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White Noise Approach to Multiparameter Stochastic Integration

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In this paper we will set up the Hida theory of generalized Wiener functionals using $\mathscr{S}^*(\mathbb{R}^d)$, the space of tempered distributions on $\mathbb{R}^d$, and apply the theory to multiparameter stochastic integration. With the partial ordering on $\mathbb{R}^d_+$: $(s_1, ..., s_d) < (t_1, ..., t_d)$ if $s_i < t_i$, $1 \leqslant i \leqslant d$, the Wiener process

$$W((t_1, ..., t_d), x) = \langle x, 1_{[0,t_1) \times \cdots \times [0,t_d)} \rangle, \; x \in \mathscr{S}^*(\mathbb{R}^d)$$

is a generalization of a Brownian motion and there is the Wiener–Ito decomposition: $L^2(\mathscr{S}^*(\mathbb{R}^d)) = \sum_{n=0}^{\infty} \oplus K_n$, where K_n is the space of n-tuple Wiener integrals. As in the one-dimensional case, there are the continuous inclusions

$$(L^2)^+ \subset L^2(\mathscr{S}^*(\mathbb{R}^d)) \subset (L^2)^-,$$

and $(L^2)^-$ is considered the space of generalized Wiener functionals. We prove that the multidimensional Ito stochastic integral is a special case of an element of $(L^2)^-$. For $d=2$ the Ito integral is not sufficient for representing elements of $L^2(\mathscr{S}^*(\mathbb{R}^2))$. We show that the other stochastic integral involved can also be realized in the Hida setting. For $F \in \mathscr{S}^*(\mathbb{R})$ we will define $F(W(s, t), x)$ as an element of $(L^2)^-$ and obtain a generalized Ito formula.

1. INTRODUCTION

Using the Hida theory of generalized Brownian functionals, Kubo and Takenaka in [9], have shown that for a nonanticipating process φ such that $E \int_a^b |\varphi(t, \omega)|^2 \, dt < \infty$,

$$\int_a^b \varphi(t, \omega) \, dB(t, \omega) = \int_a^b \partial_t^* \varphi(t) \, dt,$$

where the integral on the left is the Ito stochastic integral with respect to the Brownian motion $B(t, \omega)$. In [8] Kubo defined the composition

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$F(B(t))$, $t>0$, for the tempered distribution F and obtained an Ito formula for these generalized Brownian functionals:

$$F(B(b))-F(B(a))=\int_a^b \partial_t^* F'(B(t))\,dt+\tfrac{1}{2}\int_a^b F''(B(t))\,dt.$$

The purpose of this paper is to set up the Hida theory on $\mathscr{S}^*(\mathbb{R}^d)$ for $d>1$ and exhibit an Ito formula for the generalized Wiener functional $F(W(s,t))$, $(s,t)\in\mathbb{R}_+^2$, where W is a two-dimensional parameter Wiener process and $F\in\mathscr{S}^*(\mathbb{R})$. Towards this end we will show that

$$\int_{[a,b]^d}\varphi(\mathbf{t},\omega)\,dW(\mathbf{t},\omega)=\int_{[a,b]^d}\partial_{\mathbf{t}}^*\varphi(\mathbf{t})\,d\mathbf{t}.$$

Here the integral on the left is the d-dimensional Ito integral. In dimension two another stochastic integral, $[\int_{[a,b]^2\times[a,b]^2}]\,\varphi(\mathbf{z},\mathbf{z}',\omega)\;dW(\mathbf{z})\,dW(\mathbf{z}')$, defined by Wong and Zakai [13], is required for the Ito formula. This integral exists provided

(1) φ is measurable with respect to the σ-field generated by $\{W(\mathbf{u}))$; $(a,a)<\mathbf{u}<\mathbf{z}\vee\mathbf{z}'\}$, and

(2) $E\int_{[a,b]^2}\int_{[a,b]^2}|\varphi(\mathbf{z},\mathbf{z}',\omega)|^2\,d\mathbf{z}\,d\mathbf{z}'<\infty$.

We will show that

$$\left[\int_{[a,b]^2\times[a,b]^2}\right]\varphi(\mathbf{z},\mathbf{z}',\omega)\,dW(\mathbf{z})\,dW(\mathbf{z}')$$

$$=\int_{[a,b]^2}\int_{[a,b]^2}\partial_{\mathbf{z}'}^*\partial_{\mathbf{z}}^* 1_G(\mathbf{z},\mathbf{z}')\,\varphi(\mathbf{z},\mathbf{z}')\,d\mathbf{z}\,d\mathbf{z}',$$

where $G=\{\mathbf{z},\mathbf{z}'\in[a,b]^2;\ \mathbf{z}\text{ and }\mathbf{z}'\text{ are unordered}\}$.

Finally we will prove the Ito formula: For $0<a_1<b_1$ and $0<a_2<b_2$,

$$F(W(b_1,b_2))-F(W(b_1,a_2))-(F(W(a_1,b_2))-F(W(a_1,a_2)))$$

$$=\int_{a_2}^{b_2}\int_{a_1}^{b_1}\partial_{(s,t)}^* F'(W(s,t))\,ds\,dt$$

$$+\tfrac{1}{2}\Bigg[\int_0^{b_2}\int_0^{b_1}\int_0^{b_2}\int_0^{b_1}-\int_0^{a_2}\int_0^{b_1}\int_0^{a_2}\int_0^{b_1}$$

$$+\int_0^{a_2}\int_0^{a_1}\int_0^{a_2}\int_0^{a_1}-\int_0^{b_2}\int_0^{a_1}\int_0^{b_2}\int_0^{a_1}$$

$$\times\,\partial_{(u,v)}^*\partial_{(s,t)}^* 1_G((s,t),(u,v))\,F''(W((s,t)\vee(u,v))\,ds\,dt\,du\,dv\Bigg]$$

$$+\frac{b_1}{2}\int_{a_2}^{b_2} F''(W(b_1, t))\,dt - \frac{a_1}{2}\int_{a_2}^{b_2} F''(W(a_1, t))\,dt$$

$$+\int_{a_1}^{b_1}\left[\int_{a_2}^{b_2}\int_0^u \partial^*_{(s,t)}\frac{t}{2}F'''W(u, t))\,ds\,dt\right]du.$$

Section 1 develops the parts of the multidimensional Hida theory which are pertinent to our work. We will define the space of generalized Wiener functionals $(L^2)^-$ and the operators $\partial_{\mathbf{t}}$ and $\partial^*_{\mathbf{t}}$. In Section 2 we will recall the definitions of the d-dimensional Ito stochastic integral and the Wong–Zakai integral. We show in Section 3 that both these integrals can be realized in the Hida setting. Section 4 includes the definition of $F(W(\mathbf{t}))$ and the proof of the Ito formula.

2. The Hida Theory of Generalized Wiener Functionals

Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth real valued functions on $\mathbb{R}^d$. The dual space $\mathscr{S}^*(\mathbb{R}^d)$ of $\mathscr{S}(\mathbb{R}^d)$ consists of the tempered distributions. Thus we have the continuous inclusions $\mathscr{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathscr{S}^*(\mathbb{R}^d)$. The cannonical bilinear form connecting $\mathscr{S}$ and $\mathscr{S}^*$ will be denoted by $\langle x, \xi\rangle$, $x \in \mathscr{S}^*$ and $\xi \in \mathscr{S}$. Also, $\mathscr{S}$ is a countably Hilbert nuclear space [7], i.e., $\mathscr{S}$ is topologized by a Family $\{\|\cdot\|_p;\ p = 1, 2, ...\}$ of Hilbertian norms with the following structure: Let $\mathscr{S}_p$ be the completion of $\mathscr{S}$ with respect to the norm $\|\cdot\|_p$. Then

$$\mathscr{S} = \bigcap_p \mathscr{S}_p \subset \cdots \mathscr{S}_2 \subset \mathscr{S}_1 \subset \mathscr{S}_0 = L^2(\mathbb{R}^d) \subset \mathscr{S}^*_1 \subset \mathscr{S}^*_2 \subset \cdots \subset \bigcup_p \mathscr{S}^*_p = \mathscr{S}^*,$$

where the inclusions $\mathscr{S}_{p+1} \subset \mathscr{S}_p$ are Hilbert–Schmidt, and the inclusions $\mathscr{S} \subset \mathscr{S}_p$ and $\mathscr{S}^*_p \subset \mathscr{S}^*$ are continuous. For the nth Hermite polynomial $H_n(x) = (-1)^n \exp(x^2)\, D^n_x \exp(-x^2)$, let $h_n(x) = (2^n n!\,\sqrt{\pi})^{-1/2}$ $H_n(x)\exp(-x^2/2)$. Then $\{h_n\}_{n=0}^\infty$ is a complete orthonormal system in $L^2(\mathbb{R})$ and $\{h_{n_1 n_2 \cdots n_d}\}_{n_1, n_2, ..., n_d = 0}^\infty$, where $h_{n_1 n_2 \cdots n_d}(t_1, ..., t_d) = \prod_{i=1}^d h_{n_i}(t_i)$, is a c.o.n.s. in $L^2(\mathbb{R}^d)$. Let us denote this basis by $\{\xi_n\}_{n=0}^\infty$. For $f \in \mathscr{S}(\mathbb{R}^d)$ and p an integer, $\|f\|_p^2 = \sum_{n=0}^\infty (2n+1)^{2p}\,(f, \xi_n)^2$, the inner product on $L^2(\mathbb{R}^d)$ being denoted by $(\cdot, \cdot)$. Note that $\mathscr{S}_p = \{f \in L^2(\mathbb{R}^d) : \|f\|_p < \infty\}$. Also, it is true that $\mathscr{S}^*_p = \mathscr{S}_{-p}$.

The probability space $(\mathscr{S}^*(\mathbb{R}^d), \mathscr{B}, \mu)$ determined by the characteristic functional $C(\xi) = \exp(-\frac{1}{2}\|\xi\|^2)$ is called the d-dimensional white noise space. Here, $\|\cdot\|$ is the $L^2(\mathbb{R}^d)$ norm. For $\xi \in \mathscr{S}$, the random variable

$\langle x, \xi \rangle$ on $(\mathscr{S}^*(\mathbb{R}^d), \mathscr{B}, \mu)$ is normally distributed with mean 0 and variance $\|\xi\|^2$. Moreover, since $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, if $f \in L^2(\mathbb{R}^d)$, $\langle \cdot, f \rangle$ is Gaussian with mean 0 and variance $\|f\|^2$. We thus have that $\{\langle x, 1_A \rangle$: A is a Borel subset of $\mathbb{R}^d$ with finite Lebesgue measure$\}$ is a normal random measure on $(\mathscr{S}^*(\mathbb{R}^d), \mathscr{B}, \mu)$.

