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The total bondage number of grid graphs ☆

Fu-Tao Hu [a], You Lu [b], Jun-Ming Xu [c,*]

[a] School of Mathematical Sciences, Anhui University, Hefei, Anhui, 230601, China
[b] Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an Shaanxi, 710072, China
[c] School of Mathematical Sciences, University of Science and Technology of China, Wentsun Wu Key Laboratory of CAS, Hefei, Anhui, 230026, China

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ABSTRACT

The total domination number of a graph G without isolated vertices is the minimum number of vertices that dominate all vertices in G. The total bondage number $b_t(G)$ of G is the minimum number of edges whose removal enlarges the total domination number. This paper considers grid graphs. An (n, m)-grid graph $G_{n,m}$ is defined as the cartesian product of two paths P_n and P_m. This paper determines the exact values of $b_t(G_{n,2})$ and $b_t(G_{n,3})$, and establishes some upper bounds of $b_t(G_{n,4})$.

1. Introduction

For notation and graph-theoretical terminology not defined here we follow [28]. Specifically, let $G = (V, E)$ be an undirected graph without loops and multi-edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set, which is a subset of $\{xy \mid xy$ is an unordered pair of $V\}$. A graph G is *nonempty* if $E(G) \neq \emptyset$. Two vertices x and y are *adjacent* if $xy \in E(G)$. For a vertex x, we call the vertices adjacent to it the *neighbors* of x. We use P_n and C_n to denote a path and a cycle of order n throughout this paper.

A subset $D \subseteq V(G)$ is called a *dominating set* of G if every vertex not in D has at least one neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set.

The concept of domination is so an important and classic conception that it has become one of the most widely studied topics in graph theory, and also is frequently used to study properties of interconnection networks. The early results on this subject have been, in detail, surveyed in the two excellent domination books by Haynes et al. [12,11]. In the recent decade, a large number of research papers on domination as well as related topics appear in many scientific journals because of their applications in many fields such as networks, wireless communication and so on.

A dominating set D of a graph G without isolated vertices is said to be *total* if every vertex in G has at least one neighbor in D. The minimum cardinality of a total dominating set is called the *total domination number* of G, denoted by $\gamma_t(G)$. It is clear that $\gamma(G) \leqslant \gamma_t(G) \leqslant 2\gamma(G)$ for any graph G without isolated vertices.

The concept of total domination in graphs was introduced by Cockayne et al. [2] in 1980. The total domination in graphs has been extensively studied in the literature. In 2009, Henning [14] gave a survey of selected recent results on this topic.

In 1990, Fink et al. [4] introduced the bondage number as a parameter for measuring the vulnerability of the interconnection network under link failure. The minimum dominating set of sites plays an important role in the network for it dominates the whole network with the minimum cost. So we must consider whether its function remains good under

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* Corresponding author.
E-mail addresses: hufu@mail.ustc.edu.cn (F.-T. Hu), luyou@nwpu.edu.cn (Y. Lu), xujm@ustc.edu.cn (J.-M. Xu).

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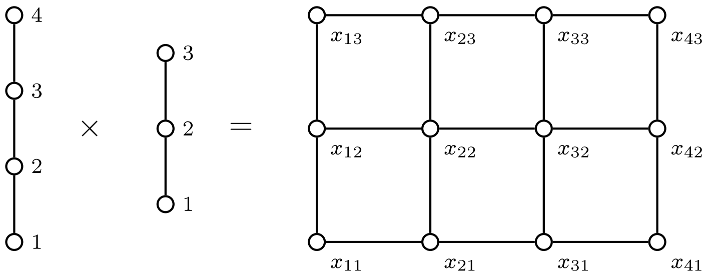

Fig. 1. A (4, 3)-grid graph $G_{4,3} = P_4 \times P_3$.

attack. Suppose that someone such as a saboteur does not know which sites in the network take part in the dominating role, but does know that the set of these special sites corresponds to a minimum dominating set in the related graph. Then how many links does he have to attack so that the cost cannot remain the same in order to dominate the whole network? That minimum number of links is just the bondage number.

A subset B of $E(G)$ is called a *bondage set* of G if its removal from G results in a graph with larger domination number than $\gamma(G)$. The *bondage number* $b(G)$ of a nonempty graph G is the minimum number of edges in a bondage set of G. Since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$, the bondage number of a nonempty graph is well defined. Many results on this topic are obtained in the literature. The exact values of the bondage numbers for some graphs are determined, for example, a complete graph, a path, a cycle, a complete t-partite graph [4], a tree [8,26,27,7], for the Cartesian product of two cycles $C_4 \times C_n$ [20] and $C_3 \times C_n$ [24], and for other graphs [16,18,19]. Some upper bounds of the bondage numbers for graphs are established; see, for example, [3,4,9,10,17,19,23,26] for general graphs, [1,5,21] for planar graphs. In particular, very recently, Hu and Xu [15] have showed that the problem of determining bondage number for general graphs is NP-hard.

Following Fink et al., Kulli and Patwari [22] proposed the concept of the total bondage number for a graph. A subset B of $E(G)$ is called a *total bondage set* of G if its removal from G results in a graph with larger total domination number than $\gamma_t(G)$. The *total bondage number* $b_t(G)$ of a nonempty graph G is the minimum number of edges in a total bondage set of G. If $b_t(G)$ does not exist, for example a star graph $K_{1,n}$, we define $b_t(G) = \infty$. Kulli and Patwari [22] calculated the exact values of $b_t(G)$ for some standard graphs such as a cycle C_n and a path P_n for $n \geqslant 4$, a complete bipartite graph $K_{m,n}$ and a complete K_n. Sridharan et al. [25] showed that for any positive integer k there exists a tree T with $b_t(T) = k$. These authors also established the upper bounds of $b_t(G)$ for a graph G in terms of its order. To the knowledge of the authors, no much research work on the total bondage number were reported in the literature except for the above-mentioned. However, Hu and Xu [15] also showed that the problem of determining total bondage number for general graphs is NP-hard.

An (n, m)-grid graph $G_{n,m}$ is the Cartesian product $P_n \times P_m$ of two paths P_n and P_m. In this paper, we consider $b_t(G_{n,m})$. Since $G_{1,m} \cong P_m$, we assume $n \geq 2$ under our discussion. In 2002, Gravier [6] determined $\gamma_t(G_{n,m})$ for any $m \in \{1, 2, 3, 4\}$. In this paper, we obtain the following results.

$$b_t(G_{n,2}) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3, \\ 2 & \text{if } n \equiv 2 \pmod 3, \\ 3 & \text{if } n \equiv 1 \pmod 3; \end{cases}$$

$$b_t(G_{n,3}) = 1; \qquad b_t(G_{6,4}) = 2, \quad \text{and}$$

$$b_t(G_{n,4}) \begin{cases} = 1 & \text{if } n \equiv 1 \pmod 5 \text{ and } n \neq 6; \\ = 2 & \text{if } n \equiv 4 \pmod 5; \\ \leq 3 & \text{if } n \equiv 2 \pmod 5; \\ \leq 4 & \text{if } n \equiv 0, 3 \pmod 5. \end{cases}$$

The proofs of these results are in Sections 3–5, respectively. In Section 2, we give three preliminary results, which are used in our proofs.

