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Continuity of the Limit Set Maps in Semidynamical Systems

STEPHEN H. SAPERSTONE

George Mason University, Fairfax, Virginia 22030

AND

MASAHIRO NISHIHAMA

University of Maryland–Baltimore County, Baltimore, Maryland 21228

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Given a semiflow on a metric space X (not necessarily locally compact), we relate notions of stability to the continuity of the orbital and limit set maps, $K(x)$ and $L(x)$, where K and L are considered as maps from X to 2^X.

1. INTRODUCTION

Let a semiflow π be given on a metric space (X, ρ). We investigate both the upper and lower semicontinuity of the set valued maps $x \to K(x)$ and $x \to L(x)$, where $K(x)$ is the positive orbit closure through x, and $L(x)$ is the positive limit set of x.

The question of continuity of these maps was prompted by a result of of Boyarsky [5]. His work concerned a characterization of the limit sets of probability measures arising from diffusion processes. In particular, he established the continuity of the map $\mu \to L(\mu)$, provided the semiflow (of probability measures μ) was Lyapunov stable. This is indeed an unnecessarily strong requirement. As our examples will show, a weaker form of stability is called for.

We assume each $K(x)$ is compact. In Section 3 we establish that the following are equivalent: π is positively stable, each $L(x)$ is positively stable, the map K is continuous on X. Our main result is in Section 6: L is continuous on X if and only if each $L(x)$ is eventually stable. We can then deduce that if K is continuous on X, then so is L. The latter is even true locally, as shown in Proposition 4.2. In Section 7 we collect some results on the continuity of prolongations.

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2. Definitions and Notations

2.1. Definition. Let $R^+ = \{t : t \geqslant 0\}$. By a *semiflow* (or a semidynamical system) on (X, ρ) we mean a mapping $\pi: X \times R^+ \to X$ so that

(i) $\pi(x, 0) = x$, $x \in X$,

(ii) $\pi(\pi(x, s), t) = \pi(x, s + t)$, $x \in X$, $s, t \in R^+$,

(iii) π is continuous.

For brevity we denote $\pi(x, t)$ by xt. For $A \subset X$, $B \subset R^+$, set

$$AB = \{xt : x \in A, t \in B\}.$$

We adopt the notation of Bhatia and Hajek [1]. $C(x) = xR^+$ is the *positive orbit* through x, $K(x) = \overline{C(x)}$ is the orbit closure and $L(x) = \bigcap\{K(xt) : t \in R^+\}$ is the *positive limit set* of x. For $r > 0$ and $A \subset X$ let $N_r(A) = \{x : \rho(x, A) < r\}$ denote the r neighborhood of A. The *positive prolongation* of x is the set $D(x) = \bigcap\{\overline{N_r(x)\, R^+} : r > 0\}$, and the *positive prolongational limit set* of x is the set $J(x) = \bigcap\{\overline{N_r(x)[t, \infty)} : r > 0, t \geqslant 0\}$.

A subset $M \subset X$ is *positively invariant* if $C(x) \subset M$ whenever $x \in M$. M is *positively minimal* if M is closed and positively invariant, but none of its nonempty proper subsets has these two properties. M is called *positively stable* if every neighborhood of M contains a positively invariant neighborhood of M. (This is called orbital stability by Bhatia and Hajek [1].) π is said to be *positively stable* if $K(x)$ is positively stable for each $x \in X$. We consider L, K, J, D as maps of X into 2^X. We say xt is *ultimately* in M if there exists $t_0 \in R^+$ such that $C(xt_0) \subset M$. A point x is said to be *attracted* to M if xt is ultimately in every neighborhood of M. As all concepts (e.g., limit sets, prolongations, stability) are positive, we shall henceforth omit the modifier "positive." Set $A(M) = \{x \in X : x \text{ is attracted to } M\}$.

Let $\mathscr{K}$ denote the collection of nonempty compact subsets of X and $h: \mathscr{K} \times \mathscr{K} \to R^+$ be the Hausdorff metric corresponding to ρ. That is, if $A, B \in \mathscr{K}$, $h(A, B) = \max\{\sup_{x \in A} \rho(x, B), \sup_{y \in B} \rho(A, y)\}$.

A map $S: X \to \mathscr{K}$ is called *upper semicontinuous* (USC) at x if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$S(y) \subset N_\epsilon(S(x)) \qquad \text{for all} \quad y \in N_\delta(x).$$

A map $S: X \to \mathscr{K}$ is called *lower semicontinuous* (LSC) at x if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$S(x) \subset N_\epsilon(S(y)) \qquad \text{for all} \quad y \in N_\delta(x).$$

Equivalently we have S is USC at x if for any sequence $x_n \to x$,

$$\sup\{\rho(y, S(x)) \colon y \in S(x_n)\} \to 0 \qquad \text{as} \quad n \to \infty.$$

S is LSC at x if for any sequence $x_n \to x$,

$$\sup\{\rho(y, S(x_n)) \colon y \in S(x)\} \to 0 \qquad \text{as} \quad n \to \infty.$$

From the definition of the Hausdorff metric on $\mathscr{K}$ one can easily show that a map S from X to $\mathscr{K}$ is continuous at x if and only if S is both USC and LSC at x.

In Sections 3, 4, 5, and 6, we will always assume that $K(x)$ is compact for each $x \in X$, and in Section 7 we will always assume that $D(x)$ is compact for each $x \in X$.

3. Continuity of the Map K

In this section we assume that $K(x)$ is compact for all $x \in X$.

3.1. Proposition. *If K is* USC *at x, then $K(x) = D(x)$. The converse is true provided $K(x)$ has a compact neighborhood.*

Proof. Suppose K is USC at x. Let $y \in D(x)$. Then there exist sequences $x_n \to x$, $t_n \in R^+$ such that $x_n t_n \to y$ as $n \to \infty$. As each $x_n t_n \in K(x_n)$,

$$\rho(x_n t_n , K(x)) \to 0.$$

Since $K(x)$ is closed, $y \in K(x)$. Hence $D(x) \subset K(x)$ and so $K(x) = D(x)$.

Conversely suppose $K(x) = D(x)$. As $K(x) = D(x) = \bigcap\{\overline{N_\delta(x) R^+} : \delta > 0\}$, for every $\epsilon > 0$ there exists $\delta > 0$ so that $\overline{N_\delta(x) R^+} \subset N_\epsilon(K(x))$. Thus $N_\epsilon(K(x)) \supset \overline{N_\delta(x) R^+} \supset K(y)$ for $y \in N_\delta(x)$. Hence K is USC at x.

We will next show that K is always LSC on X. First we obtain a new characterization of $K(x)$ and $L(x)$.

