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CLOSED CATEGORIES, LAX LIMITS AND HOMOTOPY LIMITS

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0. Introduction

The genesis of this paper lies in the proof in [20] that the nerve functor carries lax colimits in the category Cat of small categories to homotopy colimits in the category $\mathscr{S}$ of simplicial sets, up to homotopy type. In trying to extend this result I found that the main properties of homotopy limits and colimits in [3], where they are stated without proof, although very categorical in nature, did not seem to follow from any well-known categorical techniques. At first I thought what was lacking was something in the 2-categorical nature of homotopy theory, but this (apparently) was incorrect. Instead, what is involved is the closed category nature of both Cat and $\mathscr{S}$ and the fact that the nerve functor is a closed functor with a left adjoint. However, the development of the theory of such closed functors (mainly the work of Day, Eilenberg and Kelly in various combinations) has not been in the appropriate direction to yield the required formulas. In providing this development, it turned out that there is a whole aspect of the study of closed categories and closed functors that consists in applying some simple properties of what is called here a "tensor-hom-cotensor" (THC)-situation in more and more complex circumstances. The main results about closed functors are Theorems 2.4.3 and 2.5.3 which specialize to the preservation properties of the nerve functor and its left adjoint, and Theorem 2.6.4 which, via 2.7.5 leads to the replacement schemes for calculating homotopy limits and colimits.

In the brief first paragraph we describe a THC-situation and how it is affected by adjoint functors. Category-theoretically, this material is trivial. Its only virtue is that it codifies circumstances of sufficient complexity to provide a uniform description of the results that are needed later. In Section 2, this material is applied to the study of closed categories. The first three sections contain well-known results and serve mainly to fix the terminology. In the next two sections, mean cotensor and tensor products in the sense of [2] are treated and the main results mentioned above

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are proved. In 2.6, functors of several variables are treated and in 2.7 the required properties of the Yoneda imbedding and of Kan extensions are proved.

In Section 3, lax limits are discussed. After a review and reorganization of their main properties, a new proof of Street's result [19] about the construction of lax limits is given (in 3.4.2). This result is (at least psychologically) crucial since it explains how the comma category functors $\mathbb{1}/-$ arise in this subject. Finally, Section 4 is concerned with the application of these results to homotopy limits. The first three sections briefly review the closed category properties of $\mathscr{K}$ and related functor categories and the fourth section describes the required properties of the nerve functor and its left adjoint in the terminology of Section 2. In 4.5 we show how to define homotopy limits and colimits for $\mathscr{K}$-functors $F: \mathbf{A} \to \mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are both $\mathscr{K}$-categories with $\mathbf{A}$ small and $\mathbf{B}$ complete and cocomplete. If $\mathbf{A}$ is the free $\mathscr{K}$-category on an ordinary small category, then these reduce to the usual notions. The standard properties follow directly from the THC-situations involved in the definitions, and the replacement schemes follow immediately from the results in Section 2.7. In 4.6 the relations between homotopy limits and lax limits are exhaustively treated. Finally, 4.7 mentions some possible and actual generalizations.

1. An approach to closed categories

The fundamental properties of categories enriched in a closed category are encoded in tensor-hom-cotensor situations. These will be described in a generality suitable for the uses here.

1.1. Definition. A THC-situation consists of three categories, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, three functors

$$T: \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{A}_3, \quad H: \mathbf{A}_2^{\mathrm{op}} \times \mathbf{A}_3 \to \mathbf{A}_1, \quad C: \mathbf{A}_1^{\mathrm{op}} \times \mathbf{A}_3 \to \mathbf{A}_2$$

and natural isomorphisms

$$\mathbf{A}_3(T(A_1, A_2), A_3) \cong \mathbf{A}_1(A_1, H(A_2, A_3)) \cong \mathbf{A}_2(A_2, C(A_1, A_3)).$$

If Adj($\mathbf{X}$, $\mathbf{Y}$) denotes the category of functors from $\mathbf{X}$ to $\mathbf{Y}$ with chosen left adjoints and Rt Adj($\mathbf{X}$, $\mathbf{Y}$) denotes the category of contravariant functors from $\mathbf{X}$ to $\mathbf{Y}$ with chosen adjoints on the right, then evidently a THC-situation is equivalent to three functorial families of adjoint functors

$$\mathbf{A}_1 \to \mathrm{Adj}(\mathbf{A}_3, \mathbf{A}_2) : A_1 \mapsto (T(A_1, -) \dashv C(A_1, -)),$$

$$\mathbf{A}_2 \to \mathrm{Adj}(\mathbf{A}_3, \mathbf{A}_1) : A_2 \mapsto (T(-, A_2) \dashv H(A_2, -)),$$

$$\mathbf{A}_3 \to \mathrm{Rt\ Adj}(\mathbf{A}_2, \mathbf{A}_1) : A_3 \mapsto (C(-, A_3) \overset{\mathrm{rt}}{\dashv} H(-, A_3)).$$

(*Note.* $F \dashv U$ means F is left adjoint to U.) T is regarded as a generalized tensor product, H as a generalized hom functor and C as a generalized cotensor product.

Clearly any one of these bifunctors determines the other two up to unique isomorphisms. There are of course situations where only part of this structure is available, but we omit any consideration of such refinements here. There is also an interesting calculus which is derivable under mild hypotheses in special cases; e.g., if $\mathbf{A}_2=\mathbf{A}_3$, then the equation $T(A'_1, T(A_1, A_2))=T(A_1, T(A'_1, A_2))$ has useful consequences, some of which will be needed later, but these will be developed only in the contexts in which they arise.

One is frequently interested in what happens to THC-situations under the action of adjoint functors. In what follows $G_i : \mathbf{A}_i - \mathbf{A}'_i$, $i=1, 2, 3$ are functors and $[\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, T, H, C]$ and $[\mathbf{A}'_1, \mathbf{A}'_2, \mathbf{A}'_3, T', H', C']$ are THC-situations.

1.2. Proposition. *If $\check{G}_1 \dashv G_1$, $\check{G}_2 \dashv G_2$ and $G_3 \dashv \hat{G}_3$, then*

$$[\mathbf{A}'_1, \mathbf{A}'_2, \mathbf{A}'_3, G_3T(\check{G}_1-, \check{G}_2-), G_1H(\check{G}_2-, \hat{G}_3-), G_2C(\check{G}_1-, \hat{G}_3-)$$

is a THC-*situation.*

Proof. $\mathbf{A}'_3(G_3T(\check{G}_1A'_1, \check{G}_2A'_2), A'_3)=\mathbf{A}_3(T(\check{G}_1A'_1, \check{G}_2A'_2), \hat{G}_3A'_3)$ and this second expression can be written either as

$$\mathbf{A}_1(\check{G}_1A'_1, H(\check{G}_2A'_2, \hat{G}_3A'_3))=\mathbf{A}'_1(A'_1, G_1H(\check{G}_2A'_2, \hat{G}_3A'_3))$$

or as

$$\mathbf{A}_2(\check{G}_2A'_2, C(\check{G}_1A'_1, \hat{G}_3A'_3))=\mathbf{A}'_2(A'_2, G_2C(\check{G}_1A'_1, \hat{G}_3A'_3)).$$

1.3. Proposition. *Let $\check{G}_1 \dashv G_1$. Then*

(a) $[\mathbf{A}'_1, \mathbf{A}_2, \mathbf{A}_3, T(\check{G}_1-,-), G_1H(-,-), C(\check{G}_1-,-)]$ *is a* THC-*situation.*

(b) *If $G_2 \dashv \hat{G}_2$, then $G_1H(A_2, A_3)\simeq H'(G_2A_2, G_3A_3)$ if and only if $C(\check{G}_1A'_1, A_3)\simeq \hat{G}_2C'(A'_1, G_3A_3)$.*

(c) *If $\check{G}_3 \dashv G_3$, then $G_1H(A_2, A_3)\simeq H'(G_2A_2, G_3A_3)$ if and only if $T(\check{G}_1A'_1, A_2)\simeq \check{G}_3T'(A'_1, G_2A_2)$.*

Proof. (a) is a special case of 1.2, taking G_2 and G_3 as identity functors. To prove (b), observe that

$$\mathbf{A}'_1(A'_1, G_1H(A_2, A_3))=\mathbf{A}_1(\check{G}_1A'_1, H(A_2, A_3))\simeq\mathbf{A}_2(A_2, C(\check{G}_1A'_1, A_2))$$

while

$$\mathbf{A}'_1(A'_1, H'(G_2A_2, G_3A_3))\simeq\mathbf{A}'_2(G_2A_2, C'(A'_1, G_3A_3))$$

$$\simeq\mathbf{A}_2(A_2, \hat{G}_2C'(A'_1, G_3A_3))$$

from which the result follows immediately by the Yoneda lemma. The proof of (c) is similar as are the proofs of the following statements.

1.4. Proposition. *Let $\check{G}_2 \dashv G_2$. Then*

(a) $[\mathbf{A}_1, \mathbf{A}'_2, \mathbf{A}_3, T(-, \check{G}_2-), H(\check{G}_2-,-), G_2C(-,-)]$ *is a* THC-*situation.*

(b) *If $G_1 \dashv \hat{G}_1$, then $G_2C(A_1, A_3)\simeq C'(G_1A_1, G_3A_3)$ if and only if $H(\check{G}_2A'_2, A_3)\simeq \hat{G}_1H'(A'_2, G_3A_3)$.*

(c) *If* $\check{G}_3 \dashv G_3$, *then* $G_2C(A_1, A_3) \simeq C'(G_1A_1, G_3A_3)$ *if and only if* $T(A_1, \check{G}_2A'_2) \simeq \check{G}_3T'(G_1A_1, A'_2)$.

1.5. Proposition. *Let* $G_3 \dashv \hat{G}_3$. *Then*

(a) $[\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}'_3, G_3T(-,-), H(-, \hat{G}_3-), C(-, \hat{G}_3-)]$ *is a* THC-*situation.*

(b) *If* $G_1 \dashv \hat{G}_1$, *then* $G_3T(A_1, A_2) \simeq T'(G_1A_1, G_2A_2)$ *if and only if* $H(A_2, \hat{G}_3A'_3) \simeq \hat{G}_1H'(G_2A_2, A'_3)$.

(c) *If* $G_2 \dashv \hat{G}_2$, *then* $G_3T(A_1, A_2) \simeq T'(G_1A_1, G_2A_2)$ *if and only if* $C(A_1, \hat{G}_3A'_3) \simeq \hat{G}_2C'(G_1A_1, A'_3)$.

1.6. Proposition. *Let* $\mathbf{A}_1 = \mathbf{A}'_1$.

(a) *If* $G_3 \dashv \hat{G}_3$, *then* $H'(G_2A_2, A'_3) \simeq H(A_2, \hat{G}_3A'_3)$ *if and only if* $G_3T(A_1, A_2) \simeq T'(A_1, G_2A_2)$. (*I.e., the pair* (G_2, G_3) *"preserves tensors".*)

(b) *If* $\check{G}_2 \dashv G_2$, *then* $H(\check{G}_2A'_2, A_3) \simeq H'(A'_2, G_3A_3)$ *if and only if* $G_2C(A_1, A_2) \simeq C'(A_1, G_3A_3)$. (*I.e., the pair* (G_2, G_3) *"preserves cotensors"*).