2. Preliminary results

Throughout this paper, we assume that a path P_n has the vertex-set $V(P_n) = \{1, \ldots, n\}$. An (n, m)-grid graph $G_{n,m}$ is defined as the Cartesian product $G_{n,m} = P_n \times P_m$ with vertex-set $V(G_{n,m}) = \{x_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and two vertices x_{ij} and $x_{i'j'}$ being linked by an edge if and only if either $i = i' \in V(P_n)$ and $jj' \in E(P_m)$, such an edge is called a *vertical edge*, or $j = j' \in V(P_m)$ and $ii' \in E(P_n)$, such an edge is called a *horizontal edge*. The graph shown in Fig. 1 is a (4, 3)-grid graph $G_{4,3}$. It is clear, as a graphic operation, that the Cartesian product satisfies commutative associative law if identify isomorphic graphs, that is, $G_{n,m} \cong G_{m,n}$.

The following notations continually appear in our proofs. For a given integer t with $t < n$, $G_{t,m}$ is a subgraph of $G_{n,m}$. We use the notation $H_{n-t,m}$ to denote $G_{n,m} - G_{t,m}$, that is, $H_{n-t,m}$ is a subgraph of $G_{n,m}$ induced by the set of vertices

$\{x_{ij} \mid t+1 \le i \le n,\ 1 \le j \le m\}$. Clearly, $H_{n-t,m} \cong G_{n-t,m}$. For example, the graph shown in Fig. 1 by heavy lines is a subgraph $H_{2,3}$ of $G_{4,3}$, where $n = 4$, $t = 2$ and $m = 3$.

Note that both $G_{0,m}$ and $H_{n-n,m}$ are nominal graphs. For convenience of statements, we allow $G_{0,m}$ and $H_{n-n,m}$ to appear in our proofs. In this case, we always make their total dominating sets empty.

In addition, let $Y_i = \{x_{ij} \mid 1 \le j \le m\}$ for $1 \le i \le n$, called a set of *vertical vertices* in $G_{n,m}$.

We state some useful results on $\gamma_t(G_{n,m})$, which are used in our proofs.

Lemma 2.1. *Let n be a positive integer. Then*
$\gamma_t(G_{n,2}) = 2\lfloor \frac{n+2}{3} \rfloor$ *for* $n \ge 1$;
$\gamma_t(G_{1,3}) = 2$ *and* $\gamma_t(G_{n,3}) = n$ *for* $n \ge 2$;

$$\gamma_t(G_{n,4}) = \begin{cases} \lfloor \dfrac{6n+8}{5} \rfloor & \text{if } n \equiv 1, 2, 4 \pmod 5, \\ \lfloor \dfrac{6n+8}{5} \rfloor + 1 & \text{otherwise} \end{cases} \quad \text{for } n \ge 4 \ [6].$$

For $n \ge 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{n+3}{4} \rfloor$ [13].

Lemma 2.2. *Let D be a total dominating set of $G_{n,m}$. Then $\gamma_t(G_{i,m}) \le |D \cap V(G_{i+1,m})|$ for $1 \le i \le n-1$ and $m \ge 2$.*

Proof. Let $D' = D \cap V(G_{i+1,m})$. If $D' \cap Y_{i+1} = \emptyset$, then D' is a total dominating set of $G_{i,m}$, and hence $\gamma_t(G_{i,m}) \le |D'|$. Assume $D' \cap Y_{i+1} \ne \emptyset$ below and let $A = \{j \mid x_{i+1,j} \in D',\ j = 1, 2, \ldots, m\}$.

Assume $i = 1$. If $m = 2$, then $|D \cap V(G_{2,2})| \ge 2 = \gamma_t(G_{1,2})$ is obvious. Suppose $m \ge 3$ below. Consider the graph $G = G_{1,m} + x_{11}x_{1m}$. Then $G \cong C_m$ and $D'' = (D' \setminus Y_2) \cup \{x_{1(j-1)} \mid j \in A\}$ (x_{10} is replaced by x_{1n} in the case $j = 1$) is a total dominating set of G and $|D''| \le |D'|$. By Lemma 2.1, $\gamma_t(G_{1,m}) = \gamma_t(G) \le |D''| \le |D'|$.

Now, assume $i \ge 2$. Then $D''' = (D' \setminus Y_{i+1}) \cup \{x_{(i-1)j} \mid j \in A\}$ is a total dominating set of $G_{i,m}$ and $|D'''| \le |D'|$. Thus, we have $\gamma_t(G_{i,m}) \le |D'''| \le |D'|$, and so the lemma follows. □

Lemma 2.3 (*Kulli and Patwari [22]*). *For a path P_n with $n \geqslant 4$,*

$$b_t(P_n) = \begin{cases} 2 & \text{if } n \equiv 2 \pmod 4; \\ 1 & \text{otherwise}. \end{cases}$$

Since $G_{1,m} \cong P_m$ and $G_{n,1} \cong P_n$, by Lemma 2.3, we assume that if one of n and m is 1, then the other is at least 4 when we consider the existence of $b_t(G_{n,m})$.

3. The total bondage number of $G_{n,2}$

In this section, we determine the exact value of $b_t(G_{n,2})$ for $n \ge 2$. Since the computation of $b_t(G_{n,2})$ strongly depends on the value of $\gamma_t(G_{n,2})$ in Lemma 2.1, our proof consists of several lemmas according to the value of n modulo 3.

Lemma 3.1. $b_t(G_{n,2}) \le 2$ *for* $n \not\equiv 1 \pmod 3$.

Proof. By Lemma 2.1, we have

$$\gamma_t(G_{n-1,2}) = \gamma_t(G_{n,2}) \quad \text{if } n \not\equiv 1 \pmod 3. \tag{3.1}$$

Let $B = \{x_{(n-1)1}x_{n1}, x_{(n-1)2}x_{n2}\} \subset E(G_{n,2})$, and $H = G_{n,2} - B$. By (3.1), we have

$$\gamma_t(H) = 2 + \gamma_t(G_{n-1,2}) \ge 1 + \gamma_t(G_{n,2}),$$

which implies that $b_t(G_{n,2}) \le |B| = 2$. □

Lemma 3.2. *If* $n \equiv 1 \pmod 3$, *then* $\gamma_t(G_{n,2} - x_{nj}) = \gamma_t(G_{n,2}) - 1$ *for each* $j = 1, 2$.

Proof. Without loss of generality, we only consider the case $j = 1$. By the hypothesis, $n \ge 4$. It can be directly check that the lemma holds for $n = 4$. Assume $n \ge 7$ below. Let D be a minimum total dominating set of $G_{n,2} - x_{n1}$. We need to show $|D| = \gamma_t(G_{n,2}) - 1$.