3.2. Definition. Let $x \in X$, and set

$$Q(x) = \{y \in X \mid \forall x_n \to x, \exists t_n \in R^+ \text{ such that } x_n t_n \to y\},$$

$$W(x) = \{y \in X \mid \forall x_n \to x, \exists t_n \uparrow +\infty \text{ such that } x_n t_n \to y\}.$$

3.3 Lemma. *$Q(x) = K(x)$ and $W(x) = L(x)$ for each $x \in X$.*

Proof. It can be shown that $Q(x) \subset K(x)$ and $W(x) \subset L(x)$ by picking the sequence $x_n = x$. We show the reverse inclusions. For any $t \in R^+$ we claim that $xt \in Q(x)$. In fact let $x_n \to x$, and choose $t_n = t$ for every n. Then $x_n t_n =$

$x_n t \to xt \in Q(x)$. Thus $C(x) \subset Q(x) \subset K(x)$. We need only show that $Q(x)$ is closed. But this is evident from the definition of $Q(x)$. To prove that $L(x) \subset W(x)$, it will be sufficient to show that $W(x) \supset \bigcap \{Q(xt) : t \geqslant 0\} = \bigcap \{Q(xn) : n$ a positive integer$\}$. So let $y \in Q(xn)$ for each n, and let $x_k \to x$. Then $x_k n \to xn$ for each n. For each n there exists $t_k^n \in R^+$ such that $(x_k n)\, t_k^n = x_k(n + t_k^n) \to y$ as $k \to \infty$. For each n choose k_n so that $\rho((x_k n)\, t_k^n, y) < 1/n$, $k \geqslant k_n$. We can assume that $k_1 < k_2 < \cdots$. We now construct a sequence $t_k \uparrow \infty$ so that $x_k t_k \to y$. For $1 \leqslant k < k_2$ set $t_k = 1 + t_k^1$. For $k_n \leqslant k < k_{n+1}$, $n \geqslant 2$, set $t_k = n + t_k^n$. Then $t_k \uparrow \infty$, and $x_k t_k \to y$ as $k \to \infty$. Thus $y \in W(x)$ and the proof of lemma is concluded.

3.4. Proposition. *The map K is* LSC *on X.*

Proof. Assume K is not LSC at x. Then there exist $\epsilon > 0$ and a sequence $x_n \to x$ such that $\sup\{\rho(y, K(x_n)) : y \in K(x)\} \geqslant \epsilon$ for each n. From compactness of $K(x)$ there exists a sequence $y_n \in K(x)$ such that $\rho(y_n, K(x_n)) \geqslant \epsilon$. We may assume that $y_n \to y \in K(x) = Q(x)$ for some y. There is a sequence $t_n \in R^+$ with $x_n t_n \to y$. Consequently $\rho(x_n t_n, y_n) \to 0$. As $x_n t_n \in K(x_n)$ for each n, $\rho(y_n, K(x_n)) \leqslant \rho(y_n, x_n t_n) \to 0$. This is impossible. Hence K must be LSC at x.

3.5. Proposition. *If $K(x)$ is compact for some $x \in X$, then the following are equivalent.*

(i) *K is* USC *at x.*

(ii) *K is continuous at x.*

3.6. *Remark.* It is easy to see that the definition of upper semicontinuity of the map $K(x)$ at x is equivalent to the one of stability of $K(x)$ at x which is given in [3]. Using the notion of stability of $K(x)$ at x, one can show that if $K(z)$ is stable at each point $z \in K(x)$, then $K(x)$ is stable provided $K(x)$ is compact. In terms of upper semicontinuity of K, one has the following: If K is USC on $K(x)$, then $K(x)$ is stable.

Combining Proposition 3.5 and Remark 3.6 we have

3.7. Theorem. *The map K is continuous on X if and only if $K(x)$ is stable for each $x \in X$.*

3.8. Lemma. *If $L(x)$ is stable, then so is $K(x)$.*

Proof. We will show that for every $\epsilon > 0$ there exists $\delta > 0$ such that $N_\delta(K(x))\, R^+ \subset N_\epsilon(K(x))$. Since $L(x)$ is stable, there is $\delta_1 > 0$ so that $N_{\delta_1}(L(x))\, R^+ \subset N_\epsilon(L(x))$. Then there exist $T > 0$, $\delta_2 > 0$ such that

$$x[T, \infty) \subset N_{\frac{1}{2}\delta_1}(L(x)) \quad \text{and} \quad N_{\delta_2}(x)\, T \subset N_{\frac{1}{2}\delta_1}(L(x)).$$

This can be done by compactness of $K(x)$ and continuity of π. Hence

$$N_{\delta_2}(x)[T, \infty) \subset N_{\frac{1}{2}\delta_1}(L(x))\, R^+ \subset N_\epsilon(L(x)) \subset N_\epsilon(K(x)).$$

We will show that for $x[0, T]$, there exists $\delta_3 > 0$ such that $N_{\delta_3}(x[0, T])\, R^+ \subset N_\epsilon(K(x))$. Assume this is not true. Then there exist $z_n \in X$ and $t_n \in R^+$ such that $\rho(z_n, x[0, T]) \to 0$, $z_n t_n \notin N_\epsilon(K(x))$. We may assume $z_n \to xt_0$ for some $0 \leqslant t_0 \leqslant T$. If $\overline{\lim}_{n\to\infty} t_n < \infty$, we may assume $t_n \to t < \infty$. Thus $z_n t_n \to (xt_0)t = x(t_0 + t) \notin N_\epsilon(K(x))$. This is impossible since $x(t_0 + t) \in K(x)$. Thus assume $t_n \uparrow \infty$. Since $xt_0(T - t_0) = xT \in N_{\frac{1}{2}\delta_1}(L(x))$ there exists $\delta_4 > 0$ such that $N_{\delta_4}(xt_0)(T - t_0) \subset N_{\frac{1}{2}\delta_1}(L(x))$. This implies

$$N_{\delta_4}(xt_0)[T - t_0, \infty) \subset N_{\frac{1}{2}\delta_1}(L(x))\, R^+ \subset N_\epsilon(L(x)) \subset N_\epsilon(K(x)).$$

There is $N_1 > 0$ so that $z_n \in N_{\delta_4}(xt_0)$, $t_n > T - t_0$, $n \geqslant N_1$, and $z_n t_n \in N_\epsilon(K(x))$. Again this is impossible. Hence there is $\delta_3 > 0$ so that

$$N_{\delta_3}(x[0, T])\, R^+ \subset N_\epsilon(K(x)).$$

On the other hand $\overline{C(xT)} \subset \overline{N_{\frac{1}{2}\delta_1}(L(x))} \subset N_{\frac{3}{4}\delta_1}(L(x))$. Let $0 < \delta_5 < \frac{1}{4}\delta_1$. Then $N_{\delta_5}(K(xT))\, R^+ \subset N_{\delta_5}(N_{\frac{3}{4}\delta_1}(L(x))\, R^+ \subset N_{\delta_1}(L(x))\, R^+ \subset N_\epsilon(L(x)) \subset N_\epsilon(K(x))$. Setting $\delta = \min(\delta_3, \delta_5)$, we have $N_\delta(K(x))\, R^+ = N_\delta(x[0, T] \cup K(xT))\, R^+ \subset N_\delta(x[0, T])])\, R^+ \cup N_\delta(K(xT))\, R^+ \subset N_\epsilon(K(x))$. Therefore $K(x)$ is stable.