For a positive integer d, set $\mathbb{R}^d_+ = \{\mathbf{a} = (a_1, ..., a_d) \in \mathbb{R}^d : a_i \geqslant 0\}$ and denote by $\mathbb{R}_{\mathbf{a}}$ the rectangle $\prod_{i=1}^d [0, a_i[$. For $\mathbf{a} \in \mathbb{R}^d_+$ and $x \in \mathscr{S}^*(\mathbb{R}^d)$, $W(\mathbf{a}, x) = \langle x, 1_{\mathbb{R}_{\mathbf{a}}} \rangle$ is a Wiener process with d-dimensional time. For $d = 1$ we have the Brownian motion $B(t, x) = \langle x, 1_{[0,t[} \rangle$.

THEOREM 1. (*Wiener–Ito decomposition*). *$L^2(\mathscr{S}^*(\mathbb{R}^d))$ has the direct orthogonal decomposition $L^2(\mathscr{S}^*(\mathbb{R}^d)) = \sum_{n=0}^{\infty} \oplus K_n$, where $K_0 = \mathbb{R}$ and for $n > 1$, K_n is the space of n-tuple Wiener integrals based on the normal random measure $W_A = \langle x, 1_A \rangle$ mentioned above; i.e., each φ in K_n has the form*

$$\varphi(x) := I_n(f) = \int_{(\mathbb{R}^d)^n} f(\mathbf{u}_1, ..., \mathbf{u}_n)\, dW(\mathbf{u}_1, x) \cdots dW(\mathbf{u}_n, x),$$

where $f \in \hat{L}^2((\mathbb{R}^d)^n)$: the $L^2((\mathbb{R}^d)^n)$ functions which are symmetric in the $\mathbf{u}_i$'s. Moreover, $(\varphi, \psi)_{L^2(\mathscr{S}^*(\mathbb{R}^d))} = n!(f, g)_{L^2((\mathbb{R}^d)^n)}$, where $\psi \in K_n$ is the multiple Wiener integral of g.

For the proof of this theorem see Ito [6]. It is also shown there that if $\{\eta_i\}_{i=1}^k$ is an orthonormal set in $L^2(\mathbb{R}^d)$ and $p_1 + \cdots + p_k = n$, then

$$\begin{aligned} \int_{(\mathbb{R}^d)^n} & \eta_1(\mathbf{u}_1) \cdots \eta_1(\mathbf{u}_{p_1})\, \eta_2(\mathbf{u}_{p_1+1}) \cdots \eta_2(\mathbf{u}_{p_1+p_2}) \cdots \\ & \times \eta_k(\mathbf{u}_{p_1+\cdots+p_k-1}) \cdots \eta_k(\mathbf{u}_n)\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \\ & = (\sqrt{2})^{-n} \prod_{i=1}^k H_{p_i}\left(\int \eta_i\, dW/\sqrt{2}\right), \end{aligned}$$

where H_{p_i} is the Hermite polynomial of degree p_i. This product is called a Fourier–Hermite polynomial of degree n based on $\{\eta_i\}$. Then for a c.o.n.s.$\{\eta_i\}$ of $L^2(\mathbb{R}^d)$, K_n is spanned by the Fourier–Hermite polynomials of degree n based on $\{\eta_i\}$. Note that for a Borel set A with finite Lebesgue measure $\int_A dW(\mathbf{u}) = \langle x, 1_A \rangle$. Thus, for $f \in L^2(\mathbb{R}^d)$, $\langle x, f \rangle = \int_{\mathbb{R}^d} f(\mathbf{u})\, dW(\mathbf{u})$.

In the Hida theory, functionals in $L^2(\mathscr{S}^*(\mathbb{R}^d))$ are studied by means of a transformation to a space of functionals on $\mathscr{S}(\mathbb{R}^d)$. Once this transformation is made, the resulting functional has a very nice form which allows us to work in $\sum_{n=0}^{\infty} \oplus \sqrt{n!}\, \hat{L}^2((\mathbb{R}^d)^n)$.

DEFINITION 1. The S-transform on $L^2(\mathscr{S}^*(\mathbb{R}^d))$ is defined by

$$(S\varphi)(\xi)=\int_{\mathscr{S}^*}\varphi(x+\xi)\,d\mu(x),$$

where $\varphi\in L^2(\mathscr{S}^*(\mathbb{R}^d))$ and $\xi\in\mathscr{S}$.

The image of the S-transform is a space of functionals on $\mathscr{S}$ and is topologized so as to make S a Hilbert space isomorphism, see [5]. Just as in the one-dimensional case, when restricted to K_n, the S-transform has a particularly simple form.

THEOREM 2 (*Integral representation theorem*). *Suppose $\varphi\in K_n$ is of the form*

$$\varphi(x)=\int_{(\mathbb{R}^d)^n} f(\mathbf{u}_1,...,\mathbf{u}_n)\,dW(\mathbf{u}_1,x)\cdots dW(\mathbf{u}_n,x),$$

with $f\in\hat{L}^2((\mathbb{R}^d)^n)$. Then

$$(S\varphi)(\xi)=\int_{(\mathbb{R}^d)^n} f(\mathbf{u}_1,...,\mathbf{u}_n)\,\xi(\mathbf{u}_1)\cdots\xi(\mathbf{u}_n)\,d\mathbf{u}_1\cdots d\mathbf{u}_n.$$

Remark. The proof is essentially that of the one-dimensional case found in [5] for the transformation $\mathscr{T}$.

Generalized Wiener functionals arise in the following way. For $\alpha\in\mathbb{R}$, let $H^\alpha(\mathbb{R}^{nd})$ be the Sobolev space of order α over $\mathbb{R}^{nd}$, i.e., $H^\alpha(\mathbb{R}^{nd})=\{f\in\mathscr{S}^*(\mathbb{R}^{nd})\colon \int_{\mathbb{R}^{nd}}(1+|\lambda|^2)^\alpha\,|(\mathscr{F}f)(\lambda)|^2\,d\lambda<\infty\}$, $\mathscr{F}$ being the Fourier transform. This is a Hilbert space with dual $H^{-\alpha}(\mathbb{R}^{nd})$.

Define $K_n^{(n)}$ to be the elements of K_n which are n-tuple Wiener integrals of functions in $\hat{H}^{(nd+1)/2}(\mathbb{R}^{nd})$. Here $\hat{H}^{(nd+1)/2}(\mathbb{R}^{nd})=H^{(nd+1)/2}(\mathbb{R}^{nd})\cap\hat{L}^2((\mathbb{R}^d)^n)$. We then have

$$\begin{array}{ccccc}
K_n^{(n)} & \hookrightarrow & K_n & \hookrightarrow & K_n^{(-n)}\\
\updownarrow & & \updownarrow & & \updownarrow\\
\sqrt{n!}\,\hat{H}^{(nd+1)/2}(\mathbb{R}^{nd}) & \hookrightarrow & \sqrt{n!}\,\hat{L}^2((\mathbb{R}^d)^n) & \hookrightarrow & \sqrt{n!}\,\hat{H}^{-(nd+1)/2}(\mathbb{R}^{nd})
\end{array}$$

and $K_n^{(-n)}$ is defined as the space of generalized n-tuple Wiener integrals of elements in $\hat{H}^{-(nd+1)/2}(\mathbb{R}^{nd})$. More precisely, for $\varphi=I_n(f)$, $f\in\hat{L}^2((\mathbb{R}^d)^n)$, define $|||\varphi|||=(\sqrt{n!})\,\|f\|_{\hat{H}^{-(nd+1)/2}(\mathbb{R}^{nd})}$ and let $K_n^{(-n)}$ be the completion of K_n with respect to $|||\cdot|||$. We will write formally that for $\varphi\in K_n^{(-n)}$,

$$\varphi(x)=I_n(f)=\int_{(\mathbb{R}^d)^n} f(\mathbf{u}_1,...,\mathbf{u}_n)\,dW(\mathbf{u}_1,x)\cdots dW(\mathbf{u}_n,x),$$

where $f \in \hat{H}^{-(nd+1)/2}(\mathbb{R}^{nd})$. $K_n^{(-n)}$ can be viewed as the dual of $K_n^{(n)}$ with the pairing $\langle \cdot, \cdot \rangle$: For $\varphi \in K_n^{(n)}$ represented by $f \in \hat{H}^{(nd+1)/2}(\mathbb{R}^{nd})$ and $\psi \in K_n^{(-n)}$ represented by $g \in \hat{H}^{-(nd+1)/2}(\mathbb{R}^{nd})$, $\langle \psi, \varphi \rangle = n! \langle g, f \rangle$.

EXAMPLE 1. For $H_n(x, \sigma^2) = (1/n!)(-\sigma^2)^n \exp(x^2/2\sigma^2) D_x^n \exp(-x^2/2\sigma^2)$ and $\Delta = (\varepsilon, \varepsilon, ..., \varepsilon) \in \mathbb{R}_+^d$, consider $\varphi_\Delta = n! H_n[(1/\varepsilon^d)\langle x, 1_{[\mathbf{t}, \mathbf{t}+\Delta[}\rangle, 1/\varepsilon^d]$, $n > 2$, which is in K_n. In fact, φ_Δ is the multiple Wiener integral of $(1/(\varepsilon^d)^n)$ $1_{[\mathbf{t}, \mathbf{t}+\Delta[}(\mathbf{u}_1) \cdots 1_{[\mathbf{t}, \mathbf{t}+\Delta[}(\mathbf{u}_n)$, where $[\mathbf{t}, \mathbf{t}+\Delta[= \{\mathbf{x} \in \mathbb{R}_d^+;\ \mathbf{t} < \mathbf{x} < \mathbf{t} + \Delta\}$. Since $\hat{\otimes}_n (1/\varepsilon^d) 1_{[\mathbf{t}, \mathbf{t}+\Delta[} \to \hat{\otimes}_n \delta_{\mathbf{t}}$ in $\hat{H}^{-(nd+1)/2}(\mathbb{R}^{nd})$ as $\varepsilon \to 0$, $\lim_{\varepsilon \to 0} \varphi_\Delta$ is in $K_n^{(-n)}$ and is represented by $\hat{\otimes}_n \delta_{\mathbf{t}}$. In particular, for $n = 1$, $\lim_{\varepsilon \to 0} (1/\varepsilon^d)\langle x, 1_{[\mathbf{t}, \mathbf{t}+\Delta[}\rangle = I_1(\delta_{\mathbf{t}})$ is in $K_1^{(-1)}$.