We consider a subgraph $G_{n-4,2}$ and let D' be a minimum total dominating set of $G_{n-4,2}$. By Lemma 2.1, $|D'| = 2\lfloor \frac{n-4+2}{3} \rfloor = 2\lfloor \frac{n-2}{3} \rfloor$. Clearly, $D' \cup \{x_{(n-1)2}, x_{(n-2)2}, x_{(n-3)2}\}$ is a total dominating set of $G_{n,2} - x_{n1}$. Since $n \equiv 1 \pmod 3$, we have $2\lfloor \frac{n+4}{3} \rfloor = 2\lfloor \frac{n+2}{3} \rfloor = \gamma_t(G_{n,2})$ by Lemma 2.1. It follows that

$$\begin{aligned} |D| &\le |D'| + 3 = 2\left\lfloor \frac{n-2}{3} \right\rfloor + 3 \\ &= 2\left\lfloor \frac{n+4}{3} \right\rfloor - 1 = 2\left\lfloor \frac{n+2}{3} \right\rfloor - 1 \\ &= \gamma_t(G_{n,2}) - 1, \end{aligned}$$

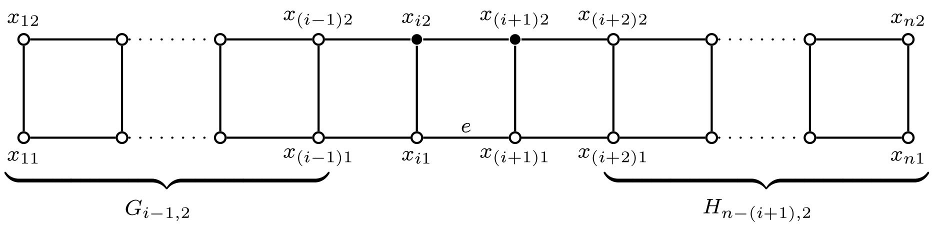

Fig. 2. Two subgraphs $G_{i-1,2}$ and $H_{n-(i+1),2}$ of $G_{n,2}$.

that is,

$$|D| \leq \gamma_t(G_{n,2}) - 1. \tag{3.2}$$

We now prove that $|D| \geq \gamma_t(G_{n,2}) - 1$. If one of $x_{(n-1)1}$ and x_{n2} belongs to D, then D is a total dominating set of $G_{n,2}$. By (3.2), we can deduce a contradiction as follows. $\gamma_t(G_{n,2}) \leq |D| \leq \gamma_t(G_{n,2}) - 1$. It follows that neither of $x_{(n-1)1}$ and x_{n2} belongs to D. Since D is a total dominating set of $G_{n,2} - x_{n1}$, the vertex $x_{(n-1)2}$ must be in D to dominate x_{n2}. Thus $D \cup \{x_{(n-1)1}\}$ is a total dominating set of $G_{n,2}$, and so

$$|D| = |D \cup \{x_{(n-1)1}\}| - 1 \geq \gamma_t(G_{n,2}) - 1.$$

The lemma follows. □

Lemma 3.3. $b_t(G_{n,2}) = 1$ *for* $n \equiv 0 \pmod 3$.

Proof. We only need to show

$$\gamma_t(G_{n,2} - x_{(n-1)1}x_{n1}) \geq \gamma_t(G_{n,2}) + 1. \tag{3.3}$$

Let $H = G_{n,2} - x_{(n-1)1}x_{n1}$ and D be a minimum total dominating set of H. Then the vertex x_{n2} must be in D otherwise D cannot dominate the vertex x_{n1} in H. Moreover, D is either a total dominating set of $G_{n+1,2}$ if x_{n1} is in D or a total dominating set of $G_{n+1,2} - x_{(n+1)1}$ if x_{n1} is not in D. Since $n \equiv 0 \pmod 3$, we have $n + 1 \equiv 1 \pmod 3$ and $\lfloor \frac{n}{3} \rfloor = \lfloor \frac{n+2}{3} \rfloor$. By Lemmas 2.1 and 3.2, we have

$$\begin{aligned}\gamma_t(H) &= |D| \geq \gamma_t(G_{n+1,2}) - 1 = 2\left\lfloor \frac{n+1+2}{3} \right\rfloor - 1 = 2\left\lfloor \frac{n}{3} \right\rfloor + 1 \\ &= 2\left\lfloor \frac{n+2}{3} \right\rfloor + 1 = \gamma_t(G_{n,2}) + 1.\end{aligned}$$

The lemma follows. □

Lemma 3.4. $b_t(G_{n,2}) = 2$ *for* $n \equiv 2 \pmod 3$.

Proof. To prove the lemma, we only need to show $b_t(G_{n,2}) \geq 2$ by Lemma 3.1. To this end, we only need to show $\gamma_t(G_{n,2} - e) = \gamma_t(G_{n,2})$ for any edge e in $G_{n,2}$. Let e be any edge in $G_{n,2}$. We only need to prove that $\gamma_t(G_{n,2} - e) \leq \gamma_t(G_{n,2})$ since $\gamma_t(G_{n,2}) \leq \gamma_t(G_{n,2} - e)$ clearly. We attain this aim by constructing a total dominating set D of $G_{n,2} - e$ such that $|D| = 2\left\lfloor \frac{n+2}{3} \right\rfloor$, which means $|D| = \gamma_t(G_{n,2})$ by Lemma 2.1.

We consider two cases according as that e is vertical or horizontal, respectively.

Suppose that e is a vertical edge $e = x_{i1}x_{i2}$, where $1 \leq i \leq n$. Let

$$D = \begin{cases} \{x_{kj} \mid k \equiv 1 \pmod 3, j = 1, 2\} & \text{if } i \equiv 2 \pmod 3; \\ \{x_{kj} \mid k \equiv 2 \pmod 3, j = 1, 2\} & \text{otherwise.} \end{cases}$$

Then D is a total dominating set of $G_{n,2} - e$ and $|D| = 2\left\lfloor \frac{n+2}{3} \right\rfloor$.

Suppose now that e is a horizontal edge, maybe $e = x_{i1}x_{(i+1)1}$ or $e = x_{i2}x_{(i+1)2}$, where $1 \leq i \leq n - 1$. Without loss of generality, set $e = x_{i1}x_{(i+1)1}$. We consider two subcases to construct D, respectively.

Assume $i \not\equiv 1 \pmod 3$. Let

$$D = \begin{cases} \{x_{kj} \mid k \equiv 2 \pmod 3, j = 1, 2\} & \text{if } i \equiv 0 \pmod 3; \\ \{x_{kj} \mid k \equiv 1 \pmod 3, j = 1, 2\} & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

Then D is a total dominating set of $G_{n,2} - e$ and $|D| = 2\lfloor \frac{n+2}{3} \rfloor$.

Assume now $i \equiv 1 \pmod 3$. We consider $G_{i-1,2}$ and $H_{n-(i+1),2}$ (see Fig. 2). Let D' and D'' be minimum total dominating sets of $G_{i-1,2}$ and $H_{n-(i+1),2}$, respectively. Then $D = D' \cup D'' \cup \{x_{i2}, x_{(i+1)2}\}$ is a total dominating set of $G_{n,2} - e$. Note $D' = \emptyset$

if $i = 1$ and $D'' = \emptyset$ if $i = n-1$. Thus, by Lemma 2.1,

$$\begin{aligned}|D| &= |D'| + |D''| + 2\\ &= 2\left\lfloor \frac{i-1+2}{3} \right\rfloor + 2\left\lfloor \frac{n-i-1+2}{3} \right\rfloor + 2\\ &= 2\left\lfloor \frac{n+2}{3} \right\rfloor.\end{aligned}$$

The lemma follows. □

Lemma 3.5. $b_t(G_{n,2}) = 3$ *for* $n \equiv 1 \pmod 3$.