The following Theorem is given in [4] for dynamical systems and does not assume the compactness of $K(x)$ for each point $x \in X$. It does assume that X is not a union of two disjoint open invariant sets and not minimal.

THEOREM. *π is positively stable if and only if the following conditions hold.*

(A) *The positive limit set $L(x)$ is nonempty compact minimal and positively stable for each $x \in X$.*

(B) *The union M of compact minimal subsets of X is a closed globally asymptotically stable set.*

For semidynamical systems, we have the following theorem with assumption $K(x)$ compact for each $x \in X$.

3.9. THEOREM. *The following are equivalent.*

(1) *π is stable.*

(2) *$L(x)$ is stable for each $x \in X$.*

(3) *The map K is continuous on X.*

Proof. From Theorem 3.7 (1) is equivalent to (3). From Lemma 3.8 (2) implies (1).

If π is stable, then each $L(x)$ is minimal. Thus $L(x)$ is stable for each $x \in X$.

4. Upper Semicontinuity of the Map L

4.1. Lemma. *Let $S: X \to \mathscr{K}$ be any map which is* USC *at x. Then for any sequence $x_n \to x$, $S(x) \cup \{\bigcup_{n=1}^{\infty} S(x_n)\}$ is compact.*

Proof. Let $x_n \to x$, and set $Y = S(x) \cup \{\bigcup_{n=1}^{\infty} S(x_n)\}$. For any sequence $\{y_n\} \subset Y$ we may assume $y_n \in S(x_n)$. Thus $\rho(y_n, S(x)) \to 0$. As $S(x)$ is compact we can find a subsequence $\{y_{n_k}\}$ and $y \in S(x)$ so that $y_{n_k} \to y$. Thus Y is compact.

4.2. Proposition. *If the map K is* USC *at x, then so is the map L.*

Proof. If $x \in L(x)$, then $L(x) = K(x)$ and hence the proposition is proved. So assume $x \notin L(x)$, and proceed by contradiction. Suppose L is not USC at x. We will first show that $x \in J(x)$.

There exist $\epsilon > 0$ and a sequence $x_n \to x$ such that

$$\sup\{\rho(w, L(x)) : w \in L(x_n)\} \geqslant 2\epsilon.$$

We may assume that $x \notin N_{2\epsilon}(L(x))$. Let $w_n \in L(x_n) - N_{2\epsilon}(L(x))$. Then for each x_n there exists $t_n > n$ with $\rho(x_n t_n, w_n) < \epsilon/n$. Hence $x_n t_n \notin N_\epsilon(L(x))$. As K is USC at x, then Proposition 3.1 and Lemma 4.1 imply that $x_n t_n$ (or some subsequence thereof) converges to some $z_1 \in K(x)$. Since for large n, $x_n t_n \notin N_\epsilon(L(x))$, then $z_1 \in K(x) - L(x)$. Thus there exists $\tau_1 \geqslant 0$ so that $z_1 = x\tau_1 \notin N_\epsilon(L(x))$. On the other hand there exists $T > \tau_1$ with $K(xT) \subset N_\epsilon(L(x))$. As $t_n \uparrow \infty$ we have $z_1 \in J(x)$. Now consider the sequence $x_n(t_n - \tau_1)$. By a similar argument to the above we may assume that $x_n(t_n - \tau_1) \to z_2$ for some $z_2 \in K(x)$. Thus $z_2 \tau_1 = z_1$. Moreover, $z_2 \in K(x) - L(x)$ so there exists some $\tau_2 \geqslant 0$ with $z_2 = x\tau_2$. Consequently $x\tau_1 = z_1 = z_2\tau_1 = (x\tau_2)\,\tau_1 = x(\tau_2 + \tau_1)$ implies $\tau_2 = 0$. In other words $x_n(t_n - \tau_1) \to z_2 = x\tau_2 = x$. Thus $x \in J(x)$.

We next show that $x \in L(x)$. As this case was previously ruled out, the proof of the proposition will be complete when $x \in L(x)$ is established. Choose $\delta > 0$ so that $N_\delta(x)T \subset N_\epsilon(L(x))$. For sufficiently large n we have $x_n, x_n(t_n - \tau_1) \in N_\delta(x)$ and $t_n - \tau_1 > T$. As $x_n T \in N_\epsilon(L(x))$, there exists s_n, $0 < s_n < t_n - \tau_1$ with $x_n s_n \in \partial N_\epsilon(L(x))$, $x_n(s_n, t_n - \tau_1] \cap N_\epsilon(L(x)) = \varnothing$. Then

$$\liminf_{n\to\infty} (t_n - \tau_1 - s_n) > 0,$$

for otherwise $x \in \partial N_\epsilon L(x))$. Let $0 < s < \liminf_{n\to\infty}(t_n - \tau_1 - s_n)$. As before we may assume that $x_n(t_n - \tau_1 - s) \to z_3$ for some $z_3 \in K(x) - L(x)$. Let $z_3 = x\tau_3$ for some $\tau_3 \geqslant 0$. Thus $x(\tau_3 + s) = z_3 s = \lim_{n\to\infty} x_n(t_n - \tau_1) = x$. This implies that $x \in L(x)$.

The following example shows that the converse of uroposition 4.2 is false.

4.3. EXAMPLE. In Fig. 4.1 every orbit approaches a point z which is a critical point. The map L is a constant map. However K is not USC at x. The space X is assumed to have the usual topology of the plane.

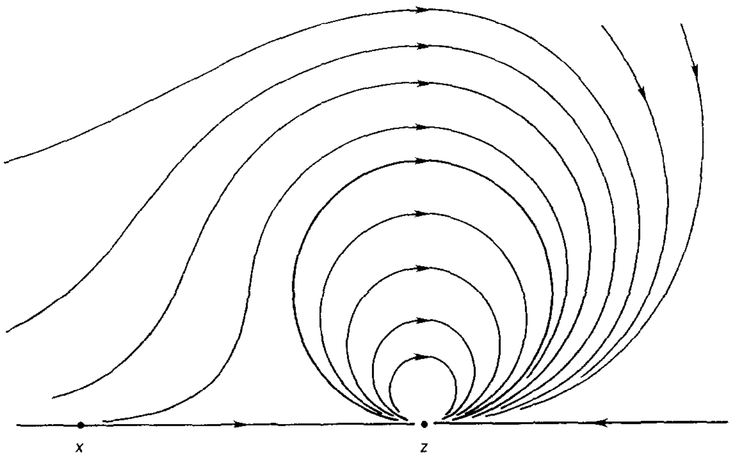


FIGURE 4.1

As one can see in Example 4.3, $L(x) = \{z\}$ is not stable. This leads us to formulate a weaker form of stability to characterize upper semicontinuity of the map L.