Define $(L^2)^+ = \sum_{n=0}^{\infty} \oplus K_n^{(n)}$ and $(L^2)^- = \sum_{n=0}^{\infty} \oplus K_n^{(-n)}$, where $K_0^{(0)}$ is the real number system. Thus we have

$$(L^2)^+ \subset L^2(\mathscr{S}^*) \subset (L^2)^-.$$

$(L^2)^+$ is called the space of test functionals and $(L^2)^-$ is called the space of generalized functionals. For $\varphi = \sum_{n=0}^{\infty} \varphi_n$ in $(L^2)^-$ and $\psi = \sum_{n=0}^{\infty} \psi_n$ in $(L^2)^+$, $\langle \varphi, \psi \rangle = \sum_{n=0}^{\infty} \langle \varphi_n, \psi_n \rangle$. We can see that the S-transform extends to $(L^2)^-$.

In the Hida theory $\{\dot{B}(t), t \in \mathbb{R}\} := \{\delta_t; t \in \mathbb{R}\}$ is viewed as a coordinate system in $\mathscr{S}^*$ so as to take time into account. Derivatives are then taken with respect to this coordinate system. This idea can be carried over to higher dimensions.

DEFINITION 2. Let U be the S-transform of $\varphi \in (L^2)^-$. Suppose the first variation of U at ξ is given by

$$(\delta U)_\xi(\eta) = \int_{\mathbb{R}^d} U'(\xi, \mathbf{u})\, \eta(\mathbf{u})\, d\mathbf{u};\ \eta \in \mathscr{S}(\mathbb{R}^d).$$

If $U'(\cdot, \mathbf{t})$ is an S-transform, then $\partial_{\mathbf{t}} \varphi$ is defined to be the generalized Wiener functional with S-transform $U'(\cdot, \mathbf{t})$, i.e.,

$$S(\partial_{\mathbf{t}} \varphi)(\xi) = U'(\xi, \mathbf{t}).$$

The adjoint $\partial_{\mathbf{t}}^*$ of $\partial_{\mathbf{t}}$ is defined by $\langle \partial_{\mathbf{t}}^* \psi, \varphi \rangle = \langle \psi, \partial_{\mathbf{t}} \varphi \rangle$, $\psi \in (L^2)^-$, $\varphi \in (L^2)^+$. Thus $\partial_{\mathbf{t}}$ is a linear operator whose domain is not all of $(L^2)^-$. The following, however, can easily be shown.

THEOREM 3. (A) *For* $\varphi = \int_{(\mathbb{R}^d)^n} f(\mathbf{u}_1, ..., \mathbf{u}_n)\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n)$ *in* $K_n^{(n)}$,

$$\partial_{\mathbf{t}} \varphi = n \int_{(\mathbb{R}^d)^{n-1}} f(\mathbf{t}, \mathbf{u}_1, ..., \mathbf{u}_{n-1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_{n-1}).$$

(B) *For* $\varphi = \int_{(\mathbb{R}^d)^n} f(\mathbf{u}_1, ..., \mathbf{u}_n)\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n)$ *in* $K_n^{(-n)}$,

$$\partial_{\mathbf{t}}^* \varphi = \int_{(\mathbb{R}^d)^{n+1}} (\delta_{\mathbf{t}} \hat{\otimes} f)(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_{n+1}).$$

Here, $\hat{\otimes}$ *indicates symmetric tensor product. Consequently,*

(1) $\partial_{\mathbf{t}}: K_n^{(n)} \to K_{n-1}^{(n-1)}$ *and* $\partial_{\mathbf{t}}^*: K_n^{(-n)} \to K_{n+1}^{-(n+1)}$ *and,*

(2) *for* $\varphi = \sum_{n=0}^{\infty} \varphi_n$ *in* $(L^2)^-$,

$$S(\partial_{\mathbf{t}}^* \varphi)(\xi) = \sum_{n=0}^{\infty} S(\partial_{\mathbf{t}}^* \varphi_n)(\xi) = \sum_{n=0}^{\infty} \xi(\mathbf{t})\, S\varphi_n(\xi) = \xi(\mathbf{t})\, S\varphi(\xi).$$

DEFINITION 3. Suppose $\varphi(\boldsymbol{\tau}) \in (L^2)^-$, $\boldsymbol{\tau} \in [a, b]^d$, and it is true that

(1) For every ψ in $(L^2)^+$, $\boldsymbol{\tau} \to \langle \varphi(\boldsymbol{\tau}), \psi \rangle$ is measurable and integrable, and

(2) $\psi \to \int_{[a,b]^d} \langle \varphi(\boldsymbol{\tau}), \psi \rangle\, d\boldsymbol{\tau}$ is a continuous linear functional on $(L^2)^+$.

Then $\int_{[a,b]^d} \varphi(\boldsymbol{\tau})\, d\boldsymbol{\tau}$ is defined as the element of $(L^2)^-$ such that

$$\left\langle \int_{[a,b]^d} \varphi(\boldsymbol{\tau})\, d\boldsymbol{\tau}, \psi \right\rangle = \int_{[a,b]^d} \langle \varphi(\boldsymbol{\tau}), \psi \rangle\, d\boldsymbol{\tau},$$

where $\psi \in (L^2)^+$.

One can similarly define $\int_{([a,b]^d)^n} \varphi(\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_n)\, d\mathbf{t}_1\, d\mathbf{t}_2 \cdots d\mathbf{t}_n$. These are just integrals of the Pettis type [3]. It is easy to show that if $\int_{[a,b]^d} \|\varphi(\boldsymbol{\tau})\|_-^2\, d\boldsymbol{\tau} < \infty$, then $\int_{[a,b]^d} \varphi(\boldsymbol{\tau})\, d\boldsymbol{\tau}$ exists and

$$\left\| \int_{[a,b]^d} \varphi(\boldsymbol{\tau})\, d\boldsymbol{\tau} \right\|_-^2 \leqslant (b-a)^d \int_{[a,b]^d} \|\varphi(\boldsymbol{\tau})\|_-^2\, d\boldsymbol{\tau}.$$

EXAMPLE 2. Suppose that $\varphi(\mathbf{t}) = \int_{(\mathbb{R}^d)^n} f(\mathbf{t}, \mathbf{u}_1, ..., \mathbf{u}_n)\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n)$, where f is in $L^2(([a, b]^d \times (\mathbb{R}^d)^n)$. The proof of existence of $\int_{[a,b]^d} \partial_{\mathbf{t}}^* \varphi(\mathbf{t})\, d\mathbf{t}$ and the calculation of its value goes as follows:

$$\partial_{\mathbf{t}}^* \varphi(\mathbf{t}) = \int_{(\mathbb{R}^d)^{n+1}} (\delta_{\mathbf{t}} \hat{\otimes} f(\mathbf{t}))(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_{n+1})$$

and

$$\int_{[a,b]^d} \|\partial_{\mathbf{t}}^* \varphi(\mathbf{t})\|_-^2\, d\mathbf{t} = \int_{[a,b]^d} (n+1)!\, \|\delta_{\mathbf{t}} \hat{\otimes} f(\mathbf{t})\|_-^2\, d\mathbf{t}.$$

But,

$$\|\delta_{\mathbf{t}} \hat{\otimes} f(\mathbf{t})\|_-^2 = \int_{\mathbb{R}^{d(n+1)}} \frac{|\mathscr{F}(\delta_{\mathbf{t}} \hat{\otimes} f(\mathbf{t}))(\boldsymbol{\lambda})|^2}{(1+|\boldsymbol{\lambda}|^2)^{(d(n+1)+1)/2}}\, d\boldsymbol{\lambda}$$

$$= \int_{\mathbb{R}^{d(n+1)}} \frac{|\exp(-2\pi i(\lambda_1, ..., \lambda_d)\cdot \mathbf{t})\, \mathscr{F}(f(\mathbf{t}))(\lambda_{d+1}, ..., \lambda_{nd+d})|^2}{(1+|\boldsymbol{\lambda}|^2)^{((n+1)d+1)/2}}\, d\boldsymbol{\lambda}$$

$$= \int_{\mathbb{R}^{d(n+1)}} \frac{|\mathscr{F}(f(\mathbf{t}))(\lambda_{d+1}, ..., \lambda_{nd+d})|^2}{(1+|\boldsymbol{\lambda}|^2)^{((n+1)d+1)/2}}\, d\boldsymbol{\lambda}$$

$$\leqslant \text{const} \times \int_{(\mathbb{R}^d)^n} \frac{|\mathscr{F}(f(\mathbf{t}))(\lambda_1, ..., \lambda_{nd})|^2}{(1+|\boldsymbol{\lambda}|^2)^{(nd+1)/2}}\, d\boldsymbol{\lambda}$$

$$= \text{const} \times \|\varphi(\mathbf{t})\|_-^2.$$

Thus,

$$\int_{[a,b]^d} \|\delta_{\mathbf{t}} \hat{\otimes} f(\mathbf{t})\|_-^2 \, d\mathbf{t} < \infty$$

and $\int_{[a,b]^d} \partial_{\mathbf{t}}^* \varphi(\mathbf{t})\, d\mathbf{t}$ exists. Furthermore, we have that

$$S\left(\int_{[a,b]^d} \partial_{\mathbf{t}}^* \varphi(\mathbf{t})\, d\mathbf{t}\right)(\xi)$$

$$= \int_{[a,b]^d} S(\partial_{\mathbf{t}}^* \varphi(\mathbf{t}))(\xi)\, d\mathbf{t} = \int_{[a,b]^d} \xi(\mathbf{t})\, S(\varphi(\mathbf{t}))(\xi)\, d\mathbf{t}$$

$$= \int_{[a,b]^d} \xi(\mathbf{t}) \int_{(\mathbb{R}^d)^n} f(\mathbf{t}, \mathbf{u}_1, ..., \mathbf{u}_n)\, \xi(\mathbf{u}_1) \cdots \xi(\mathbf{u}_n)\, d\mathbf{u}_1 \cdots d\mathbf{u}_n\, d\mathbf{t}$$

$$= \int_{(\mathbb{R}^d)^{n+1}} 1_{[a,b]^d}(\mathbf{t}) f(\mathbf{t}, \mathbf{u}_1, ..., \mathbf{u}_n)\, \xi(\mathbf{t})\, \xi(\mathbf{u}_1) \cdots \xi(\mathbf{u}_n)\, d\mathbf{t}\, d\mathbf{u}_1 \cdots d\mathbf{u}_n.$$

Letting $h(\mathbf{u}_1, ..., \mathbf{u}_{n+1})$ be the symmetrization of $1_{[a,b]^d}(\mathbf{u}_1) f(\mathbf{u}_1, ..., \mathbf{u}_{n+1})$, we then have that

$$\int_{[a,b]^d} S(\partial_{\mathbf{t}}^* \varphi(\mathbf{t}))(\xi)\, d\mathbf{t}$$

$$= \int_{(\mathbb{R}^d)^{n+1}} h(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, \xi(\mathbf{u}_1) \cdots \xi(\mathbf{u}_{n+1})\, d\mathbf{u}_1 \cdots d\mathbf{u}_{n+1}$$

$$= S\left[\int_{(\mathbb{R}^d)^{n+1}} h(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_{n+1})\right](\xi).$$

Hence,

$$\int_{[a,b]^d} \partial_{\mathbf{t}}^* \varphi(\mathbf{t})\, d\mathbf{t} = \int_{(\mathbb{R}^d)^{n+1}} h(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_{n+1}).$$

3. Stochastic Integrals of Processes with Multidimensional Time Parameter

Let $<$ be the partial ordering on $\mathbb{R}_+^d$ where $(x_1, ..., x_d) < (y_1, ..., y_d)$ if and only if $x_i \leqslant y_i$; $1 \leqslant i \leqslant d$. Assume $\{W(\mathbf{t}, \omega);\ \mathbf{t} \in \mathbb{R}_+^d\}$ is a Wiener process on a probability space $\{\Omega, \mathscr{F}, P)$ and denote by $\mathscr{F}_{\mathbf{a}}$ the σ-field generated by $\{W(\mathbf{t});\ \mathbf{t} < \mathbf{a}\}$.

