in \Phi_R
\]

and consequently \( \lambda^* > 0 \).

For any \( v \in B \), let \( u' \) be the solution of

\[
\det(D^2u) = t\lambda f(x, u^* + v, Du^* + v) + (1 - t) \lambda f(x, u^*, Du^*) \quad \text{in} \ \Omega,
\]

\[
D_\gamma u + \alpha(x) u = \psi(x) \quad \text{on} \ \partial\Omega.
\]

(5.16)
Set $S_t(v) = u' - u^*$. Then $S_t$ is completely continuous from $[0, 1] \times B$ to $B$. $v$ is a fixed point of $S_t$ if and only if $u = u^* + v$ is a solution of

$$
\det(D^2u) = t\lambda f(x, u, Du) + (1-t) \lambda f(x, u^*, Du^*) \quad \text{in } \Omega,
$$

$$
D_j u + \alpha(x) u = \psi(x) \quad \text{on } \partial \Omega.
$$

(5.17)

If $u$ is a solution of (5.17) with the form (5.15), then $u^* \leq u \leq K$, where $K = \|\psi\|_\infty / \alpha_0$ (see (2.5)). Hence $\|u\|_3 \leq M$ for some $M$ independent of $t \in (0, 1)$. Let $R > M + \|u^*\|_3$, then there is no solution $u = u^* + v$ of (5.17) with $\|v\|_3 = R$. We claim that there is no solution $u = u^* + v$ of (5.17) with $v$ on the remaining part of $\partial \Phi_R$. For if $v$ is such a solution, we have $u \geq u^*$, and $u = u^*$ at some point $x_0 \in \bar{\Omega}$ which, together with $D_j u + \alpha(x) u = D_j u^* + \alpha(x) u^* = 0$ on $\partial \Omega$, imply $Du = Du^*$, $D^2 u \geq D^2 u^*$ at $x_0$. But from (5.14) we also have

$$
\det(D^2u) = \det(D^2u^*) < 0;
$$

this is a contradiction. Therefore the topological degree

$$
\deg(I - S_t, \Phi_R, 0) = \gamma
$$

(5.18)

is well defined and independent of $t \in [0, 1]$. By the maximum principle we have $S_0(v) = w - u^* \in \Phi_R$ for all $v \in B$; hence $\gamma = 1$, where $w$ is the solution of

$$
\det(D^2w) = \lambda f(x, u^*, Du^*) \quad \text{in } \Omega,
$$

$$
D_j w + \alpha(x) w = \psi(x) \quad \text{on } \partial \Omega.
$$

Let $t = 1$ in (5.18). We obtain a solution of (5.13) given by (5.15). Thus $\lambda^* > 0$.

Similar to the proof of Theorem 5.1 we see the existence of a solution $u^*$ of (5.13) for $\lambda = \lambda^*$. We now show that (5.13) admits at least two solutions for $\lambda \in (0, \lambda^*)$. Indeed, for any fixed $\lambda \in (0, \lambda^*)$, (5.14) still holds for the solution $u^*$; thus (5.18) holds with $\gamma = 1$. From Theorem 2.1 and Remark 2.2 it follows that $\|u_{\lambda}\|_3 \leq M_\lambda$ for any solution $u_{\lambda}$ of (5.13) provided $\lambda \in (\epsilon, \lambda^* + 1)$. Hence there is no solution of $S_\lambda(v) = v$ on $\partial B_M$ for $M > M_\lambda + \|u^*\|_3$, where $B_M = \{u \in B; \|u\|_3 < M\}$. Observing that the compact mapping $S_\lambda = S_{\lambda, \lambda}$ depends continuously on $\lambda$, we have

$$
\deg(I - S_{\lambda, \lambda}, B_M, 0) = \deg(I - S_{\lambda, \lambda^* + 1}, B_M, 0) = 0
$$

for $\lambda \in [\epsilon, \lambda^* + 1]$. Let $M > R$. From (5.18) we obtain two solutions of $S_\lambda(v) = v$ in $\Phi_R$ and $B_M \setminus \Phi_R$, respectively. This completes the proof of Theorem 5.3. \[\Box\]
Remark 5.2. Because the tool we used in the proof of Theorem 5.1 is
topological degree, it is not hard to see that the set of solutions \((\lambda, u)\) of
(5.1) contains an unbounded component with the end-point \((0, 0)\).