Proof. Since $n \equiv 1 \pmod 3$, $n-1 \equiv 0 \pmod 3$. By (3.3), for the edge $e_0 = x_{(n-2)1}x_{(n-1)1}$, we have

$$\gamma_t(G_{n-1,2} - e_0) \geq \gamma_t(G_{n-1,2}) + 1. \tag{3.4}$$

Choose other two edges e_1, e_2 in $G_{n,2}$, where $e_1 = x_{(n-1)1}x_{n1}$ and $e_2 = x_{(n-1)2}x_{n2}$. Let $H = G_{n,2} - \{e_0, e_1, e_2\}$. Then $H = (G_{n-1,2} - e_0) + H_{n-(n-1),2}$ and any total dominating set of H must contain vertices x_{n1} and x_{n2}. By (3.4) and Lemma 2.1, we have

$$\begin{aligned}\gamma_t(H) &= \gamma_t(G_{n-1,2} - e_0) + 2\\ &\geq \gamma_t(G_{n-1,2}) + 1 + 2\\ &= 2\left\lfloor \frac{n-1+2}{3} \right\rfloor + 3\\ &= 2\left\lfloor \frac{n+4}{3} \right\rfloor + 1 = 2\left\lfloor \frac{n+2}{3} \right\rfloor + 1\\ &= \gamma_t(G_{n,2}) + 1,\end{aligned}$$

which implies $b_t(G_{n,2}) \leq 3$.

Now we prove $b_t(G_{n,2}) \geq 3$. To the end, let e_1 and e_2 be any two edges in $G_{n,2}$, and $H = G_{n,2} - \{e_1, e_2\}$. We only need to prove $\gamma_t(H) \leq \gamma_t(G_{n,2})$. We consider three cases, respectively.

Case 1 Both e_1 and e_2 are vertical edges.

Let $e_1 = x_{i1}x_{i2}$, $e_2 = x_{j1}x_{j2}$, $i < j$, and let

$$D = \begin{cases}\{x_{kl} \mid k \equiv 1 \pmod 3,\ l = 1, 2\} & \text{if } i, j \not\equiv 1 \pmod 3;\\ \{x_{kl} \mid k \equiv 2 \pmod 3,\ l = 1, 2\} \cup \{x_{(n-1)1}, x_{(n-1)2}\} & \text{if } i, j \not\equiv 2 \pmod 3;\\ \{x_{kl} \mid k \equiv 0 \pmod 3,\ l = 1, 2\} \cup \{x_{21}, x_{22}\} & \text{otherwise.}\end{cases}$$

Then D is a total dominating set of H and $\gamma_t(H) \leq |D| = 2\left\lfloor \frac{n+2}{3} \right\rfloor$. By Lemma 2.1, $|D| = \gamma_t(G_{n,2})$. Thus, for two vertical edges e_1 and e_2, we have

$$\gamma_t(H) = \gamma_t(G_{n,2} - \{e_1, e_2\}) \leq \gamma_t(G_{n,2}). \tag{3.5}$$

Case 2 One of e_1 and e_2 is horizontal and the other is vertical.

Without loss of generality, suppose that e_1 is horizontal and e_2 is vertical, and let $e_1 = x_{i1}x_{(i+1)1}$ and $e_2 = x_{j1}x_{j2}$, $1 \leq i \leq n-1$ and $1 \leq j \leq n$. We will prove $\gamma_t(H) \leq \gamma_t(G_{n,2})$.

Consider $G_{i,2}$ and $H_{n-i,2}$. Then both $G_{i,2}$ and $H_{n-i,2}$ do not contain the edge e_1. There are several subcases.

If $i \equiv 2 \pmod 3$, then $n - i \equiv 2 \pmod 3$ since $n \equiv 1 \pmod 3$. By Lemma 3.4, $b_t(G_{i,2}) = 2 = b_t(H_{n-i,2})$, which implies $\gamma_t(G_{i,2}) = \gamma_t(G_{i,2} - e_2)$ if e_2 is in $G_{i,2}$, and $\gamma_t(H_{n-i,2}) = \gamma_t(H_{n-i,2} - e_2)$ if e_2 is in $H_{n-i,2}$. No matter which case arises, by Lemma 2.1, we have

$$\begin{aligned}\gamma_t(H) &\leq \gamma_t(G_{i,2}) + \gamma_t(H_{n-i,2})\\ &= 2\left\lfloor \frac{i+2}{3} \right\rfloor + 2\left\lfloor \frac{n-i+2}{3} \right\rfloor\\ &\leq 2\left\lfloor \frac{n+2}{3} \right\rfloor = \gamma_t(G_{n,2}).\end{aligned} \tag{3.6}$$

If $i \equiv 1 \pmod 3$ and $j \leq i$, then e_2 is in $G_{i,2}$. By (3.5), $\gamma_t(G_{i,2} - e_2) \leq \gamma_t(G_{i,2})$. Thus, the inequalities (3.6) hold.

If $i \equiv 0 \pmod 3$ and $j \geq i + 1$, then $n - i \equiv 1 \pmod 3$ and e_2 is in $H_{n-i,2}$. Since $H_{n-i,2} \cong G_{n-i,2}$, by (3.5), we have $\gamma_t(H_{n-i,2} - e_2) \leq \gamma_t(H_{n-i,2})$. Thus, the inequalities (3.6) hold.

The remainder is the case either $i \equiv 1 \pmod 3$ and $j \geq i+1$ or $i \equiv 0 \pmod 3$ and $j \leq i$. Since $f : x_{st} \to x_{(n-s+1)t}$ for $s = 1, 2, \ldots, n$ is an automorphism of $G_{n,m}$, the two edges e_1 and e_2 can be considered as $x_{(n-i)1}x_{(n-i+1)1}$ and

$x_{(n-j+1)1}x_{(n-j+1)2}$, respectively. Then the two cases are the same since $n-i \equiv 0 \pmod 3$ and $n-j+1 \le n-i$. We only consider the latter case, that is, $i \equiv 0 \pmod 3$ and $j \le i$.

If $j = i$, let $D = \{x_{kl} \mid k \equiv 2 \pmod 3, l = 1, 2\} \cup \{x_{n1}, x_{n2}\}$, then D is a total dominating set of H, and so,

$$\gamma_t(H) \le |D| = 2\left\lfloor \frac{n+2}{3} \right\rfloor = \gamma_t(G_{n,2}).$$

We now assume $j < i$. Consider $G_{i-1,2}$ and $H_{n-(i+1),2}$. Let D' be a minimum total dominating set of $G_{i-1,2} - e_2$, and D'' be a minimum total dominating set of $H_{n-(i+1),2}$. Then $D = D' \cup D'' \cup \{x_{i2}, x_{(i+1)2}\}$ is a total dominating set of H. Since $i-1 \equiv 2 \pmod 3$, $\gamma_t(G_{i-1,2}) = \gamma_t(G_{i-1,2} - e_2)$ by Lemma 3.4. $H_{n-(i+1),2}$ contains neither e_1 nor e_2. By Lemma 2.1, we have

$$\begin{aligned}\gamma_t(H) &\le |D| = |D'| + |D''| + 2\\ &\le 2 + \gamma_t(G_{i-1,2}) + \gamma_t(H_{n-(i+1),2})\\ &= 2 + 2\left\lfloor \frac{i+1}{3} \right\rfloor + 2\left\lfloor \frac{n-i+1}{3} \right\rfloor\\ &= 2\left\lfloor \frac{n+2}{3} \right\rfloor = \gamma_t(G_{n,2}).\end{aligned}$$

Case 3 Both e_1 and e_2 are horizontal edges.