Let $T = [a, b] \subset \mathbb{R}_+^1$ and consider the Wiener process $\{W(\mathbf{t});\ \mathscr{F}_{\mathbf{t}},\ \mathbf{t} \in T^d\}$. Cairoli [1] defined the following Ito-type stochastic integral for $d = 2$ which extends to any $d > 2$. Assume $\varphi(\mathbf{t}, \omega)$ satisfies the following conditions:

(1) $\varphi(\mathbf{t}, \omega)$ is a bimeasurable function of $(\mathbf{t}, \omega)$ with respect to $\mathscr{G} \otimes \mathscr{F}$, where $\mathscr{G}$ denotes the σ-field of Borel sets in T^d.

(2) For each $\mathbf{t} \in T^d$, $\varphi(\mathbf{t}, \omega)$ is $\mathscr{F}_{\mathbf{t}}$-measurable. In this case φ is said to be non-anticipating.

(3) $\int_{T^d} E\varphi^2(\mathbf{t}, \omega)\, d\mathbf{t} < \infty$.

First suppose that φ is simple, i.e., $\varphi(\mathbf{t}, \omega) = \varphi_v(\omega)$, $\mathbf{t} \in \Delta_v$, $v = 1, 2, ..., k$, and $\varphi = 0$ elsewhere, and that Δ_v are disjoint rectangles $\Delta_v = \prod_{i=1}^d [a_i^v, b_i^v[\subset T^d$. Then the Ito integral of φ is

$$\int_{T^d} \varphi(\mathbf{t}, \omega)\, dW(\mathbf{t}, \omega) := \sum_v \varphi_v(\omega)\, \Delta_v W(\omega),$$

where for a rectangle $\Delta = \prod_{i=1}^d [a_i, b_i[$, $\Delta W(\omega) = \sum_{\mathbf{x}} (-1)^{\pi(\mathbf{t})} W(\mathbf{t}, \omega)$, the sum being taken over the 2^d vertices $\{\mathbf{t};\ t_i = a_i \text{ or } b_i\}$ and $\pi(\mathbf{t})$ is the number of b_i's in $\mathbf{t}$. The definition of $\int_{T^d} \varphi(\mathbf{t})\, dW(\mathbf{t})$ is then extended to non-simple φ by a standard completion argument.

Wong and Zakai [13] have defined a second type of stochastic integral which we will now describe. Both of these are necessary to represent elements of $L^2(\mathscr{S}^*(\mathbb{R}^2))$.

For (s, t), $(u, v) \in T^2$ we will use $(s, t) \vee (u, v)$ to denote $(\max\{s, u\}, \max\{v, t\})$. Let $G = \{(\mathbf{z}, \mathbf{z}') \in T^2 \times T^2;\ \mathbf{z} \text{ and } \mathbf{z}' \text{ are unordered}\}$. Suppose that $\psi(\omega, \mathbf{z}, \mathbf{z}')$ is a function defined on $\Omega \times T^2 \times T^2$ satisfying

(1) $\psi(\omega, \mathbf{z}, \mathbf{z}')$ is jointly measurable with respect to $\mathscr{F} \otimes \mathscr{G} \otimes \mathscr{G}$.

(2) For each $\mathbf{z}, \mathbf{z}' \in T^2$, the function $\psi(\omega, \mathbf{z}, \mathbf{z}')$ is measurable with respect to $\mathscr{F}_{\mathbf{z} \vee \mathbf{z}'}$.

(3) $E \int_{T^2 \times T^2} \psi^2(\mathbf{z}, \mathbf{z}')\, d\mathbf{z}\, d\mathbf{z}' < \infty$.

Assume that $\psi(\omega, \mathbf{z}, \mathbf{z}')$ is simple: $\psi(\omega, \mathbf{z}, \mathbf{z}') = \alpha(\omega)$ for $\mathbf{z} \in \Delta_1$ and $\mathbf{z}' \in \Delta_2$ and zero elsewhere. For $n = 1, 2, ...,$ partition T into segments of length $(b-a)/2^n$ and let P_n be the partition induced on T^2, with partition points $\{\mathbf{z}_{ij}\}_{i,j=1}^{2^n-1}$, and let $\Delta_{ij} = [\mathbf{z}_{ij}, \mathbf{z}_{i+1,j+1}[$. For $\Delta_{ij} W = W(\mathbf{z}_{i+1,j+1}) - W(\mathbf{z}_{i+1,j}) + W(\mathbf{z}_{ij}) - W(\mathbf{z}_{i,j+1})$, define

$$I_2^n(\psi) = \sum_{\substack{i,j=1 \\ k,m=1}}^{2^n-1} \psi(\mathbf{z}_{ij}, \mathbf{z}_{km})\, 1_G(\mathbf{z}_{ij}, \mathbf{z}_{km})\, \Delta_{ij} W \Delta_{km} W.$$

It is shown in [13], that $I_2^n(\psi)$ converges in $L^2(\Omega)$ as $n \to \infty$. The integral is then defined to be this limit:

$$\left[\int_{T^2 \times T^2}\right] \psi(\mathbf{z}, \mathbf{z}')\, dW(\mathbf{z})\, dW(\mathbf{z}') := \lim_{n \to \infty} \text{in q.m. } I_2^n(\psi).$$

Note that if $\Delta_1 \times \Delta_2 \subset G$, then $I_2(\psi) = \alpha \Delta_1 W \Delta_2 W$. The definition can now be extended to all functions satisfying the above conditions by approximating with linear combinations of simple functions.

4. Stochastic Integrals Using the Hida Theory

For a non-anticipating process $\varphi(t, x)$, $t \in [a, b] \subset \mathbb{R}_+^1$ and $x \in \mathscr{S}^*(\mathbb{R})$, such that $E \int_a^b |\varphi(t)|^2\, dt < \infty$, the Ito integral $\int_a^b \varphi(t, x)\, dB(t, x)$ with respect to the Brownian motion $B(t, x) = \langle x, 1_{[0,t[} \rangle$ is defined, and Kubo and Takenaka [9] have shown that

$$\int_a^b \varphi(t, x)\, dB(t, x) = \int_a^b \partial_t^* \varphi(t)\, dt.$$

The result is also true for d-dimensional time, where $B(t, x)$ is replaced with the Wiener process $W(\mathbf{t}, x) = \langle x, 1_{[\mathbf{0}, \mathbf{t}[} \rangle$, $\mathbf{t} \in \mathbb{R}_+^d$ and $x \in \mathscr{S}^*(\mathbb{R}^d)$. The integral on the right, however, may exist whether or not φ is non-anticipating.

Theorem 4. *Suppose $\varphi(\boldsymbol{\tau})$ is a stochastic process on $L^2(\mathscr{S}^*)$ such that $E \int_{[a,b]^d} |\varphi(\boldsymbol{\tau})|^2\, d\boldsymbol{\tau} < \infty$ and $E \int_{[a,b]^d} \int_{[a,b]^d} |\partial_{\boldsymbol{\tau}'} \varphi(\boldsymbol{\tau})\, \overline{\partial_{\boldsymbol{\tau}} \varphi(\boldsymbol{\tau}')}|\, d\boldsymbol{\tau}\, d\boldsymbol{\tau}' < \infty$, then $\int_{[a,b]^d} \partial_{\boldsymbol{\tau}}^* \varphi(\boldsymbol{\tau})\, d\boldsymbol{\tau}$ exists and $E |\int_{[a,b]^d} \partial_{\boldsymbol{\tau}}^* \varphi(\boldsymbol{\tau})\, d\boldsymbol{\tau}|^2 = E \int_{[a,b]^d} |\varphi(\boldsymbol{\tau})|^2\, d\boldsymbol{\tau} + E \int_{[a,b]^2} \int_{[a,b]^2} \partial_{\boldsymbol{\tau}'} \varphi(\boldsymbol{\tau})\, \overline{\partial_{\boldsymbol{\tau}} \varphi(\boldsymbol{\tau}')}\, d\boldsymbol{\tau}\, d\boldsymbol{\tau}'$.*

Remark. The proof of this theorem for the one-dimensional case can be found in [11] and it requires no change for higher dimensions.

THEOREM 5. *If* $\varphi(\tau, x)$, $\tau \in \mathbb{R}^d$, $x \in \mathcal{S}^*(\mathbb{R}^d)$, $d > 1$ *is a nonanticipating process such that* $E\int_{[a,b]^d} |\varphi(\tau)|^2\, d\tau < \infty$, *then*

$$\int_{[a,b]^d} \varphi(\tau)\, dW(\tau) = \int_{[a,b]^d} \partial_\tau^* \varphi(\tau)\, d\tau,$$

where the integral on the left is the d-dimensional Ito integral.

Note. Since $\varphi(\tau)$ is nonanticipating, $E\int_{[a,b]^2}\int_{[a,b]^2} \partial_{\tau'}\varphi(\tau)\, \overline{\partial_\tau \varphi(\tau')}\, d\tau\, d\tau' = 0$ and we have that $\int_{[a,b]^d} \partial_\tau^* \varphi(\tau)\, d\tau$ exists.