2. Examples of THC-situations

2.1. Closed categories

Let $\mathbf{V}$ be a complete and cocomplete, closed (symmetric monoidal) category, as in [7], [8], or [11], so that $\mathbf{V}$ is equipped with a tensor product $\otimes$ and an internal hom functor $\mathbf{V}(-,-)$ satisfying suitable properties, as well as an "underlying" set functor $U : \mathbf{V} \to \text{Sets}$ such that $U\mathbf{V}(-,-) = \mathbf{V}_0(-,-)$, the latter denoting the Set-valued hom functor. Let $\mathbf{A}_i = \mathbf{V}$, $i = 1, 2, 3$, and set $T = \otimes$, $H = C = \mathbf{V}(-,-)$. Then the relations

$$\mathbf{V}_0(V \otimes W, Z) \cong \mathbf{V}_0(V, \mathbf{V}(W, Z)) \cong \mathbf{V}_0(W, \mathbf{V}(V, Z))$$

show that this ia a THC-situation. Here $\mathbf{V}_0(-,-)$ can be replaced by $\mathbf{V}(-,-)$.

2.1.1. Proposition. *Let* $\Phi : \mathbf{V} \to \mathbf{V}'$ *be a functor between closed categories which has an (ordinary) left adjoint* $\check{\Phi} \to \Phi$. *Then*

(a) $\Phi\mathbf{V}(V, W) \cong \mathbf{V}'(\Phi V, \Phi W)$ *if and only if* $\check{\Phi}V' \otimes W \cong \check{\Phi}(V' \otimes' \Phi W)$.

(b) $\check{\Phi}(V' \otimes' W') \cong \check{\Phi}V' \otimes \check{\Phi}W'$ *if and only if* $\Phi\mathbf{V}(\check{\Phi}V', W) \cong \mathbf{V}'(V', \Phi W)$.

Proof. Let $G_i = \Phi$, $i = 1, 2, 3$ and use 1.3(c) for (a) and 1.5(b) for (b).

2.2. *Categories enriched in a closed category*

The basic reference for all of this section is [11]. Let $\mathbf{V}$ be a closed category as in the preceding section. A $\mathbf{V}$-category is an ordinary category $\mathbf{B}$ together with a factorization of its hom functor through $U : \mathbf{V} \to \text{Sets}$, denoted hereafter by $\mathbf{B}(-,-)$, and composition operations $\mathbf{B}(B, B') \otimes \mathbf{B}(B', B'') \to \mathbf{B}(B, B'')$ satisfying suitable properties. A **V**-functor $F:\mathbf{B}\to\mathbf{B}'$ is an ordinary functor between the underlying categories, hereafter denoted by $\mathbf{B}_0$, together with maps $F_{B,C}:\mathbf{B}(B,C)\to\mathbf{B}'(F(B),F(C))$ in **V** which are compatible with composition and units. The product $\mathbf{B}\otimes\mathbf{B}'$ of **V**-categories **B** and **B**′ is the **V**-category whose objects are pairs (B,B') and whose **V**-valued hom functor is given by

$$\mathbf{B}\otimes\mathbf{B}'((B,B'),(C,C'))=\mathbf{B}(B,C)\otimes\mathbf{B}'(B',C').$$

A **V**-category **B** is called complete and cocomplete if it has limits and colimits preserved by the **V**-valued representable functors and if it is *tensored* and *cotensored*. This means that there are **V**-functors

$$T=\bar{\otimes}:\mathbf{V}\otimes\mathbf{B}\to\mathbf{B}\quad\text{and}\quad C=\pitchfork:\mathbf{V}^{\mathrm{op}}\otimes\mathbf{B}\to\mathbf{B}$$

with isomorphisms $\mathbf{B}(V\bar{\otimes}B,D)\simeq\mathbf{V}(V,\mathbf{B}(B,D))\simeq\mathbf{B}(B,V\pitchfork D)$. This would be the standard THC-situation with $\mathbf{A}_1=\mathbf{V}$, $\mathbf{A}_2=\mathbf{A}_3=\mathbf{B}$ and $H=\mathbf{B}(-,-):\mathbf{B}^{\mathrm{op}}\otimes\mathbf{B}\to\mathbf{V}$ except that the hom functors $\mathbf{A}_i(-,-)$ are **V**-valued instead of set valued. Clearly all of Section 1 holds in this case, providing ordinary adjoints are replaced by **V**-adjoints (i.e., $F\overset{\mathbf{V}}{\dashv}F$ means $\mathbf{B}(F(B'),D)\simeq\mathbf{B}'(B',F(D))$ in **V**.) We assume these results from here on. Note then that the **V**-valued representable functors $\mathbf{B}(B,-)$ and $\mathbf{B}(-,D)$ have **V**-adjoints as indicated

$$-\bar{\otimes}B\overset{\mathbf{V}}{\dashv}\mathbf{B}(B,-)\quad\text{and}\quad -\pitchfork D\overset{\mathbf{V}.\mathrm{rt.}}{\dashv}\mathbf{B}(-,D).$$

The situation in 2.1 is a special case of this, taking $\mathbf{B}=\mathbf{V}$, $\bar{\otimes}=\otimes$ and $\pitchfork=\mathbf{V}(-,-)$. If $\mathbf{V}=$ Sets, then $V\bar{\otimes}B=\coprod_V B$ and $V\pitchfork B=\prod_V B$.

One can view $\bar{\otimes}$ and $\pitchfork$ as describing operations of **V** on **B**. These operations do not commute but they have the following properties.

2.2.1. Proposition. (i) $V\pitchfork(W\pitchfork B)\cong(V\otimes W)\pitchfork B$.
(ii) $V\bar{\otimes}(W\bar{\otimes}B)\cong(V\otimes W)\bar{\otimes}B$.

Proof. Part (i) follows from 1.4(b) by taking $G_1=V\otimes(-)$, $\hat{G}_1=\mathbf{V}(V,-)$; $\check{G}_2=V\bar{\otimes}-$, $G_2=V\pitchfork -$ and $G_3=\mathrm{Id}$, since $\mathbf{B}(V\bar{\otimes}B,C)\simeq\mathbf{V}(V,\mathbf{B}(B,C))$. Part (ii) follows similarly from 1.5(c) taking G_1 as before, $G_2=\mathrm{Id}$, and $G_3=V\bar{\otimes}-$, which shows that (i) is equivalent to (ii).

2.2.2. Proposition. *Let $F:\mathbf{B}\to\mathbf{B}'$ be a* **V***-functor between complete and cocomplete* **V***-categories which has an (ordinary) left adjoint $\check{F}\dashv F$. Then the following are equivalent:*
(a) *$\check{F}$ is a* **V***-left adjoint.*
(b) *$\check{F}$ preserves tensors, i.e., $\check{F}(V\bar{\otimes}B')\cong V\bar{\otimes}\check{F}B'$.*
(c) *F preserves cotensors; i.e., $F(V\pitchfork B)\cong V\pitchfork FB$.*

Proof. Let $G_1=\mathrm{Id}:\mathbf{V}\to\mathbf{V}$, $G_2=G_3=F$ and use 1.6.

If $\mathbf{V}$ and $\mathbf{V}'$ are closed categories, then a closed functor from $\mathbf{V}$ to $\mathbf{V}'$ consists of an ordinary functor $\Phi : \mathbf{V} \to \mathbf{V}'$ together with a natural transformation $\varphi_{V,W} : \Phi(V) \otimes' \Phi(W) \to \Phi(V \otimes W)$ and a map $\varphi_0 : I' \to \Phi(I)$ satisfying suitable coherence conditions (cf. [8, 11]). Using adjoints, φ determines another natural transformation $\tilde{\varphi}_{V,W} : \Phi(\mathbf{V}(V, W)) \to \mathbf{V}'(\Phi(V), \Phi(W))$. These structures play a central role in this paper.

Let $\mathbf{V}$-Cat denote the category of (small) $\mathbf{V}$-categories and $\mathbf{V}$-functors. Then a closed functor $\Phi : \mathbf{V} \to \mathbf{V}'$ determines a functor $\Phi_* : \mathbf{V}\text{-Cat} \to \mathbf{V}'\text{-Cat}$ as follows. If $\mathbf{B} \in \mathbf{V}$-Cat, then $\Phi_*\mathbf{B} \in \mathbf{V}$-Cat has the same objects as $\mathbf{B}$, with $\Phi_*\mathbf{B}(B, C) = \Phi(\mathbf{B}(B, C))$, composition being defined using φ. If $F : \mathbf{B} \to \mathbf{B}'$ is a $\mathbf{V}$-functor, then $\Phi_*F : \Phi_*\mathbf{B} \to \Phi_*\mathbf{B}'$ is given by $\Phi_*F(B) = F(B)$ and $(\Phi_*F)_{B,C} = \Phi(F_{B,C})$. (*Note.* $\mathbf{V}$-Cat is actually a 2-category and Φ_* is a 2-functor.) There is one additional important piece of data. Φ determines a $\mathbf{V}'$-functor $\bar{\Phi} : \Phi_*\mathbf{V} \to \mathbf{V}'$ by the rule $\bar{\Phi}(V) = \Phi(V)$ and

$$\bar{\Phi}_{V,W} = \tilde{\varphi}_{V,W} : \Phi(\mathbf{V}(V, W)) \to \mathbf{V}'(\Phi(V), \Phi(W)).$$

Φ is called *normal* if it commutes with the underlying set functors; i.e., $U'\Phi = U$, so that in particular $(\Phi_*\mathbf{B})_0(-,-) = \mathbf{B}_0(-,-)$.

We shall speak of $\Phi : \mathbf{V} \to \mathbf{V}'$ having an (ordinary) *left adjoint* $\check{\Phi}$ if $\check{\Phi} : \mathbf{V}'_0 \to \mathbf{V}_0$ is a left adjoint in the usual sense. If Φ is normal, then $\check{\Phi}$ can equally well be regarded as a functor $\check{\Phi} : \mathbf{V}'_0 \to (\Phi_*\mathbf{V})_0$. If $\check{\Phi}$ in this sense has an enrichment to a $\mathbf{V}$-functor which is $\mathbf{V}'$-left adjoint to $\bar{\Phi}$ (i.e., $\Phi\mathbf{V}(\check{\Phi}V', W) \simeq \mathbf{V}'(V', \Phi(W))$) then we shall say $\check{\Phi}$ is a $\mathbf{V}'$-*left adjoint* to Φ.

2.2.3. Proposition. *Let Φ have an (ordinary) left adjoint $\check{\Phi} \dashv \Phi$.*

(a) *If $\mathbf{B}$ is complete and cocomplete then so is $\Phi_*\mathbf{B}$; in particular, $V' \bar{\otimes}' B = \check{\Phi}V' \bar{\otimes} B$ and $V' \pitchfork' B = \check{\Phi}V' \pitchfork B$.*

(b) *Suppose in addition that Φ is normal. Then the following are equivalent and imply that $\check{\Phi}$ is a closed functor.*

(i) *$\check{\Phi}$ is $\mathbf{V}'$-left adjoint to Φ.*

(ii) *$\check{\Phi} : \mathbf{V}' \to \mathbf{V}$ preserves tensor products.*

Proof. (a) follows from 1.3(a) and (b) follows from 2.1.1(b). Note that the $\mathbf{V}'$-structure of $\check{\Phi}$; i.e., the maps

$$\mathbf{V}'(V', W') \to \Phi\mathbf{V}(\check{\Phi}(V'), \check{\Phi}(W'))$$

correspond by adjointness to the maps $\tilde{\check{\varphi}}_{V',W'}$ which in turn correspond by adjointness to the structure isomorphisms $\check{\varphi}_{V',W'} : \check{\Phi}(V') \otimes \check{\Phi}(W') \to \check{\Phi}(V' \otimes' W')$ which make $\check{\Phi}$ a closed functor.