Without loss of generality, let $e_1 = x_{i1}x_{(i+1)1}$ and $e_2 = x_{kj}x_{(k+1)j}$ are two distinct horizontal edges, where $1 \le j \le 2$ and $i \le k < n$, and $j = 2$ if $i = k$. To prove $\gamma_t(H) \le \gamma_t(G_{n,2})$, we consider three subcases, respectively.

Subcase 3.1 $k = i$.

In this subcase, $e_2 = x_{i2}x_{(i+1)2}$, H is disconnected and has exactly two connected components $G_{i,2}$ and $H_{n-i,2}$. Since both $G_{i,2}$ and $H_{n-i,2}$ contain neither of e_1 and e_2, we have $\gamma_t(H) = \gamma_t(G_{i,2}) + \gamma_t(H_{n-i,2}) = \gamma_t(G_{n,2})$ by Lemma 2.1.

Subcase 3.2 $k = i + 1$.

In this subcase, $G_{i,2}$ and $H_{n-i-1,2}$ contain neither e_1 nor e_2.

If $i \equiv 0$ or $1 \pmod 3$, let D' be a minimum total dominating set of $G_{i-1,2}$, and D'' be a minimum total dominating set of $H_{n-i-1,2}$, then $D = D' \cup D'' \cup \{x_{i2}, x_{(i+1)2}\}$ is a total dominating set of H. Note $D' = \emptyset$ if $i = 1$. By Lemma 2.1, we have

$$\begin{aligned}\gamma_t(H) &\le |D| \le 2 + \gamma_t(G_{i-1,2}) + \gamma_t(H_{n-i-1,2})\\ &= 2 + 2\left\lfloor \frac{i+1}{3} \right\rfloor + 2\left\lfloor \frac{n-i+1}{3} \right\rfloor\\ &= 2\left\lfloor \frac{n+2}{3} \right\rfloor = \gamma_t(G_{n,2}).\end{aligned}$$

If $i \equiv 2 \pmod 3$, let D' be a minimum total dominating set of $G_{i-2,2}$, and D'' be a minimum total dominating set of $H_{n-i-2,2}$, then $D = D' \cup D'' \cup \{x_{(i-1)p}, x_{ip}, x_{(i+1)p}, x_{(i+2)p}\}$, where $p = 3 - j$, is a total dominating set of H. Note that $D' = \emptyset$ if $i = 2$ and $D'' = \emptyset$ if $i = n-2$. By Lemma 2.1, we have

$$\begin{aligned}\gamma_t(H) &\le |D| \le 4 + \gamma_t(G_{i-2,2}) + \gamma_t(H_{n-i-2,2})\\ &= 4 + 2\left\lfloor \frac{i}{3} \right\rfloor + 2\left\lfloor \frac{n-i}{3} \right\rfloor\\ &= 2\left\lfloor \frac{n+2}{3} \right\rfloor = \gamma_t(G_{n,2}).\end{aligned}$$

Subcase 3.3 $k > i + 1$.

In this case, e_2 is in $H_{n-i,2}$.

If $i \equiv 2 \pmod 3$, then $n - i \equiv 2 \pmod 3$ as $n \equiv 1 \pmod 3$. Thus, $b_t(H_{n-i,2}) = 2$ by Lemma 3.4, which implies $\gamma_t(H_{n-i,2} - e_2) = \gamma_t(H_{n-i,2})$. If $i \equiv 0 \pmod 3$, then $n - i \equiv 1 \pmod 3$. Subcase 3.1 shows that $\gamma_t(G_{n,2} - \{e'_1, e'_2\}) = \gamma_t(G_{n,2})$ for two horizontal edges $e'_1 = x_{l1}x_{(l+1)1}$ and $e'_2 = x_{l2}x_{(l+1)2}$ for any $l = 1, 2, \ldots, n-1$ in $G_{n,2}$ when $n \equiv 1 \pmod 3$. Since e_2 is a horizontal edge in $H_{n-i,2}$ and $H_{n-i,2} \cong G_{n-i,2}$, we immediately have $\gamma_t(H_{n-i,2} - e_2) = \gamma_t(H_{n-i,2})$. Thus, when $i \not\equiv 1 \pmod 3$, we have

$$\begin{aligned}\gamma_t(H) &\le \gamma_t(G_{i,2}) + \gamma_t(H_{n-i,2} - e_2)\\ &= \gamma_t(G_{i,2}) + \gamma_t(H_{n-i,2})\\ &= \gamma_t(G_{n,2}).\end{aligned}$$

If $k \not\equiv 0 \pmod 3$ then, by replacing $G_{k,2}$ and $H_{n-k,2}$ by $H_{n-i,2}$ and $G_{i,2}$, respectively, we still have

$$\begin{aligned}\gamma_t(H) &\le \gamma_t(G_{k,2} - e_1) + \gamma_t(H_{n-k,2})\\ &= \gamma_t(G_{k,2}) + \gamma_t(H_{n-k,2})\\ &= \gamma_t(G_{n,2}).\end{aligned}$$

Now, we assume $i \equiv 1 \pmod 3$ and $k \equiv 0 \pmod 3$. We consider three subgraphs $G_{i-1,2}$, $H_{n-(k+1),2}$ and $H_{k-1-(i+1),2}$.

Let D' be a minimum total dominating set of $G_{i-1,2}$, D'' be a minimum total dominating set of $H_{n-(k+1),2}$, and D''' be a minimum total dominating set of $H_{k-1-(i+1),2}$. Then $D = D' \cup D'' \cup D''' \cup \{x_{i2}, x_{(i+1)2}, x_{kp}, x_{(k+1)p}\}$, where $p = 3 - j$, is a total dominating set of H. Note $D' = \emptyset$ if $i = 1$, $D'' = \emptyset$ if $k = n-1$ and $D''' = \emptyset$ if $k = i+2$. By Lemma 2.1, we have

$$\begin{aligned}\gamma_t(H) &\le |D| \le 4 + \gamma_t(G_{i-1,2}) + \gamma_t(H_{n-k-1,2}) + \gamma_t(H_{k-i-2,2})\\ &= 4 + 2\left\lfloor \frac{i+1}{3} \right\rfloor + 2\left\lfloor \frac{n-k-1+2}{3} \right\rfloor + 2\left\lfloor \frac{k-i-2+2}{3} \right\rfloor\\ &= 4 + 2\left\lfloor \frac{n-4}{3} \right\rfloor = 2\left\lfloor \frac{n+2}{3} \right\rfloor\\ &= \gamma_t(G_{n,2}).\end{aligned}$$

Summing up all cases, we prove the lemma. □

According to the above lemmas, we can state our results in this section as follows.

Theorem 3.1. *For any integer $n \ge 2$,*

$$b_t(G_{n,2}) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3,\\ 2 & \text{if } n \equiv 2 \pmod 3,\\ 3 & \text{if } n \equiv 1 \pmod 3.\end{cases}$$

4. The total bondage number of $G_{n,3}$

In this section, we will determine $b_t(G_{n,3}) = 1$ for $n \ge 2$. In this case, $Y_i = \{x_{ij} \mid 1 \le j \le 3\}$ for $1 \le i \le n$.