4.4. DEFINITION. A set $M \subset X$ is *eventually stable* if for every neighborhood W of M, there exists a neighborhood V of M such that for every $y \in V$, there is $T = T(y) \geqslant 0$ such that $C(yT) \subset W$. If T does not depend on $y \in V$, then M is said to be *uniformly eventually stable*.

It should be noted that sets M which are either stable or attractors are eventually stable. (M is an attractor if $A(M)$ is a neighborhood of M.) However, neither of the properties is necessary for eventual stability, as shown by the following example.

4.5. EXAMPLE. Consider a flow on the upper half plane given in Fig. 4.2 which is similar to Fig. 4.1. The set of critical points consists of $\{(x, y) : x = 0, 0 \leqslant y \leqslant \frac{1}{2}\}$. Let $P_0 = (-1, 0)$. Then $L(P_0) = \{(0, 0)\}$. This flow shows that $L(P_0)$ is eventually stable but neither stable nor an attractor.

4.6. PROPOSITION. *Given $x \in X$, the map L is* USC *on $L(x)$ if and only if $L(x)$ is minimal and eventually stable.*

Proof. Suppose L is USC on $L(x)$. Let $y \in L(x)$. We shall show $L(x) \subset L(y)$. For $y \in L(x)$ there exists a sequence $t_n \uparrow \infty$ such that $xt_n \to y$. Let $z \in L(x)$.

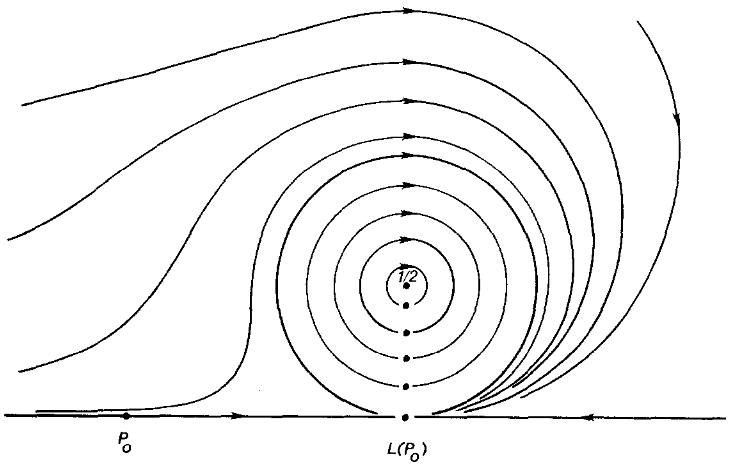


FIGURE 4.2

Then $z \in L(xt_n) = L(x)$ for each n. Since

$$\sup\{\rho(w, L(y)) \colon w \in L(xt_n)\} \to 0 \qquad \text{as} \quad n \to \infty,$$

$z \in L(y)$. Hence $L(x) \subset L(y)$, so $L(x)$ is minimal.

We now prove that $L(x)$ is eventually stable. Let $\epsilon > 0$ and $y \in L(x)$. Then there is $\delta_y > 0$ so that $N_\epsilon(L(y)) \supset L(z)$ for $z \in N_{\delta_y}(y)$. Thus $N_\epsilon(L(x)) \supset L(z)$ for $z \in N_{\delta_y}(y)$. By compactness of $L(x)$, there is a finite set of points $\{y_1, y_2, ..., y_n\} \subset L(x)$ such that $\bigcup_{i=1}^n N_{\delta_{y_i}}(y_i) \supset L(x)$. Setting $V = \bigcup_{i=1}^n N_{\delta_{y_i}}(y_i)$, $N_\epsilon(L(x)) \supset L(z)$ for each $z \in V$. This shows $L(x)$ is eventually stable.

Conversely let $L(x)$ be minimal and eventually stable. Let $y \in L(x)$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for each $z \in N_\delta(L(x))$ there is $\tau \geqslant 0$ with $C(y\tau) \subset N_{\frac{1}{2}\epsilon}(L(x))$. Thus for $z \in N_\delta(y) \subset N_\delta(L(x))$, $L(z) \subset \overline{N_{\frac{1}{2}\epsilon}(L(x))} \subset N_\epsilon(L(x)) = N_\epsilon(L(y))$. Hence L is USC on $L(x)$.

The next lemma is needed in order to establish global upper semicontinuity of the map L.

4.7. LEMMA. *The map L is* USC *on $L(x)$ if and only if L is* USC *on $A(L(x))$.*

Proof. As $L(x) \subset A(L(x))$, we need only prove that L is USC on $A(L(x))$ whenever L is USC on $L(x)$. So let $y \in A(L(x))$. According to [1, Theorem 8.12], $L(y) \subset L(x)$. As $L(x)$ is minimal from Proposition 4.6, $L(y) = L(x)$. Thus it will be sufficient to show that L is USC at $x \in A(L(x))$. As $L(x)$ is eventually stable, for every $\epsilon > 0$ there exists $\delta_1 > 0$ such that for each $z \in N_{\delta_1}(L(x))$, there is $T = T(z) \geqslant 0$ so that $C(zT) \subset N_{\frac{1}{2}\epsilon}(L(x))$. On the other hand there exist $\tau \geqslant 0$, $\delta > 0$ such that $x\tau \in N_{\delta_1}(L(x))$ and $N_\delta(x)\tau \subset N_{\delta_1}(L(x))$. Thus for every $y \in N_\delta(x)$, $L(y) = L(y\tau) \subset \overline{N_{\frac{1}{2}\epsilon}(L(x))} \subset N_\epsilon(L(x))$. Hence L is USC at x.

4.8. Lemma. *If $L(y)$ is eventually stable for each $y \in L(x)$, then $L(x)$ is positively minimal.*

Proof. Suppose $L(x)$ is not minimal. Then there exists a minimal set M in $L(x)$. Obviously, $x \notin M \subsetneq L(x)$. Let $y \in L(x) - M$, $0 < \epsilon < \rho(y, M)$ and $z \in M$. By assumption, there exists $\delta > 0$ such that for every $y' \in N_\delta(L(z))$, there exists $T = T(y') \geqslant 0$ such that $C(y'T) \subset N_{\frac{1}{2}\epsilon}(M) = N_{\frac{1}{2}\epsilon}(L(z))$. Since $z \in M \subset L(x)$, there is $\tau_1 \geqslant 0$ with $x\tau_1 \in N_\delta(L(z))$. Thus there is $T_1 = T_1(x\tau_1) \geqslant 0$ so that $x\tau_1[T_1, \infty) = x[\tau_1 + T_1, \infty) \subset N_{\frac{1}{2}\epsilon}(L(z))$. Hence $L(x) \subset \overline{N_{\frac{1}{2}\epsilon}(L(z))} \subset N_\epsilon(L(z))$. This is impossible since $y \in L(x)$ and $y \notin N_\epsilon(L(z))$. Thus $L(x)$ must be minimal.