2.2.4. Remarks. When necessary for clarity, we shall write $\Phi_0 : \mathbf{V}_0 \to \mathbf{V}'_0$ for the functor part of a closed functor $\Phi : \mathbf{V} \to \mathbf{V}'$. If $\check{\Phi}$ is a $\mathbf{V}'$-left adjoint to Φ, then we must distinguish several aspects of it:

(i) The underlying functor $\check{\Phi}_0 : \mathbf{V}'_0 \to \mathbf{V}_0$.

(ii) The closed functor $\check{\Phi}_1 : \mathbf{V}' \to \mathbf{V}$.

(iii) The $\mathbf{V}'$-functor $\check{\Phi}_2 : \mathbf{V}' \to \Phi_* \mathbf{V}$.

(iv) It is immediate that $\check{\Phi}_*$ is left adjoint to Φ_* (see 2.3.3), so $\check{\Phi}_2$ corresponds to a $\mathbf{V}$-functor $\check{\Phi}_2^\# : \check{\Phi}_* \mathbf{V}' \to \mathbf{V}$ which coincides with $\bar{\check{\Phi}}_1$ as described before 2.2.3.

2.3. *Ends, coends and functor categories*

Let $\mathbf{V}$ be a complete and cocomplete closed category. We shall make $\mathbf{V}$-Cat into a closed category as well in such a way that if Φ is a closed functor then so is Φ_*. After recalling the relevant definitions, we shall see that the main properties of this situation follow from Section 1.

The essential ingredient for the closed structure on $\mathbf{V}$-Cat is $\mathbf{V}$-functor categories. In order to describe them and related notions, we need ends and coends as described in [11]. If $F : \mathbf{A} \to \mathbf{B}$ is a $\mathbf{V}$-functor, where $\mathbf{B}$ is tensored and cotensored, then the structure maps $F_{A,B} : \mathbf{A}(A, B) \to \mathbf{B}(F(A), F(B))$ can equally well be described in terms of right action maps $r_{A,B} : \mathbf{A}(A, B) \bar{\otimes} F(A) \to F(B)$ or right coaction maps $\bar{r}_{A,B} : F(A) \to \mathbf{A}(A, B) \pitchfork FB$. Similarly a contravariant $\mathbf{V}$-functor is described by left action or coaction maps. Now, if $\mathbf{A}$ is a small $\mathbf{V}$-category, $\mathbf{B}$ a complete and cocomplete $\mathbf{V}$-category and $T : \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathbf{B}$ is a $\mathbf{V}$-functor, then the *end* of T is the equalizer

$$\textstyle\int_A T(A, A) \to \prod_A T(A, A) \underset{g}{\overset{f}{\rightrightarrows}} \prod_{B,C} \mathbf{A}(B, C) \pitchfork T(B, C)$$

where $\mathrm{pr}_{B,C} f = \bar{r}_{B,C} \mathrm{pr}_B$ and $\mathrm{pr}_{B,C} g = \bar{l}_{B,C} \mathrm{pr}_C$. Similarly, the *coend* of T is the coequalizer

$$\coprod_{B,C} \mathbf{A}(B, C) \bar{\otimes} T(C, B) \underset{g}{\overset{f}{\rightrightarrows}} \coprod_A T(A, A) \to \textstyle\int^A T(A, A)$$

where $f\ \mathrm{in}_{B,C} = \mathrm{in}_C\, r_{B,C}$ and $g\ \mathrm{in}_{B,C} = \mathrm{in}_B\, l_{B,C}$. Being particular limits and colimits, these are preserved and collectively created by the $\mathbf{V}$-valued representable functors, and there is a Fubini theorem about changing the order of "integration" when $\mathbf{A}$ is a product category. (Cf. [11].)

If $\mathbf{A}$ is a small $\mathbf{V}$-category, let $[\mathbf{A}, \mathbf{B}]$ be the $\mathbf{V}$-category whose objects are the $\mathbf{V}$-functors from $\mathbf{A}$ to $\mathbf{B}$, whose morphisms are $\mathbf{V}$-natural transformations and in which the $\mathbf{V}$-valued hom functor is given by

$$[\mathbf{A}, \mathbf{B}](F, G) = \textstyle\int_A \mathbf{B}(F(A), G(A)).$$

It is shown in [6] that $\otimes$ and $[-,-]$ determine a closed category structure on $\mathbf{V}$-Cat. In particular, $[\mathbf{A} \otimes \mathbf{A}', \mathbf{B}] \simeq [\mathbf{A}, [\mathbf{A}', \mathbf{B}]]$. The underlying set functor $U : \mathbf{V} \to$ Sets satisfies $U\mathbf{V}(V, W) = \mathbf{V}_0(V, W)$ so, by [8; I, 3.11], we can regard U as a normal closed functor. Hence it induces the forgetful functor $U_* : \mathbf{V}\text{-Cat} \to \mathrm{Cat}$, which will concern us later. Clearly $U_* \mathbf{B} = \mathbf{B}_0$ for any $\mathbf{V}$-category $\mathbf{B}$, and if $\Phi : \mathbf{V} \to \mathbf{V}'$ is a normal closed functor, then $U'_* \Phi_* = U_*$.

2.3.1. Proposition. *If* $\mathbf{A}$ *is a small* $\mathbf{V}$*-category and* $\mathbf{B}$ *is a complete and cocomplete* $\mathbf{V}$*-category, then* $[\mathbf{A}, \mathbf{B}]$ *is a complete and cocomplete* $\mathbf{V}$*-category.*

Proof. Limits and colimits are computed objectwise as usual and tensors and cotensors are given by $(V\bar{\otimes}F)(A)=V\bar{\otimes}F(A)$ and $(V\pitchfork F)(A)=V\pitchfork F(A)$.

2.3.2. Proposition. *If* $\Phi:\mathbf{V}\to\mathbf{V}'$ *is a closed functor, then* $\Phi_*:\mathbf{V}\text{-Cat}\to\mathbf{V}'\text{-Cat}$ *is a normal closed functor.*

Proof. The natural transformation φ_* with components

$$(\varphi_*)_{(A,B)}:\Phi_*\mathbf{A}\otimes'\Phi_*\mathbf{B}\to\Phi_*(\mathbf{A}\otimes\mathbf{B})$$

takes an object (A,B) to (A,B) and on the hom object from (A,B) to (A',B') it is the map $\varphi_{\mathbf{A}(A,A'),\mathbf{B}(B,B')}$. The induced dual natural transformation is of more interest to us; namely $\tilde{\varphi}_*:\Phi_*[\mathbf{A},\mathbf{B}]\to[\Phi_*\mathbf{A},\Phi_*\mathbf{B}]'$ takes a $\mathbf{V}$-functor $F:\mathbf{A}\to\mathbf{B}$ to $\Phi_*F:\Phi_*\mathbf{A}\to\Phi_*\mathbf{B}$. If also $G:\mathbf{A}\to\mathbf{B}$, then $\Phi_*[A,B](F,G)=\Phi(\int_A\mathbf{B}(F(A),G(A))$ while $[\Phi_*\mathbf{A},\Phi_*\mathbf{B}]'(\Phi_*F,\Phi_*G)=\int_A\Phi\mathbf{B}(F(A),G(A))$. (Note that the integrals over A have different meanings.) The map from the first to the second is induced by the canonical map

$$\Phi\left(\prod_A\mathbf{B}(F(A),G(A))\right)\to\prod_A\Phi\mathbf{B}(F(A),G(A))$$

(which follows by properties of φ) and hence commutes with the "evaluation" projections onto each $\mathbf{B}(F(A),G(A))$. For details, see [6] and [8].

2.3.3. Proposition. *Let* $\Phi:\mathbf{V}\to\mathbf{V}'$ *be a normal closed functor with a* $\mathbf{V}'$*-left adjoint* $\check{\Phi}$. *Then*

(i) $\check{\Phi}_*$ *is a* ($\mathbf{V}'$-Cat)*-left adjoint to* Φ_*,

(ii) $\check{\Phi}_*$ *preserves tensor products.*

Proof. It is immediate that $\check{\Phi}\dashv\Phi_*$. To be a ($\mathbf{V}'$-Cat)-left adjoint says that

$$\Phi_*[\check{\Phi}_*\mathbf{B}',\mathbf{B}]\simeq[\mathbf{B}',\Phi_*\mathbf{B}]'.$$

The underlying sets of these $\mathbf{V}'$-categories are isomorphic, so consider $F,G:\Phi_*\mathbf{B}'\to\mathbf{B}$ in $\mathbf{V}$-Cat. Then

$$\int_{B'}\mathbf{B}(F(B'),G(B'))\to\prod_{B'}\mathbf{B}(F(B'),G(B'))\rightrightarrows\prod_{C',D'}\check{\Phi}_*\mathbf{B}'(C',D')\pitchfork B(F(C'),G(D'))$$

is exact. Since Φ has a left adjoint, it preserves the equalizers and products. Furthermore, the cotensor in the last term is in $\mathbf{V}$, so it is the internal hom. Hence Φ applied to the last terms gives

$$\Phi\mathbf{V}(\check{\Phi}\mathbf{B}'(C',D'),\mathbf{B}(F(C'),F(D')))=\mathbf{V}'(\mathbf{B}'(C',D'),\Phi\mathbf{B}(F(C'),G(D')))$$

which shows that

$$\Phi(\int_{B'}\mathbf{B}(F(B'),G(B'))\cong\int_{B'}\Phi\mathbf{B}(F(B'),G(B')).$$

If F^b, $G^b : \mathbf{B}' \to \Phi_* \mathbf{B}$ correspond to F and G, then they agree with F and G on objects and their actions are given by adjointness, so that

$$\Phi[\check{\Phi}_* \mathbf{B}', \mathbf{B}](F, G) \simeq [\mathbf{B}', \Phi_* \mathbf{B}](F^b, G^b)$$

which yields the desired isomorphism. Part (ii) follows from (i) by 2.2.2. □

A particular case of this is given by $U : \mathbf{V} \to \mathrm{Sets}$ since if $\check{U}$ is an ordinary left adjoint to U, then

$$U\mathbf{V}(\check{U}(X), W) \simeq \mathbf{V}_0(\check{U}(X), W) \simeq \mathrm{Sets}(X, U(W))$$

so $\check{U}$ is Sets-left adjoint to U.