Lemma 4.1. *Let D be a minimum total dominating set of $G_{n,3}$. Then $|D \cap Y_1| \le 2$ and $|D \cap Y_n| \le 2$.*

Proof. Without loss of generality, we only show $|D \cap Y_n| \le 2$. By contradiction, suppose that there exists a minimum total dominating set D of $G_{n,3}$ such that $|D \cap Y_n| = 3$. Then D is still a total dominating set of $G_{n+1,3}$. By Lemma 2.1, $n = \gamma_t(G_{n,3}) = |D| = \gamma_t(G_{n+1,3}) = n+1$, a contradiction. Therefore, $|D \cap Y_n| \le 2$. □

Lemma 4.2. *Let D be a total dominating set of $G_{n,3}$. If at least one of x_{n1} and x_{n3} is in D, then $|D| \ge n+1$.*

Proof. Let D be a total dominating set of $G_{n,3}$.

We first consider that both x_{n1} and x_{n3} are in D. If $x_{n2} \in D$, then $|D \cap Y_n| = 3$. By Lemma 4.1, D is not a minimum total dominating set of $G_{n,3}$. Thus, by Lemma 2.1, we have $|D| \ge \gamma_t(G_{n,3}) + 1 = n+1$.

Assume $x_{n2} \notin D$ below. Since x_{n2} is not in D, both $x_{(n-1)1}$ and $x_{(n-1)3}$ must be in D. Let $D' = (D \setminus \{x_{n1}, x_{n3}\}) \cup \{x_{(n-1)2}\}$. Then D' is still a total dominating set of $G_{n,3}$, and $|D'| < |D|$. By Lemma 2.1, $|D| \ge |D'| + 1 \ge \gamma_t(G_{n,3}) + 1 = n+1$.

We now consider that only one of x_{n1} and x_{n3} is in D. Without loss of generality, we can assume $x_{n1} \in D$ and $x_{n3} \notin D$. We prove $|D| \ge n+1$ by induction on the first subscript $n \ge 2$ of $G_{n,3}$.

It is clear that $|D| \ge 3$ for $n = 2$. Suppose $|D| \ge k+1$ for any integer $k < n$. We prove that $|D| \ge n+1$ for $n \ge 3$. We can assume that $|D \cap Y_i| \le 2$ for each $i = 2, 3, \ldots, n-1$ since if $|D \cap Y_i| = 3$ for some i with $2 \le i \le n-1$, then $(D \setminus \{x_{i1}, x_{i3}\}) \cup \{x_{(i-1)2}, x_{(i+1)2}\}$ is still a total dominating set of $G_{n,3}$ with the cardinality at most $|D|$.

If $x_{i2} \notin D$ for each $i = 2, 3, \ldots, n-1$, then each vertex u in $D \setminus \{x_{n1}\}$ can totally dominate at most three vertices since u has at most three neighbors, and x_{n1} can totally dominate only two vertices. Thus, D can totally dominate at most $2 + 3(|D|-1)$ vertices. On the other hand, D can totally dominate all $3n$ vertices. From the two facts, we can deduce $3n \le 2 + 3(|D|-1)$, which yields $|D| \ge n+1$.

Now assume $x_{i2} \in D$ for some i with $2 \le i \le n-1$. Let i_0 be the largest index such that $x_{i_0 2} \in D$ for $2 \le i_0 \le n-1$. If $x_{n2} \in D$ and $i_0 = n-1$, then $D \setminus \{x_{n1}\}$ is still a total dominating set of $G_{n,3}$. Thus, $|D| \ge \gamma_t(G_{n,3}) + 1 = n+1$ by Lemma 2.1. We assume $i_0 \ne n-1$ if $x_{n2} \in D$ in the following discussion. There are two cases.

Case 1 $D \cap Y_{i_0} = \{x_{i_0 2}\}$.

In this case, since $x_{(i_0+1)2} \notin D$ by the maximality of i_0, $x_{(i_0-1)2} \in D$. Let $D_1 = D \cap V(G_{i_0,3})$. Then D_1 is a total dominating set of $G_{i_0,3}$.

If $D \cap Y_{i_0-1} = \{x_{(i_0-1)2}\}$, then $D_2 = D \cap V(H_{n-(i_0-2),3})$ is a total dominating set of $H_{n-(i_0-2),3}$, and $|D_1 \cap D_2| = |\{x_{(i_0-1)2}, x_{i_0 2}\}| = 2$ (see Fig. 3). Since $H_{n-(i_0-2),3} \cong G_{n-(i_0-2),3}$ and D_2 satisfies the condition in the lemma (i.e., $x_{n1} \in D_2$), by the induction hypothesis $|D_2| \ge (n - i_0 + 2) + 1 = n - i_0 + 3$. By Lemma 2.1, $|D_1| \ge i_0$. Thus,

$$|D| \ge |D_1| + |D_2| - 2 \ge i_0 + n - i_0 + 3 - 2 = n+1.$$

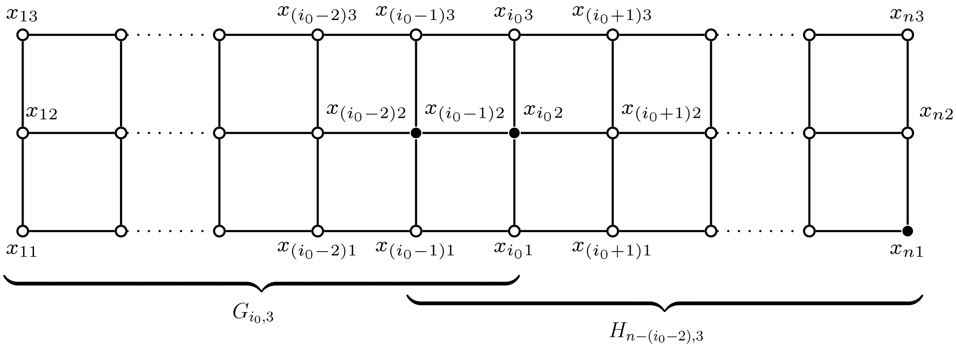

Fig. 3. Two subgraphs $G_{i_0,3}$ and $H_{n-(i_0-2),3}$ of $G_{n,3}$.

If $|D \cap Y_{i_0-1}| = 2$, then $D_3 = D \cap V(H_{n-(i_0-2),3})$ is a total dominating set of $H_{n-(i_0-2),3}$. Let $D_4 = D \cap V(G_{i_0-1}, 3)$. Then $|D_4 \cap D_3| = 2$. Since both of D_3 and D_4 satisfy the condition in the lemma, by the induction hypothesis, $|D_4| \geq i_0$ and $|D_3| \geq n - (i_0 - 2) + 1$. Thus,

$$|D| \geq |D_4| + |D_3| - 2 \geq i_0 + n - i_0 + 3 - 2 = n + 1.$$

Case 2 If $|D \cap Y_{i_0}| = 2$, then $D_5 = D \cap V(G_{i_0,3})$ is a total dominating set of $G_{i_0,3}$, $D_6 = D \cap V(H_{n-(i_0-1),3})$ is a total dominating set of $H_{n-(i_0-1),3}$ and $|D_5 \cap D_6| = 2$. Since both D_4 and D_5 satisfy the condition in the lemma, by the induction hypothesis, $|D_5| \geq i_0 + 1$ and $|D_6| \geq n - i_0 + 1 + 1 = n - i_0 + 2$. Thus,

$$|D| \geq |D_5| + |D_6| - 2 \geq i_0 + 1 + n - i_0 + 2 - 2 = n + 1.$$

The proof of the lemma is complete. □

Theorem 4.1. $b_t(G_{n,3}) = 1$ *for* $n \geq 2$.