The proof of this theorem for $d = 1$ hinges on being able to write a multiple Wiener integral as an iterated stochastic integral. This, of course, cannot be done in general for higher dimensions because $<$ is not a linear ordering. We do, however, have the following

LEMMA 1. *Let* $g(\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{n+1}) = 1_{[(a,...,a)\mathbf{u}_{n+1}[^n}(\mathbf{u}_1, ..., \mathbf{u}_n)\, f(\mathbf{u}_{n+1}, \mathbf{u}_1, ..., \mathbf{u}_n)$, *where* $f \in L^2(([a, b]^d)^{n+1})$ *and* $\hat{g}$ *is the symmetrization of* g. *Then*

$$\int_{([a,b]^d)^{n+1}} g(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_{n+1})$$
$$= (n+1) \int_{[a,b]^d} \left[\int_{[(a,...,a),\mathbf{u}_{n+1}[^n} \hat{g}(\mathbf{u}_1, ..., \mathbf{u}_{n+1}) \times dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \right] dW(\mathbf{u}_{n+1}).$$

Proof. Because of the cumbersome notation we prove the lemma only for $d = 2$. By [6] we may assume that

$$f(\mathbf{u}_{n+1}, \mathbf{u}_1, ..., \mathbf{u}_n) = 1_{A_{n+1} \times A_1 \times \cdots \times A_n}(\mathbf{u}_{n+1}, \mathbf{u}_1, ..., \mathbf{u}_n),$$

where the A_i's are disjoint rectangles in $[a, b]^2$. Let

$$D = \{(x, y);\ (x, y) \in [a, b]^2 \text{ and } x \leqslant y\}$$
$$D_i = [c_{i-1}, c_i[, \qquad i = 1, ..., 2^m - 1, \text{ where } c_i = a + i(b-a)/2^m,$$
$$E_i = [c_i, b], \qquad i = 1, ..., 2^m - 1.$$

Then $\lim_{m \to \infty} \sum_{i=1}^{2^m - 1} 1_{D_i \times E_i} = 1_D$, and since

$$1_{[(a,a),(s,t)]^n}((x_1, y_1), ..., (x_n, y_n))$$
$$= 1_D(x_1, s) \cdots 1_D(x_n, s)\, 1_D(y_1, t) \cdots 1_D(y_n, t),$$

we have that, pointwise and in $L^2([a,b]^2)^{n+1}$, $g((x_1, y_1), ..., (x_n, y_n), (s, t))$ is the limit of $\sum_{i_1,...,i_n,j_1,...,j_n} [\prod_{p=1}^n 1_{D_{i_p}\times E_{i_p}}(x_p, s)\, 1_{D_{j_p}\times E_{j_p}}(y_p, t)]\, 1_{A_1}(x_1, y_1) \cdots 1_{A_{n+1}}(s, t)$ or

$$\sum_{\substack{i_1,...,i_n \\ j_1,...,j_n}} \left[\prod_{k=1}^n 1_{(D_{i_k}\times D_{j_k})\cap A_k}(x_k, y_k)\right] 1_{(E_{i_1}\times E_{j_1})\cap \cdots \cap (E_{i_n}\times E_{j_n})\cap A_{n+1}}(s, t).$$

For sets A and B, $A < B$ indicates that for any x in A, $x < y$ for every y in B. Note that $(D_{i_k}\times D_{j_k})\cap A_k < (E_{i_1}\times E_{j_1})\cap \cdots \cap (E_{i_n}\times E_{j_n})\cap A_{n+1}$ for $k = 1, ..., n$. Therefore, we can see that

$$g(\mathbf{u}_1, ..., \mathbf{u}_{n+1}) = \lim \sum_{\mathbf{z}_{i_1},...,\mathbf{z}_{i_{n+1}}} 1_{\Delta_{\mathbf{z}_{i_1}}}(\mathbf{u}_1) \cdots 1_{\Delta_{\mathbf{z}_{i_n}}}(\mathbf{u}_n)\, 1_{\Delta_{\mathbf{z}_{i_{n+1}}}}(\mathbf{u}_{n+1}),$$

where $\mathbf{z}_{i_k}$ is the lower left-hand corner of $\Delta_{\mathbf{z}_{i_k}}$, all the rectangles $\Delta_{\mathbf{z}_{i_1}}, ..., \Delta_{\mathbf{z}_{i_{n+1}}}$ are disjoint, and $\Delta_{\mathbf{z}_{i_k}} < \Delta_{\mathbf{z}_{i_{n+1}}}$ for $k = 1, ..., n$. Let

$$h(\mathbf{u}_1, ..., \mathbf{u}_{n+1}) = 1_{\Delta_{\mathbf{z}_1}}(\mathbf{u}_1) \cdots 1_{\Delta_{\mathbf{z}_n}}(\mathbf{u}_n)\, 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{u}_{n+1})$$

with the above conditions. Then

$$\hat{h}(\mathbf{u}_1, ..., \mathbf{u}_{n+1}) = \frac{1}{(n+1)!} \sum_\pi 1_{\Delta_{\mathbf{z}_{\pi(1)}}}(\mathbf{u}_1) \cdots 1_{\Delta_{\mathbf{z}_{\pi(n)}}}(\mathbf{u}_n)\, 1_{\Delta_{\mathbf{z}_{\pi(n+1)}}}(\mathbf{u}_{n+1})$$

and the multiple weiner integral

$$\begin{aligned}\int_{[(a,a),\mathbf{u}_{n+1}[^n} &\hat{h}(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \\ &= \frac{1}{(n+1)!} 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{u}_{n+1})\, n!\Delta_{\mathbf{z}_1} W \Delta_{\mathbf{z}_2} W \cdots \Delta_{\mathbf{z}_n} W \\ &= \frac{1}{(n+1)} \Delta_{\mathbf{z}_1} W \cdots \Delta_{\mathbf{z}_n} W 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{u}_{n+1})\end{aligned}$$

is measurable with respect to $\mathscr{F}(W(\mathbf{u}_{n+1}))$ and in $L^2([a,b]^2 \times \mathscr{S}^*)$. We then have the iterated stochastic integral

$$\begin{aligned}(n+1)&\int_{[a,b]^2} \left[\int_{[(a,a),\mathbf{u}_{n+1}[^n} \hat{h}(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n)\right] dW(\mathbf{u}_{n+1}) \\ &= (n+1)\int_{[a,b]^2} \left[\frac{1}{(n+1)} \Delta_{\mathbf{z}_1} W \cdots \Delta_{\mathbf{z}_n} W 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{u}_{n+1})\right] dW(\mathbf{u}_{n+1}) \\ &= \Delta_{\mathbf{z}_1} W \cdots \Delta_{\mathbf{z}_n} W \Delta_{\mathbf{z}_{n+1}} W \\ &= \int_{([a,b]^2)^{n+1}} h(\mathbf{u}_1, ..., \mathbf{u}_{n+1})\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_{n+1}).\end{aligned}$$

We now have that the result holds for functions of the form h and thus sums of such functions. By taking limits, we get the general result.

Proof of Theorem 5. By the Wiener–Ito decomposition of $L^2(\mathscr{S}^*(\mathbb{R}^d))$, it suffices to assume that $\varphi(\tau)$ is a multiple Wiener integral. Let $T=[a,b]^d$ and suppose

$$\varphi(\tau)=\int_{(\mathbb{R}^d)^n} f(\tau;\mathbf{u}_1,...,\mathbf{u}_n)\,dW(\mathbf{u}_1)\cdots dW(\mathbf{u}_n),\qquad f\in L^2(T\times(\mathbb{R}^d)^n).$$

Since $\varphi(\tau)$ is nonanticipating,

$$\varphi(\tau)=\int_{[(a,...,a),\tau[^n} f(\tau;\mathbf{u}_1,...,\mathbf{u}_n)\,dW(\mathbf{u}_1)\cdots dW(\mathbf{u}_n).$$

Let $g(\mathbf{u}_1,...,\mathbf{u}_n,\tau)=1_{[(a,...,a),\tau[^n}(\mathbf{u}_1,...,\mathbf{u}_n)f(\tau,\mathbf{u}_1,...,\mathbf{u}_n)$. Then

$$\begin{aligned}\int_T \partial_\tau^*\varphi(\tau)\,d\tau &= \int_{(T)^{n+1}} \hat{g}(\mathbf{u}_1,...,\mathbf{u}_n,\tau)\,dW(\mathbf{u}_1)\cdots dW(\mathbf{u}_n)\,dW(\tau)\\ &=(n+1)\int_T\left[\int_{[(a,...,a),\tau[^n}\hat{g}(\mathbf{u}_1,...,\mathbf{u}_n,\tau)\,dW(\mathbf{u}_1)\cdots dW(\mathbf{u}_n)\right]dW(\tau).\end{aligned}$$

But for $\mathbf{u}_1<\tau,...,\mathbf{u}_n<\tau$,

$$\begin{aligned}\hat{g}(\mathbf{u}_1,...,\mathbf{u}_n,\tau)&=\frac{1}{(n+1)!}\sum_{\pi_{1,...,n}} f(\tau,\mathbf{u}_{\pi(1)},...,\mathbf{u}_{\pi(n)})\\ &=\frac{1}{(n+1)!}\,n!\,\hat{f}(\tau;\mathbf{u}_1,...,\mathbf{u}_n)\end{aligned}$$

(where $\hat{f}$ is the symmetrization of f in the variables $\mathbf{u}_1,...,\mathbf{u}_n$). Therefore,

$$\begin{aligned}\int_T \partial_\tau^*\varphi(\tau)\,d\tau&=\int_T\left[\int_{[(a,...,a),\tau[^n} f(\tau,\mathbf{u}_1,...,\mathbf{u}_n)\,dW(\mathbf{u}_1)\,dW(\mathbf{u}_n)\right]dW(\tau)\\ &=\int_{[a,b]^d}\varphi(\tau)\,dW(\tau).\end{aligned}$$

We will next consider the integral defined by Wong and Zakai [13]. This integral requires measurability with respect to $\mathscr{F}_{\mathbf{z}\vee\mathbf{z}'}$ for $\mathbf{z},\mathbf{z}'\in[a,b]^2$.

We show that it also can be expressed in the white noise setting. We first, however, consider the existence of $\int_{(([a,b]^2)^2} \partial_{\mathbf{z}'}^* \partial_{\mathbf{z}}^* \varphi(\mathbf{z}, \mathbf{z}')\, d\mathbf{z}\, d\mathbf{z}'$, where $\varphi(\mathbf{z}, \mathbf{z}')$ may not be measurable with respect to $\mathscr{F}_{\mathbf{z} \vee \mathbf{z}'}$.