2.3.4. Proposition. *Let $U : \mathbf{V} \to \mathrm{Sets}$ have a left adjoint $\check{U}$. Then*

(1) *$\check{U}$ preserves tensor products; i.e., $\check{U}(X \times Y) \simeq \check{U}X \otimes \check{U}Y$.*

(2) *$\check{U}_* \dashv U_*$ and $\check{U}_*$ preserves tensor products; i.e., $\check{U}_*(\mathbb{C} \times \mathbb{D}) \simeq \check{U}_* \mathbb{C} \otimes \check{U}_* \mathbb{D}$ where $\mathbb{C}$ and $\mathbb{D}$ are small categories.*

(3) *As a Cat-category, $\mathbf{V}$-Cat is complete and cocomplete; in particular $\mathbb{I} \tilde{\otimes} \mathbf{B} = \check{U}_* \mathbb{I} \otimes \mathbf{B}$ and $\mathbb{I} \pitchfork \mathbf{B} = [\check{U}_* \mathbb{I}, \mathbf{B}]$.*

(4) *If $\check{\Phi} \dashv \Phi : \mathbf{V} \to \mathbf{V}'$ as in 2.2.3 and if U and U' have left adjoints, then Φ_* preserves cotensors and $\check{\Phi}_*$ preserves tensors; i.e.*

$$\Phi_*[\check{U}_* \mathbb{I}, \mathbf{B}] \simeq [\check{U}'_* \mathbb{I}, \Phi_* \mathbf{B}] \quad \text{and} \quad \check{\Phi}_*(\check{U}'_* \mathbb{I} \otimes \mathbf{B}'] \simeq \check{U}_* \mathbb{I} \otimes \check{\Phi}_* \mathbf{B}'.$$

(5) *If $T : (\check{U}_* \mathbb{I})^{\mathrm{op}} \times \check{U}_* \mathbb{I} \to \mathbf{B}$, then there are exact sequences*

$$\textstyle\int_i T(i, i) \to \prod_i T(i, i) \rightrightarrows \prod_{j,k} \prod_{\varphi : j \to k} T(j, k),$$

$$\coprod_{j,k} \coprod_{\varphi : j \to k} T(j, k) \rightrightarrows \coprod_i T(i, i) \to \textstyle\int^i T(i, i)$$

Proof. (1) follows from 2.2.2. (2) is a special case of 2.3.2.

(3) It is well-known that if $\mathbf{V}$ has limits and colimits, then so does $\mathbf{V}$-Cat. We make $\mathbf{V}$-Cat into a Cat-category (see Section 3) by taking the Cat-valued hom to be

$$\mathbf{V}\text{-Cat}(\mathbf{B}, \mathbf{B}') = U_*[\mathbf{B}, \mathbf{B}'].$$

Since U_* has a left adjoint, it is immediate that limits and colimits are Cat-enriched. It follows from 1.3(a), that the tensors and cotensors in this situation are as indicated.

(4) Since $U'\Phi = U$, it follows that $\check{U} = \check{\Phi}\check{U}'$ and hence $\check{U}_* = \check{\Phi}_* \check{U}'_*$. Thus by 2.3.2, 2,

$$\Phi_*[\check{U}_* \mathbb{I}, \mathbf{B}] = \Phi_*[\check{\Phi}_* \check{U}'_* \mathbb{I}, \mathbf{B}] \cong [\check{U}'_* \mathbb{I}, \Phi_* \mathbf{B}]'.$$

The second equation holds because $\check{\Phi}_*$ preserves tensor products.

(5) Since $\check{U}_*$ preserves tensor products, $(\check{U}_* \mathbb{I})^{\mathrm{op}} \otimes \check{U}_* \mathbb{I} = \check{U}_*(\mathbb{I}^{\mathrm{op}} \times \mathbb{I})$, so in fact $T : \check{U}_*(\mathbb{I}^{\mathrm{op}} \times \mathbb{I}) \to \mathbf{B}$, which by adjointness is the same as an ordinary functor

$T^b : \mathbb{I}^{op} \times \mathbb{I} \to U_* \mathbf{B}$ which has the same values on objects as T. We must show that

$$\prod_{j,k} \check{U}_* \mathbb{I}(j, k) \pitchfork T(j, k) = \prod_{j,k} \prod_{\varphi : j \to k} T(j, k)$$

or, in general, that $\check{U}X \pitchfork B = \prod_X B$ in $\mathbf{B}$. This follows from 1.3(a), since for any $\mathbf{V}$-category $\mathbf{B}$, we have $U\mathbf{B}(-,-) \simeq \mathbf{B}_0(-,-)$, while, in 2.2, we calculated tensors and cotensors over Sets. ☐

For any small category $\mathbb{I}$, the $\mathbf{V}$-category $\check{U}_* \mathbb{I}$ is called the *free* $\mathbf{V}$-category on $\mathbb{I}$. It has the property that $\mathbf{V}$-functors $\check{U}_* \mathbb{I} \to \mathbf{B}$ correspond bijectively to ordinary functors $\mathbb{I} \to U_* \mathbf{B}$. It follows from (4) above that if $\Phi : \mathbf{V} \to \mathbf{V}'$ has a left adjoint, then there is a bijection between $\mathbf{V}$-functors $\check{U}_* \mathbb{I} \to \mathbf{B}$ and $\mathbf{V}'$-functors $\check{U}'_* \mathbb{I} \to \Phi_* \mathbf{B}$. Also, we get a description of $\mathbf{V}$-natural transformations in this case, since they are ordinary functors $\mathbf{2} \to U_* [\mathbf{A}, \mathbf{B}]$ which can be regarded as $\mathbf{V}$-functors in three different ways: $\check{U}_* \mathbf{2} \to [\mathbf{A}, \mathbf{B}]$, or $\check{U}_* \mathbf{2} \otimes \mathbf{A} \to \mathbf{B}$, or $\mathbf{A} \to [\check{U}_* \mathbf{2}, \mathbf{B}]$.

2.3.5. Remarks. In this section we have discussed several operations of a closed functor $\Phi : \mathbf{V} \to \mathbf{V}'$ on $\mathbf{V}$-functors. These are the following.

(i) A $\mathbf{V}$-functor $F : \mathbf{A} \to \mathbf{B}$ determines a $\mathbf{V}'$-functor $\Phi_* F : \Phi_* \mathbf{A} \to \Phi_* \mathbf{B}$ and this is the operation on objects of the components of the $\mathbf{V}'$-natural transformation given by $\tilde{\varphi}_* : \Phi_* [\mathbf{A}, \mathbf{B}] \to [\Phi_* \mathbf{A}, \Phi_* \mathbf{B}]'$.

(ii) If Φ is normal with a $\mathbf{V}'$-left adjoint $\check{\Phi}$, then a $\mathbf{V}$-functor $F : \check{\Phi}_* \mathbf{A}' \to \mathbf{B}$ corresponds by adjointness to a $\mathbf{V}'$-functor $F^b : \mathbf{A}' \to \Phi_* \mathbf{B}$. The correspondence going the other way is denoted by $(\ )^{\#}$, so $F = (F^b)^{\#}$.

(iii) The $\mathbf{V}'$-functor $\bar{\Phi} : \Phi_* \mathbf{V} \to \mathbf{V}'$ induces $\mathbf{V}'$-functors $\Phi^d_{\mathbf{A}} = [\Phi_* \mathbf{A}, \bar{\Phi}] \circ (\tilde{\varphi}_*)_{\mathbf{A},\mathbf{V}} : \Phi_* [\mathbf{A}, \mathbf{V}] \to [\Phi_* \mathbf{A}, \mathbf{V}']'$ which are natural in $\mathbf{A}$. On objects, if $F : \mathbf{A} \to \mathbf{V}$, then $\Phi^d_A(F) = \bar{\Phi} \circ \Phi_* (F)$. See 2.7.7 for an important property of this operation which plays a central role in the applications of this theory.

2.3.6. Proposition. *Let $\Phi : \mathbf{V} \to \mathbf{V}'$ be a normal closed functor with a $\mathbf{V}'$-left adjoint $\check{\Phi}$. Then $\check{\Phi}^d_{\mathbf{A}} : \check{\Phi}_* [\mathbf{A}, \mathbf{V}']' \to [\check{\Phi}_* \mathbf{A}, \mathbf{V}]$ is given by $\check{\Phi}^d_{\mathbf{A}}(F) = (\check{\Phi}_2 \circ F)^{\#}$.*

Proof. In the notation of 2.2.4 $\check{\Phi}_2 : \mathbf{V}' \to \Phi_* \mathbf{V}$ is the $\mathbf{V}'$-left adjoint of $\bar{\Phi}$. By naturality

$$(\check{\Phi}_2 \circ F)^{\#} = (\check{\Phi}_2)^{\#} \circ \check{\Phi}_* (F) = \bar{\check{\Phi}}_1 \circ \check{\Phi}_* (F) = \check{\Phi}^d_{\mathbf{A}}(F),$$

The last equality by definition of $\check{\Phi}^d_{\mathbf{A}}$ and the third equality by 2.2.4.

2.4. Mean cotensor products

Let $\mathbf{V}$ be a closed category, $\mathbf{A}$ a small $\mathbf{V}$-category and $\mathbf{B}$ a complete and cocomplete $\mathbf{V}$-category. Define $\{-,-\}_{\mathbf{B}} : [\mathbf{A}, \mathbf{V}]^{op} \otimes [\mathbf{A}, \mathbf{B}] \to \mathbf{B}$ by the formula

$$\{F, G\}_{\mathbf{B}} = \int_A F(A) \pitchfork G(A)$$

This is the cotensor product for the THC-situation in which $\mathbf{A}_1=[\mathbf{A},\mathbf{V}]$, $\mathbf{A}_2=\mathbf{B}$ and $\mathbf{A}_3=[\mathbf{A},\mathbf{B}]$. Here $T:[\mathbf{A},\mathbf{V}]\otimes\mathbf{B}\to[\mathbf{A},\mathbf{B}]$ is given by $T(F,B)(A)=F(A)\bar{\otimes}B$ and $H:\mathbf{B}\otimes[\mathbf{A},\mathbf{B}]\to[\mathbf{A},\mathbf{V}]$ is given by $H(B,G)(A)=\mathbf{B}(B,G(A))$.

2.4.1. Proposition (Cf. [2]). If $F:\mathbf{A}\to\mathbf{V}$ and $G:\mathbf{A}\to\mathbf{B}$, then

$$[\mathbf{A},\mathbf{B}](F(-)\bar{\otimes}B,G(-))\simeq[\mathbf{A},\mathbf{V}](F(-),\mathbf{B}(B,G(-)))\simeq\mathbf{B}(B,\{F,G\}_{\mathbf{B}}).$$

Proof.

$$\mathbf{B}(B,\{F,G\}_{\mathbf{B}})\cong\mathbf{B}(B,\textstyle\int_A F(A)\pitchfork G(A))\cong\int_A\mathbf{B}(B,F(A)\pitchfork G(A)).$$

This last expression can be written either as

$$\textstyle\int_A\mathbf{V}(F(A),\mathbf{B}(B,G(A))\cong[\mathbf{A},\mathbf{V}](F,\mathbf{B}(B,G(-)))$$

or as

$$\textstyle\int_A\mathbf{B}(F(A)\bar{\otimes}B,G(A))\cong[\mathbf{A},\mathbf{B}](F(-)\bar{\otimes}B,G).\quad\square$$

In [2], $\{F,G\}_{\mathbf{B}}$ is called the *mean cotensor product* of F and G and in [19] it is called the indexed limit of G over F. As usual, there are adjunctions

$$F(\cdot)\bar{\otimes}(-)\dashv\{F,-\}_{\mathbf{B}}\quad\text{and}\quad\{-,G\}_{\mathbf{B}}\overset{\text{op}}{\dashv}\mathbf{B}(-,G(\cdot)).$$

We note that if $\mathbf{B}=\mathbf{V}$, then

$$\{F,G\}_{\mathbf{V}}\cong\textstyle\int_A F(A)\pitchfork G(A)=\int_A\mathbf{V}(F(A),G(A))=[\mathbf{A},\mathbf{V}](F,G),$$

so this special THC-situation has the same relation to the general one here as 2.1 has to 2.2.

If $R:\mathbf{B}\to\mathbf{B}'$ is a $\mathbf{V}$-functor, we write $R^{\mathbf{A}}$ for the functor $[\mathbf{A},R]:[\mathbf{A},\mathbf{B}]\to[\mathbf{A},\mathbf{B}']$ which is composition on the left with R.