Combining Proposition 4.6, Lemma 4.7, and Lemma 4.8 we have

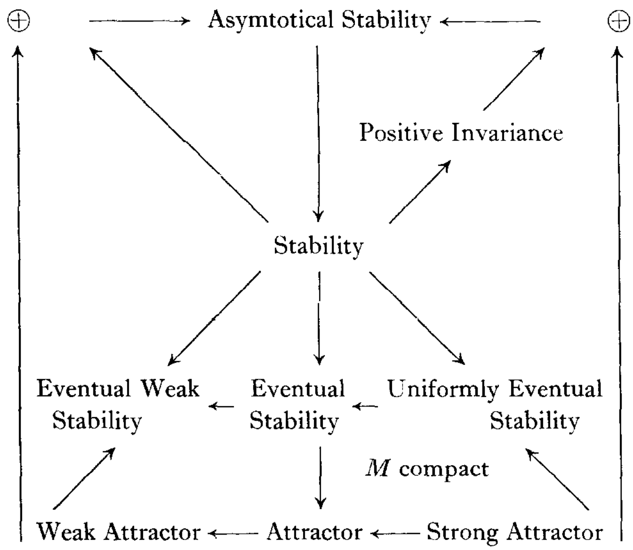
4.9. Theorem. *The map L is* USC *on X if and only if $L(x)$ is eventually stable for each $x \in X$.*

5. Lower Semicontinuity of the Map L

5.1. Definition. A set $M \subset X$ is *eventually weakly stable* if for every neighborhood W of M, there exists a neighborhood V of M such that for each $y \in V$ there is a sequence $t_n \uparrow \infty$ such that $yt_n \in W$.

There is a relationship between [1, Definition 8.7] and Definitions 4.4 and 5.1.

5.2. Proposition. *For any $M \subset X$ the following diagram holds.*



The above diagram is an extension of [1, Diagram 8.17].

5.3. PROPOSITION. *Let $L(y)$ be minimal for each $y \in L(x)$. Then the map L is* LSC *on $L(x)$ if and only if $L(z)$ is eventually weakly stable for every $z \in L(x)$.*

Proof. Suppose L is LSC on $L(x)$. Let $z \in L(x)$. Assume $L(z)$ is not eventually weakly stable. Then there exist $\epsilon > 0$, $y_n \to y \in L(z)$ so that $y_n\tau$ is ultimately in $X - N_\epsilon(L(z))$. Accordingly $L(y_n) \subset X - N_\epsilon(L(z))$. Thus

$$\sup\{\rho(w, L(y_n)) : w \in L(y)\} \geqslant \epsilon \qquad \text{with} \quad y_n \to y.$$

This is impossible since L is LSC at $y \in L(x)$.

Conversely suppose $L(z)$ is eventually weakly stable for each $z \in L(x)$. Assume L is not LSC at $y \in L(x)$. Then there exist $\epsilon > 0$ and $y_n \to y$ so that $\sup\{\rho(w, L(y_n)) : w \in L(y)\} \geqslant \epsilon$ for each n. Compactness of $L(y)$ ensures that there exist $N_1 > 0$ and $v \in L(y)$ with $\rho(v, L(y_n)) \geqslant \frac{1}{2}\epsilon$ for $n \geqslant N_1$. Since $L(y)$ is positively minimal, then for every $z \in L(y)$ there exists $T = T(z) \geqslant 0$ such that $zT \in N_{\frac{1}{4}\epsilon}(v)$. Thus there exists $r = r(z) > 0$ so that $N_r(z)T \subset N_{\frac{1}{4}\epsilon}(v)$. As $L(y)$ is compact, there is a finite covering $\bigcup_{i=1}^n N_{r_i}(z) \supset L(y)$, $z_i \in L(y)$ and $T_i \geqslant 0$ with $N_{r_i}(z_i)\, T_i \subset N_{\frac{1}{4}\epsilon}(v)$. Choose $\delta > 0$ so that

$$N_\delta(L(y)) \subset \bigcup_{i=1}^{n} N_{r_i}(z_i).$$

According to Lemma 3.3, there exists $t_n \uparrow \infty$ such that $y_n t_n \to v$.

By eventual weak stability of $L(y)$ there is $\alpha > 0$ so that for every $z' \in N_\alpha(L(y))$ there exists $t_k' \uparrow \infty$ such that $z't_k' \in N_\delta(L(y))$. Choose $N_2 \geqslant N_1$ so that $y_n t_n \in N_\alpha(L(y))$ for $n \geqslant N_2$. Fix $n \geqslant N_2$. Then there exists $t_k^n \uparrow \infty$ such that $(y_n t_n)\, t_k^n \in N_\delta(L(y))$. Compactness of $K(y_n)$ ensures that $L(y_n t_n) \cap N_\delta(L(y)) = L(y_n) \cap N_\delta(L(y)) \neq \varnothing$. Let $w_n \in L(y_n) \cap N_\delta(L(y))$. Then $\rho(w_n, L(y)) < \delta$, and there is z_i so that $w_n \in N_{\delta_i}(z_i)$ and $w_n T_i \in N_{\frac{1}{4}\epsilon}(v)$. Thus $L(y_n) \cap N_{\frac{1}{4}\epsilon}(v) \neq \varnothing$, $n \geqslant N_2$. Hence $\rho(v, L(y_n)) < \frac{1}{4}\epsilon$. This is impossible. Thus L is LSC on $L(x)$.

5.4. COROLLARY. *If the map L is* USC *on $L(x)$, then L is* LSC *on $L(x)$.*

Proof. The proof is immediate upon noting that eventual stability and upper semicontinuity of L on $L(x)$ implies the minimality of $L(x)$.

In the proof of Proposition 5.3, we notice that the minimality of $L(y)$ for each $y \in L(x)$ is needed in order to prove the "if part." The following example shows that even if $L(z)$ is eventually weakly stable for each $z \in L(x)$, L is not necessarily LSC on $L(x)$.

5.5. EXAMPLE. Let X be a torus. The dynamical system on X is an irrational flow with a single critical point z. For each $x' \in X$, we have either (1) $L(x') = X$ or (2) $L(x') = \{z\}$. We immediately see that the space X itself and $\{z\}$ are eventually weakly stable. Let $x \in X$ with

$$L(x) = X \qquad \text{and} \qquad d = \sup\{\rho(x', z') : x', z', \in X\} \qquad \text{and} \qquad 0 < \epsilon < d.$$

Then there exists $y_n \to x$ such that $L(y_n) = \{z\}$. Obviously $N_\epsilon(L(y_n)) = N_\epsilon(z) \not\supset L(x) = X$. In other words L is not LSC at $x \in L(x)$.