THEOREM 6. *Let $T = [a, b]^2$ and $\varphi(\mathbf{z}, \mathbf{z}')$ be a stochastic process such that $\int_{T^2} E\,|\varphi(\mathbf{z}, \mathbf{z}')|^2\, d\mathbf{z}\, d\mathbf{z}' < \infty$, $\int_{T^3} E\,|\partial_{\mathbf{v}} \varphi(\mathbf{z}, \mathbf{z}')\ \ (\overline{\partial_{\mathbf{z}'} \varphi(\mathbf{z}, \mathbf{v})} + \overline{\partial_{\mathbf{z}'} \varphi(\mathbf{v}, \mathbf{z})} + \overline{\partial_{\mathbf{z}} \varphi(\mathbf{v}, \mathbf{z}')} + \overline{\partial_{\mathbf{z}} \varphi(\mathbf{z}', \mathbf{v})})|\ \ d\mathbf{v}\, d\mathbf{z}\, d\mathbf{z}' < \infty$, and $\int_{T^4} E\,|\partial_{\mathbf{u}} \partial_{\mathbf{v}} \varphi(\mathbf{z}, \mathbf{z}')\, \overline{\partial_{\mathbf{z}} \partial_{\mathbf{z}'} \varphi(\mathbf{u}, \mathbf{v})}|\ d\mathbf{u}\, d\mathbf{v}\, d\mathbf{z}\, d\mathbf{z}' < \infty$. Then $\int_{T^2} \partial_{\mathbf{z}'}^* \partial_{\mathbf{z}}^* \varphi(\mathbf{z}, \mathbf{z}')\, d\mathbf{z}\, d\mathbf{z}'$ exists and*

$$
\begin{aligned}
E\left|\int_{T^2} \partial_{\mathbf{z}'}^* \partial_{\mathbf{z}}^* \varphi(\mathbf{z}, \mathbf{z}')\, d\mathbf{z}\, d\mathbf{z}'\right|^2 = &\int_{T^2} E(|\varphi(\mathbf{z}, \mathbf{z}')|^2 + \varphi(\mathbf{z}, \mathbf{z}')\, \overline{\varphi(\mathbf{z}', \mathbf{z})})\, d\mathbf{z}\, d\mathbf{z}' \\
&+ \int_{T^3} E[\partial_{\mathbf{v}} \varphi(\mathbf{z}, \mathbf{z}')(\overline{\partial_{\mathbf{z}'} \varphi(\mathbf{z}, \mathbf{v})} + \overline{\partial_{\mathbf{z}'} \varphi(\mathbf{v}, \mathbf{z})} \\
&+ \overline{\partial_{\mathbf{z}} \varphi(\mathbf{v}, \mathbf{z}')} + \overline{\partial_{\mathbf{z}} \varphi(\mathbf{z}', \mathbf{v})})]\, d\mathbf{v}\, d\mathbf{z}\, d\mathbf{z}' \\
&+ \int_{T^4} E\, \partial_{\mathbf{u}} \partial_{\mathbf{v}} \varphi(\mathbf{z}, \mathbf{z}')\, \overline{\partial_{\mathbf{z}} \partial_{\mathbf{z}'} \varphi(\mathbf{u}, \mathbf{v})}\, d\mathbf{u}\, d\mathbf{v}\, d\mathbf{z}\, d\mathbf{z}'.
\end{aligned}
$$

Remark. The idea for this proof is the same as for the one-dimensional case [11] but there are more cases to consider.

THEOREM 7. *Given a jointly measurable function $\varphi(x, \mathbf{z}, \mathbf{z}')$ on $\mathscr{S}^*(\mathbb{R}^2) \times [a, b]^2 \times [a, b]^2$ such that*

(1) *For each pair $\mathbf{z}, \mathbf{z}'$, $\varphi(x, \mathbf{z}, \mathbf{z}')$ is measurable with respect to $\mathscr{F}_{\mathbf{z} \vee \mathbf{z}'}$*

(2) $E \int_{[a,b]^2} \int_{[a,b]^2} \varphi^2(\mathbf{z}, \mathbf{z}')\, d\mathbf{z}\, d\mathbf{z}' < \infty$,

it follows that

$$
\begin{aligned}
&\int_{[a,b]^2} \int_{[a,b]^2} \partial_{\mathbf{z}'}^* \partial_{\mathbf{z}}^*\, 1_G(\mathbf{z}, \mathbf{z}')\, \varphi(\mathbf{z}, \mathbf{z}')\, d\mathbf{z}\, d\mathbf{z}' \\
&\quad = \left[\int_{[a,b]^2 \times [a,b]^2}\right] \varphi(\mathbf{z}, \mathbf{z}')\, dW(\mathbf{z})\, dW(\mathbf{z}').
\end{aligned}
$$

Here $G = \{(\mathbf{z}, \mathbf{z}') \in [a, b]^2 \times [a, b]^2$ such that $\mathbf{z}$ and $\mathbf{z}'$ are unordered$\}$ and $[\int_{[a,b]^2 \times [a,b]^2}]$ denotes the Wong and Zakai integral.

LEMMA 2. *Let $T = [a, b]^2$. For $f \in L^2(T^{n+2})$, let*

$$
g(\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{z}, \mathbf{z}') = 1_{[(a,a), \mathbf{z} \vee \mathbf{z}'[^n}(\mathbf{u}_1, \ldots, \mathbf{u}_n)\, 1_G(\mathbf{z}, \mathbf{z}') f(\mathbf{z}, \mathbf{z}', \mathbf{u}_1, \ldots, \mathbf{u}_n).
$$

Then,

$$\int_T \cdots \int_T \hat{g}(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}')\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n)\, dW(\mathbf{z})\, dW(\mathbf{z}')$$

$$= \frac{(n+2)(n+1)}{2} \left[\int_{T \times T} \right] \left(\int_{[(a,a), \mathbf{z} \vee \mathbf{z}'[^n} \hat{g}(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}') \right.$$

$$\left. \times\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \right) dW(\mathbf{z})\, dW(\mathbf{z}').$$

Proof. As in Lemma 1 we may assume that

$$f(\mathbf{z}, \mathbf{z}', \mathbf{u}_1, ..., \mathbf{u}_n) = 1_{A_{n+2} \times A_{n+1} \times A_1 \times \cdots A_n}(\mathbf{z}, \mathbf{z}', \mathbf{u}_1, ..., \mathbf{u}_n),$$

where the A_i's are disjoint rectangles in $[a, b]^2$.

Note that $G = G_1 \cup G_2$, where $G_1 = \{((s, t), (s_1, t_1)) \mid s < s_1 \text{ and } t > t_1\}$ and $G_2 = \{((s, t), (s_1, t_1)) \mid s > s_1 \text{ and } t < t_1\}$. Let $D = \{(x, s) \mid x < s\}$ and $D' = \{(x, s) \mid x > s\}$. Then

$$1_{G_1}((s, t), (s_1, t_1)) = 1_D(s, s_1)\, 1_{D'}(t, t_1)$$

and

$$1_{G_2}((s, t), (s_1, t_1)) = 1_{D'}(s, s_1)\, 1_D(t, t_1).$$

Let $P_q = \{c_i\}$ be the partition of $[a, b]$ in segments of length $(b-a)/2^q$, $D_i = [c_{i-1}, c_i]$, $E_i = [c_i, b]$, $E'_i = [a, c_i[$, so that

$$1_D(s, s_1) = \lim_{q \to \infty} \sum 1_{D_i \times E_i}(s, s_1) \qquad \text{and} \qquad 1_{D'}(t, t_1) = \lim_{q \to \infty} \sum 1_{D_i \times E'_i}(t, t_1).$$

Thus

$$1_{](a,a),(s,t) \vee (s_1,t_1)[^n}((x_1, y_1), ..., (x_n, y_n))\, 1_G((s, t), (s_1, t_1))$$

$$= 1_{[(a,a),(s,t) \vee (s_1,t_1)[^n}((x_1, y_1), ..., (x_n, y_n))\, 1_{G_1}((s, t), (s_1, t_1))$$

$$+ 1_{[(a,a),(s,t) \vee (s_1,t_1)[^n}((x_1, y_1), ..., (x_n\, y_n))\, 1_{G_2}((s, t), (s_1, t_1))$$

$$= 1_{[(a,a),(s_1,t)[^n}((x_1, y_1), ..., (x_n, y_n))\, 1_{G_1}((s, t), (s_1, t_1))$$

$$+ 1_{[(a,a),(s,t_1)[^n}((x_1, y_1), ..., (x_n, y_n))\, 1_{G_2}((s, t), (s_1, t_1))$$

$$= 1_D(x_1, s_1) \cdots 1_D(x_n, s_1)\, 1_D(y_1, t) \cdots 1_D(y_n, t)\, 1_D(s, s_1)\, 1_{D'}(t, t_1)$$

$$+ 1_D(x_1, s_1) \cdots 1_D(x_n, y_n)\, 1_D(y_1, t) \cdots 1_D(y_n, t)\, 1_{D'}(s, s_1)\, 1_D(t, t_1).$$

Substituting this formula into the definition of g, one sees that

$$g(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}') = \lim \sum_{i_1, ..., i_{n+2}} 1_{\Delta_{\mathbf{z}_{i_1}}}(\mathbf{u}_1) \cdots 1_{\Delta_{\mathbf{z}_{i_n}}}(\mathbf{u}_n)\, 1_{\Delta_{\mathbf{z}_{i_{n+1}}}}(\mathbf{z})\, 1_{\Delta_{\mathbf{z}_{i_{n+2}}}}(\mathbf{z}'),$$

where $\Delta_{\mathbf{z}_{i_1}}, ..., \Delta_{\mathbf{z}_{i_{n+2}}}$ are disjoint rectangles, $\Delta_{\mathbf{z}_{i_{n+1}}} \times \Delta_{\mathbf{z}_{i_{n+2}}} \subset G$, and $\Delta_{\mathbf{z}_{i_k}} < \Delta_{\mathbf{z}_{i_{n+1}} \vee \mathbf{z}_{i_{n+2}}}$ for $k = 1, ..., n$. Consider

$$h(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}') = 1_{\Delta_{\mathbf{z}_1}}(\mathbf{u}_1) \cdots 1_{\Delta_{\mathbf{z}_n}}(\mathbf{u}_n)\, 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{z})\, 1_{\Delta_{\mathbf{z}_{n+2}}}(\mathbf{z}')$$