6. EQUATIONS OF PRESCRIBED GAUSS CURVATURE

In this section we discuss briefly the existence of two or more solutions
to the equation of prescribed Gauss curvature,

\[
\det(D^2u) = g(x, u)(1 + |Du|^2)^{\alpha/n} \quad \text{in } \Omega,
\]

with the boundary condition

\[
\beta(x) D_u u + \alpha(x) u = \psi(x) \quad \text{on } \partial \Omega,
\]

where \(\gamma = \gamma(x)\) is the unit outward normal to \(\partial \Omega\), \(\beta, \alpha,\) and \(g(x, u)\) are
continuous nonnegative functions, and

\[
\alpha(x) \geq \alpha_0 > 0 \quad \forall x \in \Omega.
\]

We have

THEOREM 6.1. Suppose (6.3) and

\[
\lim_{x \to -\infty} g(x, z) = +\infty \quad \text{uniformly for } x \in \Omega.
\]

Then the solution \(u\) of (6.1), (6.2) satisfies

\[
\|u\|_0 \leq M,
\]

where \(M\) depends only on \(\alpha_0, \Omega, n, g\).

Indeed, Theorem 6.1 can be extended to the equation

\[
\det(D^2u) = f(x, u, Du) \quad \text{in } \Omega
\]

provided \(f(x, z, p)\) satisfies

\[
f(x, z, p) \geq g(z)/h(p) \quad \text{for all } x \in \bar{\Omega}, z < 0, p \in R^n,
\]

where \(g, h\) are nonnegative functions, \(h(p) \in L^1(R^n)\), \(g(z)\) is non-
increasing, and \(\lim_{z \to -\infty} g(z) = +\infty\).

That is
THEOREM 6.1'. Suppose (6.3), (6.7) hold. If $u \in C^3(\Omega) \cap C^{0,1}(\overline{\Omega})$ is a solution of (6.2), (6.6), then $\|u\|_3 \leq M$, where $M$ depends only on $\Omega$, $n$, $g$, $h$, and $u_0$.

Proof. Since $g(u) \leq h(Du) \det(D^2u)$, we have

$$\int_{\Omega} g(u) \, dx \leq \int_{\Omega} h(Du) \det(D^2u) \, dx$$

$$\leq \int_{\mathbb{R}^n} h(p) \, dp = \|h\|_{L^1(\mathbb{R}^n)}.$$  \hspace{1cm} (6.8)

Let $\delta > 0$ be given. By the convexity of $u$ we have

$$u(x) \leq \sup_{\Omega} u + \frac{\delta}{\text{diam}(\Omega)} (\inf_{\Omega} u - \sup_{\Omega} u) =: d$$

for $x \in \Omega_\delta = \{x \in \Omega; \text{dist}(x, \partial \Omega) \geq \delta\}$. Thus from (6.8) and the monotonicity of $g(z)$ it follows that

$$\text{mes}(\Omega_\delta) g(d) \leq \|h\|_{L^1(\mathbb{R}^n)}.$$  \hspace{1cm} (6.9)

Since $\lim_{z \to -\infty} g(z) = +\infty$, combining (6.9) with (2.6) we obtain (6.5).

We conclude this paper by giving the following two theorems.

THEOREM 6.2. Suppose $f \in C^2(\overline{\Omega} \times \mathbb{R})$ satisfies (6.7), (3.2)–(3.4). Then (6.6) possesses a nontrivial solution $u \in C^3(\Omega) \cap C^{0,1}(\overline{\Omega})$ which satisfies $u = 0$ on $\partial \Omega$.

The proof is just the same as that of Theorem 3.1, and is omitted here. Similar to Theorem 4.1, we have

THEOREM 6.3. Suppose $f \in C^2(\overline{\Omega} \times \mathbb{R})$, $\alpha(x) \in C^3(\overline{\Omega})$, and (4.2)–(4.4), (6.7) hold. Then (6.6) possesses a nontrivial solution $u \in C^3(\overline{\Omega})$ which satisfies $D_\alpha u + \alpha(x) u = 0$ on $\partial \Omega$.

REFERENCES