Proof. Let $H = G_{n,3} - x_{(n-1)2}x_{n2}$ and D be a minimum total dominating set of H. Whether x_{n2} is in D or not, at least one of two vertices x_{n1} and x_{n3} is in D. Note that D is also a total dominating set for Gn, 3. By Lemma 4.2, $|D| \geq n + 1$. Combining this fact with Lemma 2.1, we have $\gamma_t(H) = |D| \geq n + 1 = \gamma_t(G_{n,3}) + 1$. Therefore, $b_t(G_{n,3}) = 1$. □

5. The total bondage number of $G_{n,4}$

In this section, we determine the exact value of $b_t(G_{n,4})$ for $n \equiv 1,\ 4 \pmod 5$, and establish the upper bounds of $b_t(G_{n,4})$ for $n \equiv 0, 2, 3 \pmod 5$.

Lemma 5.1. $b_t(G_{6,4}) = 2$, $b_t(G_{n,4}) = 1$ *for* $n \geq 7$ *and* $n \equiv 1 \pmod 5$.

Proof. It is easy to check that $B = \{x_{51}x_{61}, x_{52}x_{62}\}$ is a total bondage set of $G_{6,4}$ and $\gamma_t(G_{6,4} - e) = \gamma_t(G_{6,4})$ for any edge e in $G_{6,4}$. Thus $b_t(G_{6,4}) = 2$. Assume $n \geq 7$ below.

Let D be a minimum total dominating set of $G_{n,4} - x_{n2}x_{n3}$. It is easy to see that $|D \cap (Y_{n-1} \cup Y_n)| \geq 4$. Thus, $|D| \geq 4 + |D \cap V(G_{n-2,4})|$. When $n \geq 7$, $n - 3 \equiv 3 \pmod 5$ and $n - 3 \geq 4$. By Lemma 2.1, we have $\gamma_t(G_{n-3,4}) = \lfloor \frac{6(n-3)+8}{5} \rfloor + 1$. Thus, by Lemma 2.2, we have

$$\begin{aligned} |D \cap V(G_{n-2,4})| &\geq \gamma_t(G_{n-3,4}) \\ &= \left\lfloor \frac{6(n-3)+8}{5} \right\rfloor + 1 = \left\lfloor \frac{6n+8}{5} \right\rfloor - 2 \end{aligned}$$

and, hence,

$$\gamma_t(G_{n,4} - x_{n2}x_{n3}) = |D| \geq 2 + \left\lfloor \frac{6n+8}{5} \right\rfloor > \gamma_t(G_{n,4}).$$

Therefore, $b_t(G_{n,4}) = 1$. □

To determine $b_t(G_{n,4})$ for $n \equiv 4 \pmod 5$, we state two simple observations; see Fig. 4 for $n = 9$.

Proposition 5.1. *For* $n \equiv 4 \pmod 5$, *both*

$$D = \{x_{i2}, x_{i3}, x_{(i+2)1}, x_{(i+3)1}, x_{(i+2)4}, x_{(i+3)4} : i \equiv 1 \pmod 5, 1 \leq i \leq n-3\} \quad \text{and}$$
$$D' = \{x_{i2}, x_{i3}, x_{(i-3)1}, x_{(i-2)1}, x_{(i-3)4}, x_{((i-2))4} : i \equiv 4 \pmod 5, 4 \leq i \leq n\}$$

are minimum total dominating sets of $G_{n,4}$.

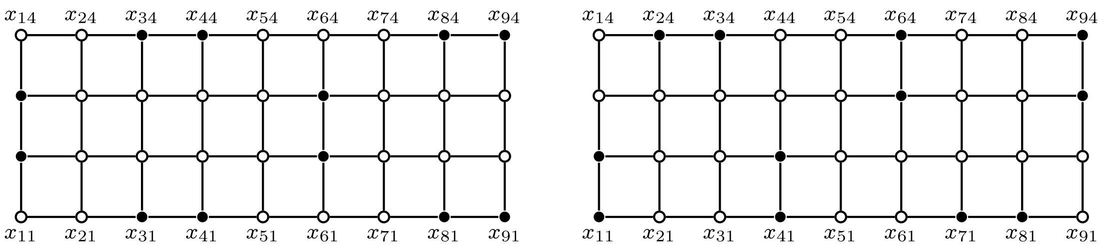

Fig. 4. Two minimum total dominating sets (bold vertices) of $G_{9,4}$ defined in Propositions 5.1 and 5.2, respectively.

Proposition 5.2. *For* $n \equiv 4 \pmod 5$, *both*

$$\begin{aligned} D \;=\; & \{x_{i1}, x_{i2}, x_{(i+3)1}, x_{(i+3)2}, x_{(i+1)4}, x_{(i+2)4} \mid i \equiv 1 \pmod{10},\ 1 \le i \le n-3\} \\ & \cup\{x_{j3}, x_{j4}, x_{(j+3)3}, x_{(j+3)4}, x_{(j+1)1}, x_{(j+2)1} \mid j \equiv 6 \pmod{10},\ 1 \le j \le n-3\} \quad \textit{and} \\ D' \;=\; & \{x_{i3}, x_{i4}, x_{(i+3)3}, x_{(i+3)4}, x_{(i+1)1}, x_{(i+2)1} \mid i \equiv 1 \pmod{10},\ 1 \le i \le n-3\} \\ & \cup\{x_{j1}, x_{j2}, x_{(j+3)1}, x_{(j+3)2}, x_{(j+1)4}, x_{(j+2)4} \mid j \equiv 6 \pmod{10},\ 1 \le j \le n-3\} \end{aligned}$$

are minimum total dominating sets of $G_{n,4}$.

Lemma 5.2. $b_t(G_{n,4}) = 2$ *for* $n \equiv 4 \pmod 5$.

Proof. It is easy to check that $B = \{x_{12}x_{13}, x_{42}x_{43}\}$ is a total bondage set of $G_{4,4}$ and $\gamma_t(G_{4,4} - e) = \gamma_t(G_{4,4})$ for any edge e in $G_{4,4}$. Thus $b_t(G_{4,4}) = 2$. Assume $n \ge 9$ below.

For any edge $e \in E(G_{n,4})$, it is easy to verify that D or D' defined in Proposition 5.1 or Proposition 5.2 is also a minimum total dominating set of $G_{n,4} - e$. Thus, $b_t(G_{n,4}) \ge 2$. We now prove that $b_t(G_{n,4}) \le 2$.

Let $H = G_{n,4} - x_{(n-1)1}x_{n1} - x_{(n-1)2}x_{n2}$ and let S be a minimum total dominating set of H. Then the vertex x_{n2} must be in S, and at least one of x_{n1} and x_{n3} must be in S in H, that is, $|Y_n \cap S| \ge 2$.