2.4.2. Proposition. *If $R:\mathbf{B}\to\mathbf{B}'$ is a $\mathbf{V}$-functor with a $\mathbf{V}$-left adjoint $\check{R}-R$, then*
(a) $R(\{F,G\}_{\mathbf{B}})\cong\{F,R^{\mathbf{A}}G\}_{\mathbf{B}'}$.
(b) $\check{R}^{\mathbf{A}}(F(-)\bar{\otimes}B')=F(-)\bar{\otimes}\check{R}B'$.

Proof. Since $\check{R}^{\mathbf{A}}$ is left adjoint to $R^{\mathbf{A}}$ and clearly $\mathbf{B}(\check{R}B',G(-))\simeq\mathbf{B}(B',R^{\mathbf{A}}G(-))$, part (a) follows from 1.6(b) by taking $(G_2,G_3)=(R,R^{\mathbf{A}})$ while (b) follows from 1.6(a) by taking $(G_2,G_3)=(\check{R},\check{R}^{\mathbf{A}})$. $\square$

2.4.3. Theorem. *Let $\Phi:\mathbf{V}\to\mathbf{V}'$ be a normal closed functor with a $\mathbf{V}'$-left adjoint $\check{\Phi}$. Let $\mathbf{A}'$ be a $\mathbf{V}'$-category, $\mathbf{B}$ a $\mathbf{V}$-category and let $F:\mathbf{A}'\to\mathbf{V}'$ and $G:\mathbf{A}'\to\Phi_*\mathbf{B}$ be $\mathbf{V}'$-functors. Then*

$$\{F,G\}_{\Phi_*\mathbf{B}}\cong\{(\check{\Phi}F)^{\#},G^{\#}\}_{\mathbf{B}}\cong\{\check{\Phi}^{d}_{\mathbf{A}'}(F),G^{\#}\}_{\mathbf{B}},$$

$$F(-)\bar{\otimes}'B\cong((\check{\Phi}F)^{\#}(-)\bar{\otimes}B)^{b}\cong(\check{\Phi}^{d}_{\mathbf{A}'}(F)(-)\bar{\otimes}B)^{b}.$$

Proof. To derive these formulas, consider the situation in 2.4.1 for $M:\check{\Phi}_*\mathbf{A}'\to\mathbf{V}$

and $G^{\#}$, apply Φ to this and use the isomorphism $\Phi_*[\check{\Phi}_*\mathbf{A}', \mathbf{B}] \cong [\mathbf{A}', \Phi_*\mathbf{B}]'$ to get a THC-situation on $[\mathbf{A}', \Phi_*\mathbf{V}]'$, $\Phi_*\mathbf{B}$, and $[\mathbf{A}', \Phi_*\mathbf{B}]'$ in which $T(M^b, B) = (M(-)\bar{\otimes}B)^b$, $H(B, G) = \mathbf{B}(B, G^{\#}(-))^b$ and $C(M^b, G) = \{M, G^{\#}\}_{\mathbf{B}}$. Now consider the $\mathbf{V}'$-adjoint functors

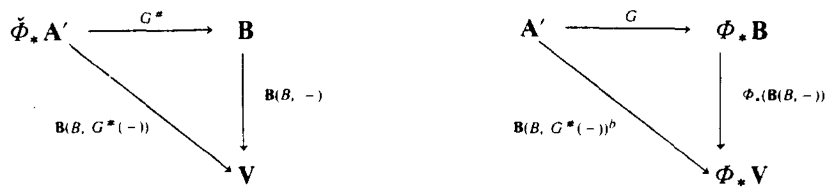
$$[\mathbf{A}', \check{\Phi}]' \dashv [\mathbf{A}', \bar{\Phi}]' : [\mathbf{A}', \Phi_*\mathbf{V}]' \to [\mathbf{A}', \mathbf{V}']'.$$

We claim that

$$[\mathbf{A}', \bar{\Phi}]'(\mathbf{B}(B, G^{\#}(-))^b) \cong \bar{\Phi} \circ \mathbf{B}(B, G^{\#}(-))^b \cong \Phi_*\mathbf{B}(B, G(-)).$$

Thus the cotensors and tensors for $\Phi_*\mathbf{B}(B, G(-))$ which are the left hand sides of the formulas in the theorem are by 1.3(a), given by the middle formulas. The right hand formulas follow from 2.3.6.

To substantiate the claim observe that in the diagrams



the commutativity of the first implies that of the second by naturality of the adjunction correspondence. Hence

$$\begin{aligned}\bar{\Phi} \circ \mathbf{B}(B, G^{\#}(-))^b &= \bar{\Phi} \circ \Phi_*(\mathbf{B}(B, -)) \circ G \\ &= \Phi^d_{\mathbf{B}}(\mathbf{B}(B, -)) \circ G \\ &\cong \Phi_*\mathbf{B}(B, -) \circ G = \Phi_*\mathbf{B}(B, G(-))\end{aligned}$$

since, by 2.7.7, $\Phi^d_{\mathbf{B}}$ commutes with the Yoneda embedding.

2.4.4. Corollary. *Under the hypotheses of 2.4.3 with* $\mathbf{B} = \mathbf{V}$,

$$\bar{\Phi}(\{\check{\Phi}^d_{\mathbf{A}'}(F), G^{\#}\}_{\mathbf{V}}) \cong \{F, \bar{\Phi}G\}'_{\mathbf{V}'},$$

$$\check{\Phi}(F(-)\bar{\otimes}' V') \cong \check{\Phi}^d_{\mathbf{A}'}(F)(-)\bar{\otimes}\check{\Phi}V'$$

Proof. Immediate from 2.4.2 and 2.4.3.

2.5. Mean tensor products

Let $\mathbf{V}$, $\mathbf{A}$ and $\mathbf{B}$ be as in 2.4. Define $- * -$: $[\mathbf{A}^{\mathrm{op}}, \mathbf{V}] \otimes [\mathbf{A}, \mathbf{B}] \to \mathbf{B}$ by the formula

$$H * G = \int^A H(A)\bar{\otimes}G(A)$$

This is the tensor product for the THC-situation in which $\mathbf{A}_1 = [\mathbf{A}^{\mathrm{op}}, \mathbf{V}]$, $\mathbf{A}_2 = [\mathbf{A}, \mathbf{B}]$ and $\mathbf{A}_3 = \mathbf{B}$. Here H : $[\mathbf{A}, \mathbf{B}]^{\mathrm{op}} \otimes \mathbf{B} \to [\mathbf{A}^{\mathrm{op}}, \mathbf{B}]$ is given by $H(G, B)(A) = \mathbf{B}(G(A), B)$ and C : $[\mathbf{A}^{\mathrm{op}}, \mathbf{B}]^{\mathrm{op}} \otimes \mathbf{B} \to [\mathbf{A}, \mathbf{B}]$ is given by $C(H, B)(A) = H(A) \pitchfork B$.

2.5.1. Proposition (Cf. [2]). *If* $H : \mathbf{A}^{op} \to \mathbf{V}$ *and* $G : \mathbf{A} \to \mathbf{B}$, *then*

$$\mathbf{B}(H * G, B) \simeq [\mathbf{A}^{op}, \mathbf{V}](H(-), \mathbf{B}(G(-), B))$$
$$\simeq [\mathbf{A}, \mathbf{B}](G(-), H(-) \pitchfork B).$$

Proof. Coends in the first variable of hom come out as ends; otherwise, it is similar to 2.4.1.

In [2], $H * G$ is called the *mean tensor product* of H and G and in [19] it is called the indexed colimit of G over H. As usual, there are adjunctions

$$H * (-) \dashv H(\cdot) \pitchfork (-) \quad \text{and} \quad (-) * G \dashv \mathbf{B}(G(\cdot), -).$$

2.5.2. Proposition. *If* $R : \mathbf{B} \to \mathbf{B}'$ *is a* $\mathbf{V}$*-functor with a* $\mathbf{V}$-left adjoint $\check{R} \dashv R$, *then*

(a) $\check{R}(H * G') \cong H * \check{R}^{\mathbf{A}} G'$,

(b) $R^{\mathbf{A}}(H(-) \pitchfork B) \cong H(-) \pitchfork RB$.

Proof. See 2.4.2.

2.5.3. Theorem. *Let* $\Phi : \mathbf{V} \to \mathbf{V}'$ *be a normal closed functor with a* $\mathbf{V}'$*-left adjoint* $\check{\Phi}$, *let* $\mathbf{B}$ *be a* $\mathbf{V}$*-category,* $\mathbf{A}'$ *a* $\mathbf{V}'$*-category and let* $H : (\mathbf{A}')^{op} \to \mathbf{V}'$ *and* $G : \mathbf{A}' \to \phi_* \mathbf{B}$ *be* $\mathbf{V}'$*-functors. Then*

$$(H *' G)_{\Phi_* \mathbf{B}} \cong ((\check{\Phi} H)^{\#} * G^{\#})_{\mathbf{B}} \cong \check{\Phi}^{d}_{(\mathbf{A}')^{op}}(H) * G^{\#}$$
$$(H(-) \pitchfork' B)_{\Phi_* \mathbf{B}} \cong (((\check{\Phi} H)^{\#}(-) \pitchfork B)_{\mathbf{B}})^{b} \cong \check{\Phi}^{d}_{(\mathbf{A}')^{op}}(H) \pitchfork B.$$

Proof. See 2.4.3.

2.5.4. Corollary. *Under the hypotheses of 2.5.3, with* $\mathbf{B} = \mathbf{V}$ *and* $G' : \mathbf{A}' \to \mathbf{V}'$,

$$\check{\Phi}((H *' G')_{\mathbf{V}'}) \cong (\check{\Phi}^{d}_{(\mathbf{A}')^{op}}(H) * \check{\Phi}^{d}_{\mathbf{A}'}(G'))_{\mathbf{V}}$$
$$\Phi(\check{\Phi}^{d}_{(\mathbf{A}')^{op}}(H)(-) \pitchfork V)^{b} \cong H(-) \pitchfork' \Phi(V)$$

2.6. Functor categories of two variables

Ends and coends are $\mathbf{V}$-functorial in any extra $\mathbf{V}$-functorial parameters in their arguments, and hence so are mean tensor and cotensor products. In order to describe this we shall use $(\cdot)$ and $(:)$ for variables which have been integrated out, and $(-)$ and $(=)$ for variables that are still present. Consider four $\mathbf{V}$-categories, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ and let $\mathbf{A}_1 = [\mathbf{A}^{op} \otimes \mathbf{B}, \mathbf{V}]$, $\mathbf{A}_2 = [\mathbf{A} \otimes \mathbf{C}, \mathbf{D}]$ and $\mathbf{A}_3 = [\mathbf{B} \otimes \mathbf{C}, \mathbf{D}]$. Let $P \in \mathbf{A}_1$, $Q \in \mathbf{A}_2$ and $R \in \mathbf{A}_3$. Let $T(P, Q) = P(\cdot, -) * Q(\cdot, =) \in \mathbf{A}_3$; i.e.,

$$T(P, Q)(B, C) = \int^{A} P(A, B) \bar{\otimes} Q(A, C).$$

Similarly $H(Q, R) = [\mathbf{C}, \mathbf{D}](Q(-, \cdot), R(=, \cdot)) \in \mathbf{A}_1$; i.e.

$$H(Q, R)(A, B) = \int_C \mathbf{D}(Q(A, C), R(B, C)).$$

Finally $C(P, R) = \{P(-,\bullet), R(\bullet, =)\} \in \mathbf{A}_2$; i.e.,

$$C(P, R)(A, C) = \int_B P(A, B) \pitchfork R(B, C).$$

2.6.1. Proposition. *This is a* THC-*situation; i.e.,*

$$[\mathbf{B}\otimes\mathbf{C}, \mathbf{D}](T(P, Q), R) \simeq [\mathbf{A}^{\mathrm{op}}\otimes\mathbf{B}, \mathbf{V}](P, H(Q, R))$$
$$\simeq [\mathbf{A}\otimes\mathbf{C}, \mathbf{D}](Q, C(P, R)).$$

Proof.

$$\begin{aligned}
[\mathbf{B}\otimes\mathbf{C}, \mathbf{C}](T(P, Q), \ ) &= \int_{B,C} \mathbf{D}(T(P, Q)(B, C), R(B, C)) \\
&\simeq \int_{B,C} \mathbf{D}\left(\int^A P(A, B)\bar{\otimes} Q(A, C), R(B, C)\right) \\
&\simeq \int_B \int_C \int_A \mathbf{D}(P(A, B)\bar{\otimes} Q(A, C), R(B, C)) \\
&= \int_A \int_B \int_C \mathbf{V}(P(A, B), \mathbf{D}(Q(A, C), R(B, C))) \\
&\simeq \int_{A,B} \mathbf{V}\left(P(A, B), \int_C \mathbf{D}(Q(A, C), R(B, C))\right) \\
&= [\mathbf{A}^{\mathrm{op}}\otimes\mathbf{B}, \mathbf{V}](P, H(Q, R)).
\end{aligned}$$

The other isomorphism is proved similarly.