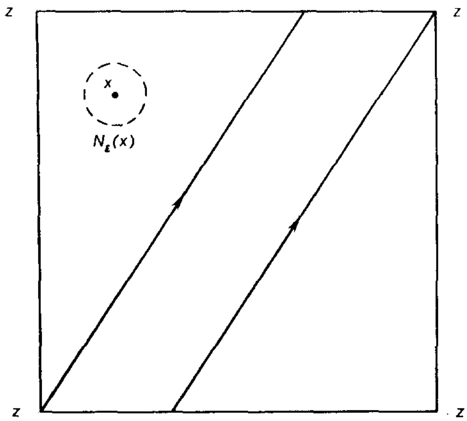


FIGURE 5.1

6. CONINUITY OF THE MAP L

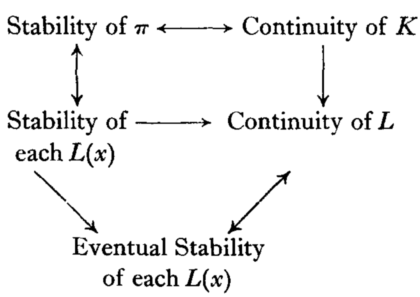
6.1. THEOREM. *A necessary and sufficient condition that the map L be continuous on $L(x)$ is that $L(x)$ is minimal and eventually stable.*

Proof. The proof is immediate from Proposition 4.6 and Corollary 5.4.

6.2. THEOREM *A necessary and sufficient condition that the map L be continuous on X is that $L(x)$ is eventually stable for each $x \in X$.*

Proof. As is given in the proof of Lemma 4.7, one can easily show the lower semicontinuity of the map L on $A(L(x))$ provided L is LSC and $L(x)$ is minimal. Thus the proof is immediate from Theorem 6.1.

6.3. *Remark.* From Theorem 3.9 and Theorem 6.2, we have the following diagram.



7. Continuity of the Maps J and D

As one can see in [3, Definition 2.3] the notion of upper semicontinuity of the map D at x is equivalent to the notion of stability of $D(x)$ at x. The following Proposition for the map D is given in [3, Theorem 2.7] in the different form.

7.1. Proposition. *If $J(x)$ has a compact neighborhood, then the maps J and D are* USC *at x.*

Proof. Let $\epsilon > 0$. Since $J(x) = \bigcap \overline{\{N_\delta(x)[t, \infty)} : \delta > 0, t \geqslant 0]$, there exist $r > 0$ and $\tau \geqslant 0$ so that $\overline{N_r(x)[\tau, \infty)} \subset N_\epsilon(J(x))$. Let $y \in N_r(x)$. There is an $\alpha > 0$ so that $N_\alpha(y) \subset N_r(x)$. Thus $J(y) \subset \overline{N_\alpha(y)[\tau, \infty)} \subset \overline{N_r(x)[\tau, \infty)} \subset N_\epsilon(J(x))$. Therefore J is USC.

Now for the map D. Let $\epsilon > 0$. Then there exist $r' > 0$, $\tau \geqslant 0$ so that $\overline{N_{r'}(x)[\tau, \infty)} \subset N_\epsilon(J(x)) \subset N_\epsilon(D(x))$. Since $x[0, \tau] \subset N_\epsilon(D(x))$, there exists r, $0 < r < r'$, so that $\overline{N_r(x[0, t])} \subset N_\epsilon(D(x))$. Thus $\overline{N_r(x)\, R^+} \subset N_\epsilon(D(x))$. The rest of the proof is similar to the one for J.

From Proposition 7.1, if the space X is locally compact, then J, D are USC on X. Without assuming any condition on X, we have the following characterization for upper semicontinuity of the maps D and J.

7.2. Lemma. *The map $D(J)$ is* USC *at x if and only if for every sequence $x_n \to x$, $D(x) \cup \{\bigcup_{n=1}^{\infty} D(x_n)\}(J(x) \cup \{\bigcup_{n=1}^{\infty} J(x_n)\})$ is compact.*

Proof. Lemma 4.1 takes care of the "only if" part. We now show the "if" part. Suppose D is not USC at x. Then there exists $\epsilon > 0$ and $x_n \to x$ so that $\sup\{\rho(w, D(x)) : w \in D(x_n)\} \geqslant \epsilon$ for each n. Thus there is $y_n \in D(x_n)$, and we may assume $y_n \to y \in Y$. Hence $\rho(y, D(x)) \geqslant \epsilon$. $y_n \in D(x_n)$ implies there exist $\{x_n{}^k\}$, $\{t_n{}^k\}$ so that $x_n{}^k \to x_n$, $x_n{}^k t_n{}^k \to y_n$ as $k \to \infty$. Thus for each n, there exists $k_n > 0$ such that $\rho(x_n^{k_n}, x_n) < 1/n$ and $\rho(x_n^{k_n} t_n^{k_n}, y_n) < 1/n$. This shows $x_n^{k_n} \to x$, $x_n^{k_n} t_n^{k_n} \to y$ as $n \to \infty$. Therefore $y \in D(x)$. This is impossible. Hence D is USC at x. The proof for J is similar.

7.3. *Remark and example.* We see from Lemma 4.1 that if a map $S: X \to \mathscr{K}$ is USC at x, then for any sequence $x_n \to x$, $S(x) \cup \{\bigcup_{n=1}^{\infty} S(x_n)\}$ is compact. However, the converse is not true. For instance, consider a flow on $X = [0, 1]$ with the usual topology of R. There are only two critical points 0, 1, so that every point $x(\neq 0, 1)$ approaches 1 positively and approaches 0 negatively. Let $0 < x_n < 1$ with $x_n \to 0$. Then $L(x_n) = \{1\}$, $L(0) = \{0\}$. Thus

$$\left\{ \bigcup_{n=1}^{\infty} L(x_n) \right\} \cup L(0) = \{0\} \cup \{1\}$$

is compact. However, L is neither USC nor LSC at 0.

Similarly

$$\left\{ \bigcup_{n=1}^{\infty} K(x_n) \right\} \cup K(0) = X$$

is compact, but K is not USC at 0.

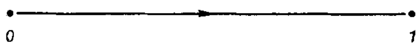


FIGURE 7.1

The following proposition shows some relationship between the map D and the map J.

7.4. PROPOSITION. *If D is* USC *at x, then so is J.*

Proof. Let $\epsilon > 0$. By the first part of Proposition 7.1 we need only show the existence of $r > 0$ and $\tau \geqslant 0$ such that $N_r(x)[\tau, \infty) \subset N_{\frac{1}{2}\epsilon}(J(x)) \subset N_\epsilon(J(x))$. Suppose this is not the case. Then there exist $x_n \to x$ and $t_n \uparrow \infty$ so that $x_n t_n \notin N_{\frac{1}{2}\epsilon}(J(x))$. Since D is USC at x, then $D(x) \cup \{\bigcup_{n=1}^{\infty} D(x_n)\}$ is compact. $\{x_n t_n\} \subset D(x) \cup \{\bigcup_{n=1}^{\infty} D(x_n)\}$ implies there exists a subsequence $\{n_k\}$ of $\{n\}$ and $y \in D(x) \cup \{\bigcup_{n=1}^{\infty} D(x_n)\}$ such that $x_{n_k} t_{n_k} \to y$ as $k \to \infty$ with $x_{n_k} \to x$, $t_{n_k} \uparrow \infty$. Hence $y \in J(x)$. This is impossible since $x_{n_k} t_{n_k} \notin N_{\frac{1}{2}\epsilon}(J(x))$. Therefore J must be USC at x.