with the above conditions. We obtain that

$$\begin{aligned}
&\int_{[(a,a),\mathbf{z} \vee \mathbf{z}'[^n} \hat{h}(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}')\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \\
&\quad = \int_{T^n} 1_{[(a,a),\mathbf{z} \vee \mathbf{z}'[^n}(\mathbf{u}_1, ...\mathbf{u}_n) \frac{1}{(n+2)!} \\
&\qquad \times \sum_\pi 1_{\Delta_{\mathbf{z}_{\pi(1)}}}(\mathbf{u}_1) \cdots 1_{\Delta_{\mathbf{z}_{\pi(n+1)}}}(\mathbf{z})\, 1_{\Delta_{\mathbf{z}_{\pi(n+2)}}}(\mathbf{z}')\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \\
&\quad = \int_{T^n} \frac{1}{(n+2)!} \left[\sum_\tau 1_{\Delta_{\mathbf{z}_{\tau(1)}}}(\mathbf{u}_1) \cdots 1_{\Delta_{\mathbf{z}_{\tau(n)}}}(\mathbf{u}_n)\, 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{z})\, 1_{\Delta_{\mathbf{z}_{n+2}}}(\mathbf{z}') \right. \\
&\qquad \left. + \sum_\tau 1_{\Delta_{\mathbf{z}_{\tau(1)}}}(\mathbf{u}_1) \cdots 1_{\Delta_{\mathbf{z}_{\tau(n)}}}(\mathbf{u}_n)\, 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{z}')\, 1_{\Delta_{\mathbf{z}_{n+2}}}(\mathbf{z}) \right] dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \\
&\quad = \frac{1}{(n+2)!} n!\, [1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{z})\, 1_{\Delta_{\mathbf{z}_{n+2}}}(\mathbf{z}') + 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{z}')\, 1_{\Delta_{\mathbf{z}_{n+2}}}(\mathbf{z})]\, \Delta_{\mathbf{z}_1} W \cdots \Delta_{\mathbf{z}_n} W,
\end{aligned}$$

which is measurable with respect to $\mathscr{F}(W(\mathbf{z} \vee \mathbf{z}'))$ and in $L^2(T \times T \times \mathscr{S}^*)$. We can then look at the Wong and Zakai integral:

$$\begin{aligned}
&\frac{(n+2)(n+1)}{2} \left[\int_{T \times T} \right] \left(\int_{[(a,a),\mathbf{z} \vee \mathbf{z}'[^n} \hat{h}(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}')\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \right) \\
&\qquad \times dW(\mathbf{z})\, dW(\mathbf{z}') \\
&\quad = \frac{1}{2} \left[\int_{T \times T} \right] \left((1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{z})\, 1_{\Delta_{\mathbf{z}_{n+2}}}(\mathbf{z}') + 1_{\Delta_{\mathbf{z}_{n+1}}}(\mathbf{z}')\, 1_{\Delta_{\mathbf{z}_{n+2}}}(\mathbf{z}))\, \Delta_{\mathbf{z}_1} W \cdots \Delta_{\mathbf{z}_n} W \right) \\
&\qquad \times dW(\mathbf{z})\, dW(\mathbf{z}') \\
&\quad = \Delta_{\mathbf{z}_1} W \cdots \Delta_{\mathbf{z}_n} W \Delta_{\mathbf{z}_{n+1}} W\, \Delta_{\mathbf{z}_{n+2}} W \\
&\quad = \int_{T^{n+2}} h(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}')\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n)\, dW(\mathbf{z})\, dW(\mathbf{z}').
\end{aligned}$$

The Lemma now follows easily for g.

Proof of Theorem 7. Here again let $T=[a,b]^2$. Also, set $[(a,a), \mathbf{z} \vee \mathbf{z}'[= D$. Assume

$$\varphi(x, \mathbf{z}, \mathbf{z}') = \int_D \cdots \int_D f(\mathbf{z}, \mathbf{z}', \mathbf{u}_1, ..., \mathbf{u}_n)\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n),\ f \in L^2(T^{n+2}).$$

Defining $g(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}') = 1_D(\mathbf{u}_1, ..., \mathbf{u}_n) \quad 1_G(\mathbf{z}, \mathbf{z}') f(\mathbf{z}, \mathbf{z}', \mathbf{u}_1, ..., \mathbf{u}_n)$, we obtain

$$\int_{T^2} \int_{T^2} \partial^*_{\mathbf{z}'} \partial^*_{\mathbf{z}} 1_G(\mathbf{z}, \mathbf{z}')\, \varphi(\mathbf{z}, \mathbf{z}')\, d\mathbf{z}\, d\mathbf{z}'$$

$$= \int_{T^{n+2}} \hat{g}(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}')\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n)\, dW(\mathbf{z})\, dW(\mathbf{z}')$$

$$= \frac{(n+2)(n+1)}{2} \left[\int_{T \times T} \right] \left(\int_{D^n} \hat{g}(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}') \right.$$

$$\left. \times\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \right) dW(\mathbf{z})\, dW(\mathbf{z}').$$

Note that for $\mathbf{u}_1, ..., \mathbf{u}_n < \mathbf{z} \vee \mathbf{z}'$,

$$\hat{g}(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{z}, \mathbf{z}')$$

$$= \frac{1}{(n+2)(n+1)} 1_G(\mathbf{z}, \mathbf{z}') [\hat{f}(\mathbf{z}, \mathbf{z}', \mathbf{u}_1, ..., \mathbf{u}_n) + \hat{f}(\mathbf{z}', \mathbf{z}, \mathbf{u}_1, ..., \mathbf{u}_n)],$$

where $\hat{f}$ is the symmetrization of f in the variables $\mathbf{u}_1, ..., \mathbf{u}_n$. Thus we have

$$\int_T \int_T \partial^*_{\mathbf{z}'} \partial^*_{\mathbf{z}} 1_G(\mathbf{z}, \mathbf{z}')\, \varphi(\mathbf{z}, \mathbf{z}')\, d\mathbf{z}\, d\mathbf{z}'$$

$$= \left[\int_{T \times T} \right] \left(\int_{D^n} 1_G(\mathbf{z}, \mathbf{z}') \frac{1}{2} (f(\mathbf{z}, \mathbf{z}', \mathbf{u}_1, ..., \mathbf{u}_n) \right.$$

$$\left. + f(\mathbf{z}', \mathbf{z}, \mathbf{u}_1, ..., \mathbf{u}_n))\, dW(\mathbf{u}_1) \cdots dW(\mathbf{u}_n) \right) dW(\mathbf{z})\, dW(\mathbf{z}')$$

$$= \left[\int_{T \times T} \right] (1_G(\mathbf{z}, \mathbf{z}') \frac{1}{2} (\varphi(\mathbf{z}, \mathbf{z}') + \varphi(\mathbf{z}', \mathbf{z}))\, dW(\mathbf{z})\, dW(\mathbf{z}')$$

$$= \left[\int_{T \times T} \right] \varphi(\mathbf{z}, \mathbf{z}')\, dW(\mathbf{z})\, dW(\mathbf{z}').$$

4. A Generalized Ito Formula for Two-Dimensional Time

In order to develop our generalized Ito formula, we will need to define the generalized Wiener functional $F(W(s,t))$, where F is a tempered distribution. This is the analogue of the one-dimensional time case given by Kubo [8], Kuo [10], and Russek [12]. We will use the Russek approach which generalizes straightforwardly to higher dimensions.

Let H_n, $n \geqslant 0$, denote the nth Hermite polynomial: $H_0 = 1$ and

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} \exp(x^2/2)\, D^n \exp(-x^2/2).$$

For $f \in L^2(\mathbb{R}^d)$ with L^2-norm not zero, let $I(f)$ represent the Wiener integral of f. The composition $F(I(f))$ is defined by

$$F(I(f)) = \sum_{n=0}^{\infty} a_n H_n(I(f)/\sigma), \tag{$*$}$$

where $\sigma = \|f\|_{L^2(\mathbb{R}^d)}$ and $a_n = \langle F, H_n(x/\sigma)\, g_{\sigma^2}(x) \rangle$; $g_{\sigma^2}(x)$ denoting $(\sqrt{2\pi}\,\sigma)^{-1} \exp(-x^2/(2\sigma^2))$.

Theorem 9. *The series* $(*)$ *is convergent in* $(L^2)^-$ *and*

$$SF(I(f))(\xi) = (F * g_{\sigma^2})(\langle f, \xi \rangle).$$

The proof of this theorem can be found in the Russek paper for $d = 1$, but the proof does not depend on the value of d.

The Ito formula below is a generalization of that given by Cairoli and Walsh [2] and Wong and Zakai [13]. For $F \in \mathscr{S}(\mathbb{R})$, one recognizes the second integral in the formula as $\int_{[(a_1,a_2),(b_1,b_2)]} F''(W(z))\, dJ(\mathbf{z})$ and we see the Cairoli–Walsh formula. The conditions that Wong and Zakai place on F to obtain their Ito formula ensure that $F(W(\mathbf{z}))$ is a martingale on every increasing staircase. In this case only the first two integrals in the formula would remain and we also see their formula.

Theorem 10. *For* $0 < a_1 < b_1$, $0 < a_2 < b_2$, *and* $F \in \mathscr{S}^*(R)$,

$$F(W(b_1,b_2)) - F(W(b_1,a_2)) - (F(W(a_1,b_2)) - F(W(a_1,a_2)))$$

$$= \int_{a_2}^{b_2} \int_{a_1}^{b_1} \partial^*_{(s,t)} F'(W(s,t))\, ds\, dt$$

$$+ \frac{1}{2} \Bigg[\int_0^{b_2}\int_0^{b_1}\int_0^{b_2}\int_0^{b_1} - \int_0^{b_2}\int_0^{a_1}\int_0^{b_2}\int_0^{a_1} + \int_0^{a_2}\int_0^{a_1}\int_0^{a_2}\int_0^{a_1} - \int_0^{a_2}\int_0^{b_1}\int_0^{a_2}\int_0^{b_1}$$

$$\times (\partial^*_{(u,v)} \partial^*_{(s,t)} 1_G((s,t),(u,v))\, F''(W((s,t) \vee (u,v)))\, ds\, dt\, du\, dv \Bigg]$$

$$+\frac{b_1}{2}\int_{a_2}^{b_2} F''(W(b_1, t))\,dt - \frac{a_1}{2}\int_{a_2}^{b_2} F''(W(a_1, t))\,dt$$

$$+\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\int_0^u \partial^*_{(s,t)} \frac{t}{2} F'''(W(u, t))\,ds\,dt\right) du,$$

where G is the set of unordered pairs in $]\mathbf{0}, (b_1, b_2)]$.

Remark. We will derive our formula by considering the S-transformation.