If $|Y_n \cap S| \ge 3$ then $|S| \ge |S \cap V(G_{n-1,4})| + 3$. By Lemmas 2.2 and 2.1,, we have

$$\begin{aligned} |S \cap V(G_{n-1,4})| &\ge \gamma_t(G_{n-2,4}) \\ &= \left\lfloor \frac{6(n-2)+8}{5} \right\rfloor = \left\lfloor \frac{6n+8}{5} \right\rfloor - 2 \end{aligned}$$

and, hence,

$$\gamma_t(H) = |S| \ge \gamma_t(G_{n-2,4}) + 3 \ge 1 + \left\lfloor \frac{6n+8}{5} \right\rfloor > \gamma_t(G_{n,4}).$$

If $|Y_n \cap S| = 2$, then $Y_n \cap S$ can totally dominate at most one vertex in $G_{n-1,4}$, that is, $x_{(n-1)3}$ if so. Thus, $(S \cap V(G_{n-1,4})) \cup \{x_{(n-1)2}\}$ is a total dominating set of $G_{n-1,4}$, which implies $|S| \ge |(S \cap V(G_{n-1,4})) \cup \{x_{(n-1)2}\}| + 1 \ge \gamma_t(G_{n-1,4}) + 1$. By Lemma 2.1, we have

$$\gamma_t(H) = |S| \ge \gamma_t(G_{n-1,4}) + 1 > \gamma_t(G_{n,4}).$$

Therefore, $b_t(G_{n,4}) \le 2$. $\square$

Lemma 5.3. $b_t(G_{4,2}) = 3$, $b_t(G_{n,4}) \le 3$ *for* $n \ge 7$ *and* $n \equiv 2 \pmod 5$.

Proof. $b_t(G_{4,2}) = 3$ by Theorem 3.1. It is easy to check that $B = \{x_{61}x_{71}, x_{62}x_{72}, x_{72}x_{73}\}$ is a total bondage set of $G_{7,4}$. Thus $b_t(G_{7,4}) \le 3$. Assume $n \ge 12$ below.

Let $H = G_{n,4} - x_{(n-1)1}x_{n1} - x_{(n-1)2}x_{n2} - x_{n2}x_{n3}$ and let S be a minimum total dominating set of H. Since $x_{n1}x_{n2}$ is an isolated edge in H, both x_{n1} and x_{n2} must be in S. To dominate the three vertices $x_{(n-1)1}$, x_{n3} and x_{n4}, we need at least three other vertices in S. In other words, $|(Y_{n-2} \cup Y_{n-1} \cup Y_n) \cap S| \ge 5$. Thus,

$$|S| \ge 5 + |S \cap V(G_{n-3,4})|. \tag{5.1}$$

By Lemmas 2.2 and 2.1, when $n \ge 8$ and $n \equiv 2 \pmod 5$, we have

$$\begin{aligned} |S \cap V(G_{n-3,4})| &\ge \gamma_t(G_{n-4,4}) \\ &= \left\lfloor \frac{6(n-4)+8}{5} \right\rfloor + 1 = \left\lfloor \frac{6n+8}{5} \right\rfloor - 4, \end{aligned}$$

that is,

$$|S \cap V(G_{n-3,4})| \ge \left\lfloor \frac{6n+8}{5} \right\rfloor - 4. \tag{5.2}$$

It follows from (5.1), (5.2) and Lemma 2.1 that

$$\gamma_t(H) = |S| \geq 1 + \left\lfloor \frac{6n+8}{5} \right\rfloor > \gamma_t(G_{n,4}).$$

Therefore, $b_t(G_{n,4}) \leq 3$. □

Lemma 5.4. $b_t(G_{n,4}) \leq 4$ *for* $n \equiv 0, 3 \pmod 5$.

Proof. It is easy to check that $B = \{x_{(n-3)1}x_{(n-2)1}, x_{(n-1)1}x_{n1}, x_{(n-1)2}x_{n2}, x_{n2}x_{n3}\}$ is a total bondage set of $G_{n,4}$ for $n = 5, 8$. Thus $b_t(G_{n,4}) \leq 4$. Assume $n \geq 10$ below.

Let $H = G_{n,4} - x_{(n-5)1}x_{(n-4)1} - x_{(n-5)2}x_{(n-4)2} - x_{(n-5)3}x_{(n-4)3} - x_{(n-5)4}x_{(n-4)4}$ and let S be a minimum total dominating set of H. Then H consists of two grid subgraphs $G_{n-5,4}$ and $H_{5,4}$, and so

$$|S| = \gamma_t(G_{n-5,4}) + \gamma_t(H_{5,4}). \tag{5.3}$$

Since $n - 5 \equiv 0, 3 \pmod 5$ and $5 \equiv 0 \pmod 5$, by Lemma 2.1, we have

$$\begin{aligned}\gamma_t(G_{n-5,4}) + \gamma_t(H_{5,4}) &= \left\lfloor \frac{6(n-5)+8}{5} \right\rfloor + 1 + 8 \\ &= \left\lfloor \frac{6n+8}{5} \right\rfloor + 3 > \gamma_t(G_{n,4}).\end{aligned} \tag{5.4}$$

Combining (5.3) with (5.4), we have

$$\gamma_t(H) = \gamma_t(G_{n-5,4}) + \gamma_t(H_{5,4}) > \gamma_t(G_{n,4}).$$

Therefore, $b_t(G_{n,4}) \leq 4$. □

Summing up the above lemmas, we can state our result, in this section, as follows.

Theorem 5.1. *For any integer* $n \geq 1$, $b_t(G_{6,4}) = 2$, *and*

$$b_t(G_{n,4}) \begin{cases} = 1 & \text{if } n \equiv 1 \pmod 5 \text{ and } n \neq 6; \\ = 2 & \text{if } n \equiv 4 \pmod 5; \\ \leq 3 & \text{if } n \equiv 2 \pmod 5; \\ \leq 4 & \text{if } n \equiv 0, 3 \pmod 5. \end{cases}$$

6. Concluding remarks

In this paper, we investigate the total bondage number $b_t(G_{n,m})$ of an (n, m)-grid graph $G_{n,m}$ for $2 \leq m \leq 4$, completely determine the exact values of $b_t(G_{n,2})$ and $b_t(G_{n,3})$. We also partially determine the exact values of $b_t(G_{n,4})$, and establishes the upper bounds of $b_t(G_{n,4})$ for otherwise. We have attempted to decrease the two upper bounds given in Theorem 5.1 for $n \equiv 2 \pmod 5$ and $n \equiv 0, 3 \pmod 5$ or to prove that they are tight when n is large enough, but not succeeded. Noting the two upper bounds are tight for some small n's, we guess that the two upper bounds are tight for $n \geq 7$. To prove this conjecture, it may be necessary to find a new method since, according to our way, the removal of any three edges results in many complicated cases. We also have tried to discuss $b_t(G_{n,m})$ for general n and m, but it strongly depends on the value of $\gamma_t(G_{n,m})$, which has not been determined as yet. Thus, it may also be necessary to determine the value of $\gamma_t(G_{n,m})$ for general n and m. These questions are our further work.

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