2.6.2. Examples. There are several special cases of this that are interesting.

(i) Let $\mathbf{A} = \mathbf{B}$, $\mathbf{C} = \mathbf{A}^{\mathrm{op}}$ and $\mathbf{D} = \mathbf{V}$. Then $[\mathbf{A}^{\mathrm{op}}\otimes\mathbf{A}, \mathbf{V}]$ is a closed category in which the tensor product is $T(P, Q)$ and the internal hom is $H(Q, R)$. For the rest of the data in this example, see [5].

(ii) Let $\mathbf{A} = \mathbf{B}$ and $\mathbf{C} = \mathbf{1}$. Then $[\mathbf{A}, \mathbf{D}]$ is an $[\mathbf{A}^{\mathrm{op}}\otimes\mathbf{A}, \mathbf{V}]$-category in which the $[\mathbf{A}^{\mathrm{op}}\otimes\mathbf{A}, \mathbf{V}]$-valued hom functor is $H(Q, R)$ and tensors and cotensors are given by $T(P, Q)$ and $C(P, R)$. In particular, $[\mathbf{A}\otimes\mathbf{C}, \mathbf{D}] \cong [\mathbf{A}, [\mathbf{C}, \mathbf{D}]]$ is an $[\mathbf{A}^{\mathrm{op}}\otimes\mathbf{A}, \mathbf{V}]$-category.

The mean tensor and cotensor products treated here are themselves functors so it is possible to iterate these operations resulting in formulas like 2.2.1.

2.6.3. Proposition. *Let* $P : \mathbf{A}^{\mathrm{op}} \to \mathbf{V}$ *and* $R : \mathbf{B} \to \mathbf{C}$ *be* $\mathbf{V}$-*functors.*

(i) *If* $Q : \mathbf{A}\otimes\mathbf{B} \to \mathbf{V}$, *then*

$$\{P(:), \{Q(:,\bullet), R(\bullet)\}\} = \{P(:) * Q(:,\bullet), R(\bullet)\}.$$

(ii) *If* $Q : \mathbf{A}\otimes\mathbf{B}^{\mathrm{op}} \to \mathbf{V}$, *then*

$$[P(:) * Q(:,\bullet)] * R(\bullet) = P(:) * [Q(:,\bullet) * R(\bullet)].$$

Proof.

$$\begin{aligned}
\{P(:), \{Q(:,\bullet), R(\bullet)\}\} &= \int_A P(A) \pitchfork \left(\int_B Q(A, B) \pitchfork R(B)\right) \\
&\simeq \int_A \int_B P(A) \pitchfork (Q(A, B) \pitchfork R(B))
\end{aligned}$$

$$\cong \int_B \int_A (P(A) \otimes Q(A, B)) \pitchfork R(B) \quad \text{(by 2.2.1)}$$

$$\cong \int_B \left(\int^A P(A) \otimes Q(A, B)\right) \pitchfork R(B)$$

$$= \{P(:) * Q(:, \cdot), R(\cdot)\}$$

Alternatively, this is a special case of 1.4(b) applied to the situation in 2.6.1 in which $\mathbf{C} = \mathbf{1}$ and $\mathbf{A}$ is replaced by $\mathbf{A}^{\mathrm{op}}$. Here $G_2 : [\mathbf{A}^{\mathrm{op}}, \mathbf{D}] \to \mathbf{D}$ is $\{P, -\}$, $G_1 : [\mathbf{A} \otimes \mathbf{B}, \mathbf{V}] \to [\mathbf{B}, \mathbf{V}]$ is $P * -$ and $G_3 = \mathrm{Id}$. The other equation is proved similarly. □

There is also a version of this involving a change of closed categories which will be needed in discussing the replacement and coreplacement schemes for homotopy limits. Let $\Phi : \mathbf{V} \to \mathbf{V}'$ be a normal closed functor with a $\mathbf{V}'$-left adjoint $\check{\Phi}$. Let $\mathbf{A}'$ and $\mathbf{B}'$ be $\mathbf{V}'$-categories, and let $P : (\check{\Phi}_* \mathbf{A}')^{\mathrm{op}} \to \mathbf{V}$ and $R : \check{\Phi}_* \mathbf{B}' \to \mathbf{C}$ be $\mathbf{V}$-functors. We use the notation of 2.3.5 in what follows.

2.6.4. Theorem. (i) *If* $Q : \mathbf{A}' \otimes' \mathbf{B}' \to \mathbf{V}'$ *is a* $\mathbf{V}'$*-functor, then*

$$\{P(:), \{Q(:, \cdot), R^b(\cdot)\}^{\#}_{\Phi.\mathbf{C}}\}_{\mathbf{C}} = \{(Q(:, \cdot) *' P^b(:))^{\#}_{\Phi.\mathbf{V}}, R(\cdot)\}_{\mathbf{C}}$$

(ii) *If* $Q : \mathbf{A}' \otimes' \mathbf{B}'^{\mathrm{op}} \to \mathbf{V}'$ *is a* $\mathbf{V}'$*-functor, then*

$$(P(:) * (Q(:, \cdot) *' R^b(\cdot))^{\#} = (Q(:, \cdot) *' P^b(:))^{\#} * R(\cdot).$$

Proof. By 2.4.3,

$$\{Q(-, \cdot), R^b(\cdot)\}^{\#}_{\Phi.\mathbf{C}} = \{(\check{\Phi} Q)^{\#}(-, \cdot), R(\cdot)\}_{\mathbf{C}}$$

and by 2.5.3,

$$(Q(:, -) *' P^b(:))^{\#}_{\Phi.\mathbf{V}} = (\check{\Phi} Q)^{\#}(:, -) * P(:).$$

Hence, using the symmetry of the tensor product in $\mathbf{V}$, the results are immediate from 2.6.3.

2.7. Aspects of Yoneda and Kan

The formulas developed in the preceding sections are very useful in expressing the Yoneda lemma and in describing Kan extensions. (See [6, 7], etc.) If $\mathbf{V}$ is a closed category and $\mathbf{A}$ is a $\mathbf{V}$-category, then we write the $\mathbf{V}$-valued Yoneda functors as

$$Y_{\mathbf{A}} : \mathbf{A} \to [\mathbf{A}^{\mathrm{op}}, \mathbf{V}] : A \mapsto \mathbf{A}(-, A),$$

$$Y^{\mathbf{A}} : \mathbf{A}^{\mathrm{op}} \to [\mathbf{A}, \mathbf{V}] : V \mapsto \mathbf{A}(A, -).$$

The Yoneda lemma (or representation theorems) can be written in a number of equivalent forms:

(i) If $F : \mathbf{A} \to \mathbf{B}$ is a $\mathbf{V}$-functor, where $\mathbf{B}$ is complete and cocomplete, then

$$\int_A \mathbf{A}(A', A) \pitchfork F(A) \cong F(A') \quad \text{and} \quad \int^A \mathbf{A}(A, A') \bar{\otimes} F(A) \cong F(A').$$

Equivalently, in the notation of 2.6,

$$\{\mathbf{A}(-,\bullet),\ F(\bullet)\}\cong F(-) \quad\text{and}\quad A(\bullet,-)*F(\bullet)\cong F(-);$$

or, writing $\mathbf{A}(-,=)=\mathrm{Y}_{\mathbf{A}}(=)(-)=Y^{\mathbf{A}}(-)(=)$,

$$\{Y_{\mathbf{A}}(\bullet)(-),\ F(\bullet)\}\cong F(-) \quad\text{and}\quad Y^{\mathbf{A}}(\bullet)(-)*F(\bullet)\cong F(-).$$

Strictly speaking, we should write $\mathrm{Ev}_{(-)}Y_{\mathbf{A}}(\bullet)$ for $Y_{\mathbf{A}}(\bullet)(-)$, etc.

(ii) If $H:\mathbf{A}^{\mathrm{op}}\to\mathbf{B}$ is a $\mathbf{V}$-functor, then there are analogous formulas derived from

$$\textstyle\int_A \mathbf{A}(A,A')\,\dot{\pitchfork}\,H(A)\cong H(A') \quad\text{and}\quad \int^A \mathbf{A}(A',A)\,\bar{\otimes}\,H(A)=H(A'). \quad \square$$

There is one more-intrinsic formula that occurs in the replacement schemes later, based on the symmetry of the tensor product in $\mathbf{V}$.

2.7.1. Proposition. *Let $F:\mathbf{A}\to\mathbf{V}$ be a $\mathbf{V}$-functor. Then*

$$(F(\bullet)*Y^{\mathbf{A}}(\bullet))_{[\mathbf{A},\mathbf{V}]}\cong F.$$

Proof. Let $A'\in\mathbf{A}$. Then

$$\begin{aligned}(F(\bullet)*Y^{\mathbf{A}}(\bullet))_{[\mathbf{A},\mathbf{V}]}(A') &= [\textstyle\int^A F(A)\,\bar{\otimes}\,Y^{\mathbf{A}}(A)](A')\\ &= \textstyle\int^A [(F(A)\,\bar{\otimes}\,Y^{\mathbf{A}}(A))(A')] = \int^A F(A)\otimes\mathbf{A}(A,A')\\ &= \textstyle\int^A \mathbf{A}(A,A')\otimes F(A)\cong F(A').\end{aligned}$$

2.7.2. Corollary. *Let $Q:\mathbf{A}\otimes\mathbf{B}\to\mathbf{V}$ be a $\mathbf{V}$-functor with $Q^{t}:\mathbf{B}\to[A,\mathbf{V}]$ its transpose under exponential adjointness. Then*

$$(Q(\bullet,-)*Y^{\mathbf{A}}(\bullet))_{[\mathbf{A},\mathbf{V}]}\cong Q^{t}(-).$$

The replacement schemes will be a special case of 2.6.4 in a form using the Yoneda imbedding. Abstractly, this comes about as follows. Suppose $[\mathbf{A},\mathbf{V}]$ has a closed category structure (e.g., from a premonoidal structure on $\mathbf{A}$, as in [5]), to be denoted by $[[\mathbf{A},\mathbf{V}]]$, in such a way that $\Phi=\underline{\lim}:[\mathbf{A},\mathbf{V}]\to\mathbf{V}$ is a closed functor with the constant functor $\nabla:\mathbf{V}\to[\mathbf{A},\mathbf{V}]$ as a $\mathbf{V}$-left adjoint. Suppose further that $\Phi_*[[\mathbf{A},\mathbf{V}]]\cong[\mathbf{A},\mathbf{V}]$. Then $Y^{\mathbf{A}}:\mathbf{A}^{\mathrm{op}}\to[\mathbf{A},\mathbf{V}]$ corresponds to a $[[\mathbf{A},\mathbf{V}]]$-functor $(Y^{\mathbf{A}})^{\#}:\nabla_*\mathbf{A}^{\mathrm{op}}\to[[\mathbf{A},\mathbf{V}]]$.