LEMMA 3. *Suppose that* $F \in \mathscr{S}^*(R)$, $0 < a_1 < b_1$, *and* $0 < a_2 < b_2$. *Then*

$$F(W(b_1, b_2)) - F(W(b_1, a_2)) - (F(W(a_1, b_2) - F(W(a_1, a_2))$$

$$= \int_{a_2}^{b_2}\int_0^{b_1} \partial^*_{(s,r)} F'(W(b_1, r))\,ds\,dr - \int_{a_2}^{b_2}\int_0^{a_1} \partial^*_{(s,r)} F'(W(a_1, r))\,ds\,dr$$

$$+\frac{1}{2}\int_{a_2}^{b_2} (b_1 F''(W(b_1, r)) - a_1 F''(W(a_1, r)))\,dr$$

Proof.

$$S(F(W(b_1, b_2)) - F(W(b_1, a_2)) - (F(W(a_1, b_2)) - F(W(a_1, a_2))(\xi)$$

$$= g_{b_1 b_2} * F\langle \xi, 1_{]0,(b_1,b_2)]}\rangle - g_{b_1 a_2} * F\langle \xi, 1_{]0,(b_1,a_2)]}\rangle$$

$$-(g_{a_1 b_2} * F\langle \xi, 1_{]0,(a_1,b_2)]}\rangle - g_{a_1 a_2} * F\langle \xi, 1_{]0,(a_1,a_2)]}\rangle)$$

$$= \int_{a_2}^{b_2} \frac{d}{dr}(g_{b_1 r} * F\langle \xi, 1_{]0,(b_1,r)]}\rangle)\,dr$$

$$-\int_{a_2}^{b_2} \frac{d}{dr}(g_{a_1 r} * F\langle \xi, 1_{]0,(a_1,r)]}\rangle)\,dr.$$

The lemma now follows by a simple computation.

LEMMA 4.

$$\int_{a_2}^{b_2}\int_0^{b_1} \partial^*_{(s,t)} F'(W(b_1, t))\,ds\,dt - \int_{a_2}^{b_2}\int_0^{a_1} \partial^*_{(s,t)} F'(W(a_1, t))\,ds\,dt$$

$$= \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\int_0^{a_1} \partial^*_{(s,t)} \frac{t}{2} F'''(W(u, t))\,ds\,dt\right) du$$

$$+\int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^t \partial^*_{(s,t)}\,\partial^*_{(u,v)} F''(W(u, t))\,dv\,du\,ds\,dt$$

$$+\int_{a_2}^{b_2}\int_{a_1}^{b_1} \partial^*_{(s,t)} F'(W(b_1, t))\,ds\,dt.$$

Proof.

$$\int_{a_2}^{b_2}\int_0^{b_1} \partial^*_{(s,t)} F'(W(b_1, t))\, ds\, dt - \int_{a_2}^{b_2}\int_0^{a_1} \partial^*_{(s,t)} F'(W(a_1, t))\, ds\, dt$$
$$= \int_{a_2}^{b_2}\int_0^{a_1} \partial^*_{(s,t)} [F'(W(b_1, t)) - F'(W(a_1, t))]\, ds\, dt$$
$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1} \partial^*_{(s,t)} F'(W(b_1, t))\, ds\, dt.$$

Now,

$$S\left(\int_{a_2}^{b_2}\int_0^{a_1} \partial^*_{(s,t)} [F'(W(b_1, t)) - F'(W(a_1, t))]\, ds\, dt\right)(\xi)$$
$$= \int_{a_2}^{b_2}\int_0^{a_1} \xi(s, t)[g_{b_1 t} * F'\langle \xi, 1_{]0,(b_1,t)]}\rangle - g_{a_1 t} * F'\langle \xi, 1_{]0,(a_1,t)]}\rangle]\, ds\, dt$$
$$= \int_{a_2}^{b_2}\int_0^{a_1} \xi(s, t)\left[\int_{a_1}^{b_1} \frac{d}{du} g_{ut} * F'\langle \xi, 1_{]0,(u,t)]}\rangle\, du\right] ds\, dt$$
$$= \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\int_0^{a_1} \xi(s, t)\frac{t}{2} g_{ut} * F'''\langle \xi, 1_{]0,(u,t)]}\rangle\, ds\, dt\right) du$$
$$+ \int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^{t} \xi(s, t)\, \xi(u, v)\, g_{ut} * F''\langle \xi, 1_{]0,(u,t)]}\rangle\, dv\, du\, ds\, dt.$$

Proof of Theorem 10. By the lemmas,

$$S([F(W(b_1, b_2)) - F(W(b_1, a_2))] - [F(W(a_1, b_2)) - F(W(a_1, a_2)]$$
$$- \frac{b_1}{2}\int_{a_2}^{b_2} F''(W(b_1, t))\, dt + \frac{a_1}{2}\int_{a_2}^{b_2} F''(W(a_1, t))\, dt)(\xi)$$
$$= \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\int_0^{a_1} \xi(s, t)\frac{t}{2} g_{ut} * F'''\langle \xi, 1_{]0,(u,t)]}\rangle\, ds\, dt\right) du$$
$$+ \int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^{t} \xi(s, t)\, \xi(u, v)\, g_{ut} * F''\langle \xi, 1_{]0,(u,t)]}\rangle\, dv\, du\, ds\, dt$$
$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1} \xi(s, t)\, g_{b_1 t} * F'\langle \xi, 1_{]0,(b_1,t)]}\rangle\, ds\, dt.$$

Note that

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} \xi(s,t)\, g_{b_1 t} * F'\langle \xi, 1_{]0,(b_1,t)]}\rangle \, ds\, dt$$

$$= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \xi(s,t)\Bigg[g_{st} * F'\langle \xi, 1_{]0,(s,t)]}\rangle + \int_s^{b_1} \frac{d}{du}(g_{ut} * F'\langle \xi, 1_{]0,(u,t)]}\rangle)\, du\Bigg] ds\, dt$$

$$= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \xi(s,t)\, g_{st} * F'\langle \xi, 1_{]0,(s,t)]}\rangle \, ds\, dt + \int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_s^{b_1} \xi(s,t)\frac{t}{2} g_{ut} * F'''\langle \xi, 1_{]0,(u,t)]}\rangle \, du\, ds\, dt + \int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_s^{b_1}\int_0^t \xi(s,t)\,\xi(u,v)\, g_{ut} * F''\langle \xi, 1_{]0,(u,t)]}\rangle \, dv\, du\, ds\, dt.$$

Thus,

$$S([F(W(b_1,b_2)) - F(W(b_1,a_2))] - [F(W(a_1,b_2)) - F(W(a_1,a_2)] - \frac{b_1}{2}\int_{a_2}^{b_2} F''(W(b_1,t))\, dt + \frac{a_1}{2}\int_{a_2}^{b_2} F''(W(a_1,t))\, dt)(\xi)$$

$$= \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\int_0^{a_1} \xi((s,t)\frac{t}{2} g_{ut} * F'''\langle \xi, 1_{]0,(u,t)]}\rangle \, ds\, dt\right) du$$

$$+ \int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^t \xi(s,t)\,\xi(u,v)\, g_{ut} * F''\langle \xi, 1_{]0,(u,t)]}\rangle \, dv\, du\, ds\, dt \tag{1}$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1} \xi(s,t)\, g_{st} * F'\langle \xi, 1_{]0,(s,t)]}\rangle \, ds\, dt + \int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_s^{b_1} \xi(s,t)\frac{t}{2} g_{ut} * F'''\langle \xi, 1_{]0,(u,t)]}\rangle \, du\, ds\, dt$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_s^{b_1}\int_0^t \xi(s,t)\,\xi(u,v)\, g_{ut} * F''\langle \xi, 1_{]0,(u,t)]}\rangle \, dv\, du\, ds\, dt. \tag{2}$$

Observing that we can insert $1_G((s,t),(u,v))$ in integrals (1) and (2) above, we see that

$$\int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^t \xi(s,t)\,\xi(u,v)\,g_{ut} * F''\langle \xi, 1_{]0,(u,t)]}\rangle\,dv\,du\,ds\,dt$$

$$+\int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_s^{b_1}\int_0^t \xi(s,t)\,\xi(u,v)\,g_{ut} * F''\langle \xi, 1_{]0,(u,t)]}\rangle\,dv\,du\,ds\,dt$$

$$=\int_0^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^t-\int_0^{a_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^t+\int_0^{b_2}\int_0^{b_1}\int_s^{b_1}\int_0^t$$

$$-\int_0^{a_2}\int_0^{b_1}\int_s^{b_1}\int_0^t+\int_0^{a_2}\int_0^{a_1}\int_s^{b_1}\int_0^t-\int_0^{b_2}\int_0^{a_1}\int_s^{b_1}\int_0^t$$

$$\times\,\xi(s,t)\,\xi(u,v)\,1_G((s,t),(u,v))\,g_{ut} * F''\langle \xi, 1_{]0,(u,t)]}\rangle\,dv\,du\,ds\,dt$$

$$=\int_0^{b_2}\int_0^{b_1}\int_s^{b_1}\int_0^t-\int_0^{b_2}\int_0^{a_1}\int_s^{a_1}\int_0^t+\int_0^{a_2}\int_0^{a_1}\int_s^{a_1}\int_0^t-\int_0^{a_2}\int_0^{b_1}\int_s^{b_1}\int_0^t$$

$$\times\,[\xi(s,t)\,\xi(u,v)\,1_G((s,t),(u,v))\,g_{(s,t)\vee(u,v)}$$

$$* F''\langle \xi, 1_{]0,((s,t)\vee(u,v))]}\rangle]\,dv\,du\,ds\,dt,$$

which, because of 1_G is

$$\int_0^{b_2}\int_0^{b_1}\int_0^{b_1}\int_0^t-\int_0^{b_2}\int_0^{a_1}\int_0^{a_1}\int_0^t+\int_0^{a_2}\int_0^{a_1}\int_0^{a_1}\int_0^t-\int_0^{a_2}\int_0^{b_1}\int_0^{b_1}\int_0^t$$

$$\times\,(\xi(s,t)\,\xi(u,v)\,1_G((s,t),(u,v))\,g_{(s,t)\vee(u,v)}$$

$$* F''\langle \xi, 1_{]0,((s,t)\vee(u,v))]}\rangle)\,dv\,du\,ds\,dt,$$

which by symmetry in the variables v and t is

$$\frac{1}{2}\left[\int_0^{b_2}\int_0^{b_1}\int_0^{b_2}\int_0^{b_1}-\int_0^{b_2}\int_0^{a_1}\int_0^{b_2}\int_0^{a_1}+\int_0^{a_2}\int_0^{a_1}\int_0^{a_2}\int_0^{a_1}-\int_0^{a_2}\int_0^{b_1}\int_0^{a_2}\int_0^{b_1}\right]$$

$$\times\,(\xi(s,t)\,\xi(u,v)\,1_G((s,t),(u,v))\,g_{(s,t)\vee(u,v)}$$

$$* F''\langle \xi, 1_{]0,((s,t)\vee(u,v))]}\rangle)\,du\,dv\,ds\,dt$$

and the theorem follows.

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