The following example shows that the converse is not true in general. The example was suggested by N. P. Bhatia.

7.5. EXAMPLE. The phase space

$$X = \{(x, y) \in R^2 : x \leqslant 0\} - \{(x, y) \in R^2 : y = 0, -1 < x < 0\}$$

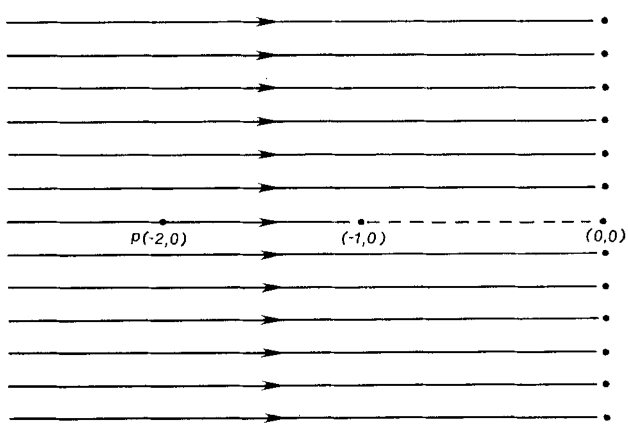


FIGURE 7.2

has the usual topology of R^2. The set $\{(x, y) : x = 0\} \cup \{(-1, 0)\}$ consists of critical points. The remaining flow is parallel in the direction indicated in Fig. 7.2.

Let $p = (-2, 0)$ and $p_n \to p$ where $p_n = (x_n, y_n)$, $y_n \neq 0$. Then $J(p) = \{(-1, 0)\} \cup \{(0, 0)\}$ and $J(p_n) = \{(0, y_n)\} \to \{(0, 0)\} \subset J(p)$. Thus J is USC at p. However, D is not USC at p.

We now characterize the lower semicontinuity of D and J.

7.6. Proposition. *If D is* LSC *at x, then for any $x_n \to x$, $\delta_n > 0$, $y \in D(x)$, there exist $x_n' \to x$, $t_n' \geqslant 0$ such that $\rho(x_n, x_n') < \delta_n$, $x_n' t_n' \to y$.*

Proof. Let $x_n \to x$, $\delta_n > 0$, $y \in D(x)$. Then $\rho(y, D(x_n)) \to 0$ as $n \to \infty$, and there exists $z_n \in D(x_n)$ so that $\rho(y, z_n) \to 0$. For each z_n there exist $x_n{}^k \to x_n$, $t_n{}^k \geqslant 0$ such that $x_n{}^k t_n{}^k \to z_n$ as $k \to \infty$. Choose $k_n > 0$ so that

$$\rho(x_n^{k_n} t_n^{k_n}, z_n) < 1/n, \qquad \rho(x_n^{k_n}, x_n) < \min(1/n, \delta_n) \qquad \text{for each} \quad n.$$

This shows that $x_n^{k_n} \to x$ and $x_n^{k_n} t_n^{k_n} \to y$ as $n \to \infty$. Setting $x_n' = x_n^{k_n}$, $t_n' = t_n^{k_n}$, we have $x_n' \to x$, $x_n' t_n' \to y$ with $\rho(x_n', x_n) < \delta_n$.

The following Proposition will give the converse of Proposition 7.6 under the assumption of upper semicontinuity of the map D.

7.7 Proposition. *Let D be* USC *on X. If for every $x_n \to x$, $\delta_n > 0$, $y \in D(x)$, there exist $x_n' \to x$, $t_n' \geqslant 0$ with $\rho(x_n, x_n') < \delta_n$, $x_n' t_n' \to y$, then D is* LSC *at x.*

Proof. Suppose D is not LSC at x. Then there exist $\epsilon > 0$, $x_n \to x$ so that $\sup\{\rho(w, D(x_n)) : w \in D(x)\} \geqslant \epsilon$. Thus there exist y, $y_n \in D(x)$, with $y_n \to y$ such that $\rho(y_n, D(x_n)) \geqslant \epsilon$. Then there is an $N > 0$ so that $\rho(y, D(x_n)) \geqslant \frac{1}{2}\epsilon$ for $n \geqslant N$. From the definition of D, $D(x_n) = \overline{\bigcap_{k=1}^{\infty} N_{k^{-1}}(x_n) R^+}$. We first show that for each n, there is $k_n > 0$ such that $N_{k_n^{-1}}(x_n) R^+ \cap N_{\frac{1}{4}\epsilon}(y) = \varnothing$.

Suppose $N_{k^{-1}}(x_n) R^+ \cap N_{\frac{1}{4}\epsilon}(y) \neq \varnothing$ for each k. Then there exist $z_k \in N_{k^{-1}}(x_n)$ and $t_k \in R^+$ so that $z_k t_k \in N_{\frac{1}{4}\epsilon}(y)$. Thus $z_k \to x_n$ as $k \to \infty$. Now $\{\bigcup_{k=1}^{\infty} D(z_k)\} \cup D(x_n)$ is compact. Hence we may assume

$$z_k t_k \to z \in \left\{ \bigcup_{k=1}^{\infty} D(z_k) \right\} \cup D(x_n) \qquad \text{for some} \quad z.$$

As $z_k \to x_n$, then $z \in D(x_n)$ so $\rho(z, y) \leqslant \frac{1}{4}\epsilon$. This shows $\rho(D(x_n), y) \leqslant \frac{1}{4}\epsilon$. This is impossible. Therefore for each n there exists $k_n > 0$ such that

$$N_{k_n^{-1}}(x_n) R^+ \cap N_{\frac{1}{4}\epsilon}(y) = \varnothing.$$

Consider $\{x_n\}$, $\{k_n^{-1}\}$, $y \in D(x)$. Then there exist $x_n' \to x$, $t_n' \in R^+$ such that $\rho(x_n, x_n') < k_n^{-1}$, $x_n' t_n' \to y$. Therefore there exists $N' \geqslant N$ so that for $n \geqslant N'$ $\rho(x_n' t_n', y) < \frac{1}{4}\epsilon$. This implies $N_{k_n^{-1}}(x_n) R^+ \cap N_{\frac{1}{4}\epsilon}(y) \neq \varnothing$ for $n \geqslant N'$. Again this is impossible. Therefore D must be LSC at x.

Combining Propositions 7.6 and 7.7, we have:

7.8. THEOREM. *Let D be* USC *on X. Then D is* LSC *at x if and only if for any sequences $x_n \to x$, $\delta_n > 0$, and any point $y \in D(x)$, there exist sequences $x_n' \to x$, $t_n' \in R^+$ such that $\rho(x_n, x_n') < \delta_n$, $x_n' t_n' \to y$.*

For the map J, we need a slightly stronger condition given in the following Proposition. Its proof is similar to that of Proposition 7.6.