2.7.3. Definition (cf. [3]). Let $\mathbf{C}$ be a complete and cocomplete $[[\mathbf{A},\mathbf{V}]]$-category.

(i) If $F:\nabla_*\mathbf{A}^{\mathrm{op}}\to\mathbf{C}$ is a $[[\mathbf{A},\mathbf{V}]]$-functor, then the *total object* of F is

$$\mathrm{Tot}\ F=\{(Y^{\mathbf{A}})^{\#},F\}_{\mathbf{C}}$$

(ii) If $H:\nabla_*\mathbf{A}\to\mathbf{C}$ is a $[[\mathbf{A},\mathbf{V}]]$-functor, then the *diagonal object* of H is

$$\mathrm{Diag}\ H=((Y^{\mathbf{A}})^{\#}*H)_{\mathbf{C}}.$$

2.7.4. Definition. Let **A** and **B** be **V**-categories and let **C** be a [[**A**, **V**]]-category. Let $R : \nabla_* \mathbf{B} \to \mathbf{C}$ be a [[**A**, **V**]]-functor.

(i) If $Q : \mathbf{A} \otimes \mathbf{B} \to \mathbf{V}$ is a **V**-functor, then set

$$\tilde{Q}(R)(-) = \{Q(-,\cdot),\ R^b(\cdot)\}^{\#}_{\Phi.\mathbf{C}} : \ \nabla_* \mathbf{A}^{\mathrm{op}} \to \mathbf{C}.$$

(ii) If $Q : \mathbf{A} \otimes \mathbf{B}^{\mathrm{op}} \to \mathbf{V}$ is a **V**-functor, then set

$$\underset{\sim}{Q}(R)(-) = (Q(-,\cdot) * R^b(\cdot))^{\#}_{\Phi.\mathbf{C}} : \ \nabla_* \mathbf{A} \to \mathbf{C}.$$

2.7.5. Proposition. (i) Tot $\tilde{Q}(R) \cong \{Q^t(\cdot)^{\#},\ R(\cdot)\}_{\mathbf{C}}$

(ii) Diag $\underset{\sim}{Q}(R) \cong (Q^t(\cdot)^{\#} * R(\cdot))_{\mathbf{C}}$.

Proof. By 2.6.4 and 2.7.2

$$\begin{aligned}\text{Tot } \tilde{Q}(R) &= \{(Y^{\mathbf{A}})^{\#}(:),\ \{Q(:,\cdot),\ R^b(\cdot)\}^{\#}_{\Phi.\mathbf{C}}\}_{\mathbf{C}} \\ &= \{(Q(:,\cdot) * Y^{\mathbf{A}}(:))^{\#}_{[\mathbf{A},\mathbf{V}]},\ R(\cdot)\}_{\mathbf{C}} \\ &= \{Q^t(\cdot)^{\#},\ R(\cdot)\}_{\mathbf{C}} \quad \Box\end{aligned}$$

Kan extensions can also be expressed by ends and coends; namely, if $F : \mathbf{A} \to \mathbf{B}$ is a **V**-functor with **A** a small **V**-category and if **D** is a complete and cocomplete **V**-category, then

$$[F,\ \mathbf{D}] : [\mathbf{B},\ \mathbf{D}] \to [\mathbf{A},\ \mathbf{D}]$$

has a **V**-left adjoint given by

$$E_F(G)(-) = \mathrm{Lan}_F G(-) = \int^A \mathbf{B}(F(A), -)\bar{\otimes} G(A) = \mathbf{B}(F(\cdot), -) * G(\cdot)$$

and a **V**-right adjoint given by

$$E^F(G)(-) = \mathrm{Ran}_F G(-) = \int_A \mathbf{B}(-,\ F(A)) \pitchfork G(A) = \{\mathbf{B}(-,\ F(\cdot),\ G(\cdot))\}.$$

Note that not only does $[F,\ \mathbf{D}]$ trivially preserve tensors and cotensors, but also, by 2.2.2, E_F preserves tensors and E^F preserves cotensors. Kan extensions along the Yoneda functors occur in the applications later. Using the Yoneda lemma, the following results are immediate.

2.7.6. Proposition. (i) *Let* $G : \mathbf{A} \to \mathbf{D}$ *and* $H : \mathbf{A}^{\mathrm{op}} \to \mathbf{V}$. *Then* $E_{Y_{\mathbf{A}}}(G)(H) = H * G$ so $E_{Y_{\mathbf{A}}}(G)$ *has a* **V**-*right adjoint given by* $\hat{E}_{Y_{\mathbf{A}}}(G)(D) = \mathbf{D}(G(-),\ D)$.

(ii) *Let* $G : \mathbf{A} \to \mathbf{D}$ *and* $F : \mathbf{A} \to \mathbf{V}$. *Then* $E^{Y^{\mathbf{A}}}(G)(F) = \{F,\ G\}$ *so* $E^{Y^{\mathbf{A}}}(G)$ *is* **V**-*adjoint on the right to* $\hat{E}^{Y^{\mathbf{A}}}(G)(D) = \mathbf{D}(D,\ G(-))$.

Proof. By Yoneda,

$$\begin{aligned}E_{Y_{\mathbf{A}}}(G)(H) &= [\mathbf{A}^{\mathrm{op}},\ \mathbf{V}](Y_{\mathbf{A}}(\cdot),\ H) * G(\cdot) \\ &= \{Y_{\mathbf{A}}(\cdot)(:),\ H(:)\} * G(\cdot) = H(\cdot) * G(\cdot)\end{aligned}$$

and, by 2.5, $-*G$ has the indicated right adjoint. Finally, we need one other property of the Yoneda functor.

2.7.7. Proposition. *Let* $\Phi : \mathbf{V} \to \mathbf{V}'$ *be a closed functor. Then* $\Phi^d_{\mathbf{B}} : \Phi_*[\mathbf{B}, \mathbf{V}] \to [\Phi_*\mathbf{B}, \mathbf{V}']'$ *(see* 2.3.5*) commutes with Yoneda; i.e.,* $\Phi^d_{\mathbf{B}} \circ \Phi_*(Y^{\mathbf{B}}) = Y^{\Phi_*\mathbf{B}}$.

Proof. If $B \in \mathbf{B}$, $\Phi_*(Y^{\mathbf{B}})(B) = \mathbf{B}(B, -) \in \Phi_*[\mathbf{B}, \mathbf{V}]$. Composing with $\Phi^d_{\mathbf{B}}$ gives $[1, \bar{\Phi}](\Phi_*(\mathbf{B}(B, -)))$ which is $\bar{\Phi} \circ \Phi_*(\mathbf{B}(B, -)) : \Phi_*\mathbf{B} \to \mathbf{V}'$. The value of this on an object B' is $\Phi(\mathbf{B}(B, B')) = \Phi_*\mathbf{B}(B, B')$ and on a hom-object $\Phi_*\mathbf{B}(B', B'')$ it is the composition

$$\begin{array}{ccc} \Phi\mathbf{B}(B', B'') & \xrightarrow{\Phi(\circ^{\#})} & \Phi\mathbf{V}(\mathbf{B}(B, B'), \mathbf{B}(B, B'')) \\ & {\scriptstyle \bar{\Phi} \circ \Phi_*(\mathbf{B}(B, -))_{B', B''}} \searrow & \downarrow {\scriptstyle \tilde{\varphi}_{\mathbf{B}(B, B'), \mathbf{B}(B, B'')}} \\ & & \mathbf{V}'(\Phi\mathbf{B}(B, B'), \Phi\mathbf{B}(B, B'')) \end{array}$$

where $\circ^{\#}$ is adjoint to the composition for B. A calculation shows that the result is adjoint to the composition for $\Phi_*\mathbf{B}$ and hence $\bar{\Phi} \circ \Phi_*(\mathbf{B}(B, -)) = \Phi_*\mathbf{B}(B, -)$.

2.7.8. Remark. One can combine this with 2.7.6 to give a functorial version of 2.4.3, but we don't need this here.

3. Lax limits

In this section we apply the preceding results to the cartesian closed category $\mathbf{V} = \mathrm{Cat}$; i.e., the category of small categories. Since $\otimes = \times$, we shall write the internal hom functor exponentially; thus $\mathbb{Y}^{\mathbb{X}}$ is the category of functors from $\mathbb{X}$ to $\mathbb{Y}$ and natural transformations between such. The prefix Cat- is unattractive and is therefore replaced everywhere by the prefix 2-. As general references, see [10] or [12].

The underlying set functor $-$: Cat→Sets, which assigns to a category its set of objects, is part of a string of adjunctions

$$\pi \dashv D \dashv - \dashv G$$

where $D(X)$ is the discrete category with objects X, $G(X)$ is the trivial groupoid on X and $\pi(\mathbb{X})$ is the set of path components of $\mathbb{X}$ (cf. [10]). Clearly $D = G = \mathrm{id}$, $\pi D = \mathrm{id}$ and $\pi G = 1$ (the one point set). Note that by 2.2, $-$ is a normal closed functor and $D = \check{-}$ is a closed functor. A trivial calculation shows that $\pi = \check{D}$ is a $\bar{D}$-left adjoint to D and hence π preserves (finite) products and is a closed functor.

These closed functors give rise to a similar string of closed functors

$$\pi_* \dashv D_* \dashv -_* \dashv G_*$$

between 2-Cat (= "Cat"-Cat) and Cat (= "Set"-Cat). In general, if $\Phi : \mathbf{V} \to \mathbf{V}'$ is closed then the closed functor $\Phi_* : \mathbf{V}\text{-Cat} \to \mathbf{V}'\text{-Cat}$ determines the ($\mathbf{V}'$-Cat)-category $(\Phi_*)_*(\mathbf{V}\text{-Cat})$ and the ($\mathbf{V}'$-Cat)-functor $\bar{\Phi}_*$. In this case $\mathbf{V} = \text{Cat}$, $\mathbf{V}' = \text{Sets}$, and $\mathbf{V}'$-Cat = Cat = $\mathbf{V}$ so 2-Cat can be regarded (via $-_{**}$) as a 2-category. Cat itself is of course the fundamental 2-category, and all of the above functors determine 2-functors which will be denoted by the same symbols.