7.9. PROPOSITION. *If J is* LSC *at x, then for any $x_n \to x$, $\delta_n > 0$, $y \in J(x)$, $t_n \uparrow \infty$, there exist $x_n' \to x$, $t_n' \geqslant t_n$ such that $\rho(x_n, x_n') < \delta_n$ and $x_n' t_n' \to y$.*
Just as Proposition 7.7 we have

7.10. PROPOSITION. *Let D be* USC *on X. If for any $x_n \to x$, $\delta_n > 0$, $t_n \uparrow \infty$, $y \in J(x)$, there exist $x_n' \to x$, $t_n' \geqslant t_n$ such that $\rho(x_n, x_n') < \delta_n$, $x_n' t_n' \to y$, then J is* LSC *at x.*

Combining Propositions 7.9 and 7.10 we have

7.11. THEOREM. *Let D be* USC *on X. Then J is* LSC *at x if and only if for any $x_n \to x$, $t_n \uparrow \infty$, $y \in J(x)$, $\delta_n > 0$, there exist $x_n' \to x$, $t_n' \geqslant t_n$ such that $\rho(x_n, x_n') < \delta$, $x_n' t_n' \to y$.*

In Propositions 7.7 and 7.10 we assumed upper semicontinuity of the map D in order to show the lower semicontinuity of the maps D and J. The following example shows that Propositions 7.7 and 7.10 do not hold without assuming upper semicontinuity of the map D.

7.12. EXAMPLE. Let

$$\begin{aligned} X = \{(x, y) \in R^2 \mid x \leqslant 0, 0 \leqslant y \leqslant 1\} \\ -\{(x, y) \in R^2 : -1 < x \leqslant 0, y = (\tfrac{1}{2})^n, n = 0, 1 \ldots\} \\ -\{(x, y) \in R^2 : -1 < x < 0, y = 0\} \end{aligned}$$

with the usual topology of R^2.

The set of critical points consists of $\{(x, y) \in R^2 : x = -1, y = (\frac{1}{2})^n, n = 0, 1 \ldots\} \cup \{(-1, 0)\} \cup \{(x, y) \in R^2 : x = 0, 0 \leqslant y \leqslant 1, y \neq (\frac{1}{2})^n, n = 0, 1 \ldots\}$. The phase portrait is given in Fig. 7.3. The flow is parallel in the direction indicated. Let $p = (-1, 0)$, $\tilde{p}_n = (-1, (\frac{1}{2})^n)$, $n = 1, 2, \ldots$. Then $D(\tilde{p}_n) = \{\tilde{p}_n\}$, $D(p) = \{p\} \cup \{(0, 0)\} = J(p)$. Thus D is not LSC at p. On the other hand for any $p_n \to p$, $\delta_n > 0$, $p \in D(p)$, $p_n' t_n' \to p$ where $p_n' = p_n$, $t_n' = 0$, or for any $p_n \to p$, $\delta_n > 0$, $(0, 0) \in D(p)$ there exist p_n', $t_n' \uparrow \infty$ such that $\rho(p_n, p_n') < \delta_n$, $p_n' \notin \{(x, y) \in R^2 : y = (\frac{1}{2})^k, k = 0, 1, \ldots\} \cup \{(x, y) \in R^2 : y = 0\}$ with $p_n' t_n' \to (0, 0)$. Similarly the map J is not LSC at p. However for any $p_n \to p$,

$\delta_n > 0$, $p \in J(p)$ (or $(0, 0) \in J(p)$), $t_n \uparrow \infty$, there exist $p_n' \to p$, $t_n' \geqslant t_n$ such that $\rho(p_n', p_n) < \delta_n$, $p_n' t_n' \to p$ (or $(0, 0)$).

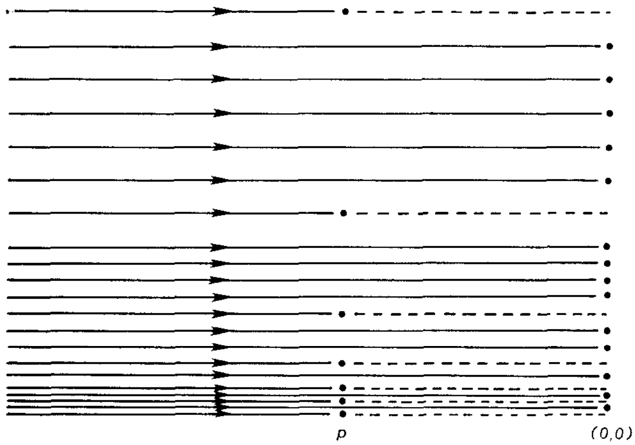


FIGURE 7.3

Finally we give the following proposition, which is similar to Proposition 3.1. The proof is similar to that of Proposition 3.4.

7.13. PROPOSITION. *If $D(x) = K(x)$ at x, then D is* LSC *at x.*

7.14. COROLLARY. *If $K(x)$ is stable, then D is* LSC *at x. Moreover if π is stable, then D is continuous on X.*

The following example shows that the converse of Corollary 7.14 is not true in general.

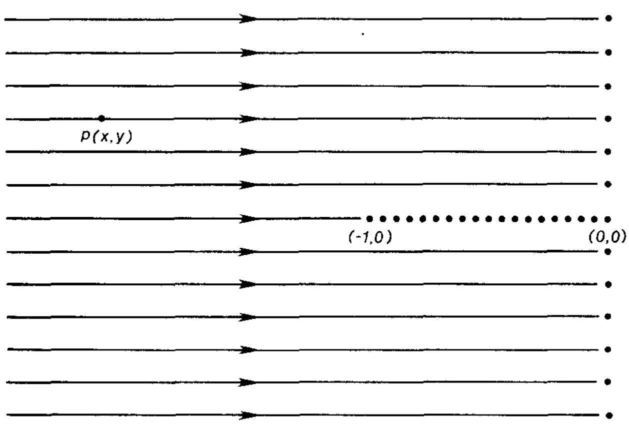


FIGURE 7.4

7.15. Example. The phase space X is given by $\{(x, y) \in R^2 : x \leqslant 0\}$ with the usual topology of R^2. The set

$$\{(x, y) \in R^2 : -1 \leqslant x \leqslant 0, y = 0\} \cup \{(x, y) \in R^2 : x = 0\}$$

consists of critical points. The remaining flow is parallel in the direction indicated. For any point $p = (x, y)$, $y \neq 0$, $D(p) = \{(x', y') : y' = y, x \leqslant x' \leqslant 0\}$. Obviously D is continuous on X. However for any $q = (x, 0)$, $-1 \leqslant x < 0$, $K(q) = \{q\}$ is not stable.

Acknowledgment

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