3.1. Lax comma categories

If $F_i : \mathbf{A}_i \to \mathbf{B}$, $i = 1, 2$, are 2-functors between 2-categories, then $F_1/\!/F_2$ denotes the *lax comma* category of F_1 and F_2 (called the 2-comma category in [10] and denoted by $[F_1, F_2]$ there). Its objects are triples (A_1, h, A_2) where $A_i \in \mathbf{A}_i$ and $h : F_1(A_1) \to F_2(A_2)$ in $\mathbf{B}$. Its morphisms are triples

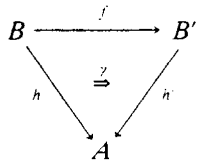
$$(f_1, \gamma, f_2) : (A_1, h, A_2) \to (A'_1, h', A'_2)$$

where $f_i : A_i \to A'_i$, $i = 1, 2$, and $\gamma : F_2(f_2)h \Rightarrow h'F_1(f_1)$ is a 2-cell in $\mathbf{B}$. Finally, its 2-cells are pairs

$$(\varphi_1, \varphi_2) : (f_1, \gamma, f_2) \Rightarrow (f'_1, \gamma', f'_2)$$

where $\varphi_i : f_i \Rightarrow f'_i$, is a 2-cell in $\mathbf{A}_i$, $i = 1, 2$ and $h'F_1(\varphi_1) \bullet \gamma = \gamma' \bullet F_2(\varphi_2)h$. (*Note.* Justaposition composition in $\mathbf{B}$ takes precedence over $\bullet$ composition within the hom categories $\mathbf{B}(B, C)$.) Usually $F_1/\!/F_2$ is regarded as a 2-category over $\mathbf{A}_1 \times \mathbf{A}_2$ via the projection 2-functor $P : F_1/\!/F_2 \to \mathbf{A}_1 \times \mathbf{A}_2$ given by $P(A_1, h, A_2) = (A_1, A_2)$, etc. The components of P are denoted by P_1 and P_2.

In particular, if $1 : \mathbf{1} \to \text{Cat}$ is the functor whose value is the category $\mathbf{1} \in \text{Cat}$ and $F : \mathbf{A} \to \text{Cat}$, then $1/\!/F$ is the "Grothendieck category" of F, denoted by $\mathbf{A}\int F$ in [12], which notation we adopt here. Another special case of interest is given by the identity 2-functor on a small 2-category $\mathbf{A}$ together with the "name of an object" 2-functor $\ulcorner A \urcorner : \mathbf{1} \to \mathbf{A}$. The lax comma category is written $\mathbf{A}/\!/A$ in this case. Note that morphisms here are diagrams



and a 2-cell $(\varphi, 1) : (F, \varphi, 1) \Rightarrow (f', \gamma', 1)$ satisfies $h'\varphi \bullet \gamma = \gamma'$. (Cf., Street [19] and [3].) If $s : A \to A'$ in $\mathbf{A}$, then composition with s determines a 2-functor $\mathbf{A}/\!/s : \mathbf{A}/\!/A \to \mathbf{A}/\!/A'$ and a 2-cell $\psi : s \Rightarrow s'$ determines a 2-natural transformation $\mathbf{A}/\!/\psi : \mathbf{A}/\!/s \Rightarrow \mathbf{A}/\!/s'$ this construction yielding a 2-functor $\mathbf{A}/\!/- : \mathbf{A} \to 2\text{-Cat}$. Dually, there is a 2-functor $-/\!/\mathbf{A} : \mathbf{A}^{\text{op}} \to 2\text{-Cat}$. We shall be interested in the composed 2-functor $\pi_* \circ (\mathbf{A}/\!/-) : \mathbf{A} \to \text{Cat}$. If $\mathbb{I}$ is a small category, then $D_*\mathbb{I}$ has no non-trivial 2-cells, so $D_*\mathbb{I}/\!/i = D_*(\mathbb{I}/i)$ where $\mathbb{I}/i$ is the ordinary comma category of objects over i.

Hence $\pi_*(D_*\mathbb{I}/\!/i) = (\mathbb{I}/i)$ and

$$\pi_* \circ (D_*\mathbb{I}/\!/-) = (\mathbb{I}/-)^{\#} : D_*\mathbb{I} \to \mathrm{Cat}$$

(see [3]). Similarly, $\pi_* \circ (-/\!/D_*\mathbb{I}) = (-/\mathbb{I})^{\#}$ which is denoted by $\mathbb{I} \backslash -$ in [3].

3.2. Lax natural transformations

2-Cat is a closed category via $\times$ and $[-,-]$ as in 2.3, so $[-,-]$ coincides with the exponentiation in the cartesian closed structure here. However, there is another "functor category" for a pair of 2-categories $\mathbf{A}$ and $\mathbf{B}$, denoted by Fun($\mathbf{A}$, $\mathbf{B}$). Its objects are 2-functors from $\mathbf{A}$ to $\mathbf{B}$ and if $F, G : \mathbf{A} \to \mathbf{B}$ are such, then Fun($\mathbf{A}$, $\mathbf{B}$)(F, G) is the category whose objects are *lax natural transformations* from F to G; i.e., 2-functors $\sigma : \mathbf{A} \to F/\!/G$ such that $P\sigma = \Delta_{\mathbf{A}}$ (the diagonal functor from $\mathbf{A}$ to $\mathbf{A} \times \mathbf{A}$). Its morphisms are 2-natural transformations $s : \sigma \Rightarrow \sigma'$ between such functors satisfying $Ps = \mathrm{id}$. (For a more explicit description, see "quasi-natural transformations" in [10] or [12].)

If $\mathbf{A}$ is a small 2-category, then $\nabla_{\mathbf{A}} : \mathbf{B} \to \mathrm{Fun}(\mathbf{A}, \mathbf{B})$ denotes the constant embedding; i.e., $\nabla_{\mathbf{A}}(B)(A) = B$ for all $A \in \mathbf{A}$. Left and right 2-adjoints (= Cat-adjoints), to $\nabla_{\mathbf{A}}$, if they exist, are called *lax colimits* and *lax limits* respectively. They are denoted by

$$\underset{\longrightarrow \mathbf{A}}{\mathrm{llim}} \dashv \nabla_{\mathbf{A}} \dashv \underset{\longleftarrow \mathbf{A}}{\mathrm{llim}}$$

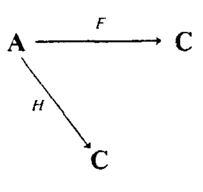
It can be shown easily by direct computation that if $\mathbf{B} = \mathrm{Cat}$ and $F : \mathbf{A} \to \mathrm{Cat}$ is a 2-functor then

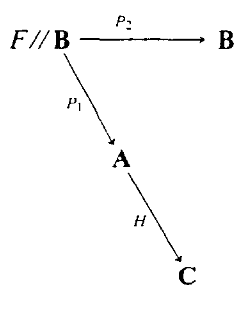
$$\underset{\longrightarrow \mathbf{A}}{\mathrm{llim}}\, F = \pi_*(\mathbf{A} \textstyle\int F) \quad \text{and} \quad \underset{\longleftarrow \mathbf{A}}{\mathrm{llim}}\, F : \Gamma(\mathbf{A} \textstyle\int F).$$

Here $\Gamma(A \int F)$ denotes the category of sections of $P : \mathbf{A} \int F \to \mathbf{A}$.

3.3. Lax Kan extensions

In [10] these are called quasi-Kan extensions. The lax left-Kan extension along a 2-functor $F : \mathbf{A} \to \mathbf{B}$ between small 2-categories is constructed as follows: If $H : \mathbf{A} \to \mathbf{C}$, replace F and H by





P_2 and HP_1 as illustrated. By definition, $l\,\mathrm{Lan}_F\, H = l\,\mathrm{Lan}_{P_2}(HP_1)$, where for any $K : F/\!/B \to \mathbf{C}$, one defines

$$l\,\mathrm{Lan}_{P_2}(K)(B) = \underleftarrow{\mathrm{llim}}_{P_2^{-1}(B)}(K \mid P_2^{-1}(B)).$$

Here $P_2^{-1}(B)$ is the fibre of $F/\!/\mathbf{B}$ over $B \in \mathbf{B}$; i.e., the sub 2-category consisting of i-cells, $i = 0, 1, 2$ which project to B.

In particular, taking F to be the identity 2-functor on $\mathbf{A}$, this gives a 2-functor

$$l\,\mathrm{Lan}_{\mathbf{A}} : \mathrm{Fun}(\mathbf{A}, \mathbf{B}) \to \mathbf{B}^{\mathbf{A}}$$

which is left adjoint to the inclusion 2-functor J going the other way and which coincides with Street's "first construction" in [18].

3.3.1. Proposition. *If* $\mathbb{C} \in \mathrm{Cat}$, *then the constant functor* $\Gamma_{\mathbf{A}}(\mathbb{C}) : \mathbf{A} \to \mathrm{Cat}$ *satisfies*

$$l\,\mathrm{Lan}_{\mathbf{A}}(\Gamma_{\mathbf{A}}(\mathbb{C}) = \pi_*(\mathbf{A}/\!/-) \times \mathbb{C}.$$

Proof. It is immediate that for $P_2 : \mathrm{Id}/\!/\mathbf{A} \to \mathbf{A}$, one has $P_2^{-1}(A) = \mathbf{A}/\!/A$, and the corresponding $K : \mathbf{A}/\!/A \to \mathrm{Cat}$ is the constant functor with value $\mathbb{C}$. But then clearly $(\mathbf{A}/\!/A) \int K = (\mathbf{A}/\!/A) \times D_*\mathbb{C}$. Since π preserves products, so does π_*; hence

$$\underleftarrow{\mathrm{llim}}_{P_2^{-1}(A)}(K \mid P_2^{-1}(A)) = \pi_*(\mathbf{A}/\!/A \times D_*\mathbb{C}) = \pi_*(\mathbf{A}/\!/A) \times \mathbb{C}.$$

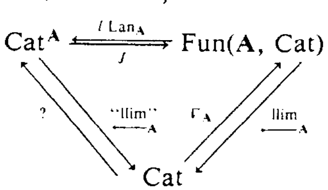
3.4.2. Theorem (Street [19]). *Let* $\mathbf{A}$ *be a small* 2-*category and* $\mathbf{B}$ *a complete and co-complete* 2-*category. If* $F : \mathbf{A} \to \mathbf{B}$ *is a* 2-*functor then*

$$\underleftarrow{\mathrm{llim}}_{\mathbf{A}}\, F = \{\pi_*(\mathbf{A}/\!/-),\ F\}_{\mathbf{B}}, \qquad \underrightarrow{\mathrm{llim}}_{\mathbf{A}}\, F = \pi_*(-/\!/A) * F.$$

Proof. The Cat-valued representable functors on $\mathbf{B}$ preserve and collectively create lax limits and colimits, so it is sufficient to prove these for $\mathbf{B} = \mathrm{Cat}$; e.g., assuming the result for Cat, one has

$$\begin{aligned}
\mathbf{B}(\underrightarrow{\mathrm{llim}}_{\mathbf{A}}\, F,\ B) &\cong \underleftarrow{\mathrm{llim}}_{\mathbf{A}^{\mathrm{op}}}\mathbf{B}(F(-),\ B) \cong \{\pi_*(\mathbf{A}^{\mathrm{op}}/\!/-),\ \mathbf{B}(F(-),\ B)\}_{\mathrm{Cat}} \\
&\cong \textstyle\int_{\mathbf{A}} \mathbf{B}(F(A),\ B)^{\pi_*(\mathbf{A}^{\mathrm{op}}/\!/A)} \cong \int_A \mathbf{B}(\pi_*(\mathbf{A}^{\mathrm{op}}/\!/A) \bar{\otimes} F(A),\ B) \\
&\cong \mathbf{B}(\textstyle\int^A \pi_*(\mathbf{A}^{\mathrm{op}}/\!/A) \bar{\otimes} F(A),\ B) \cong \mathbf{B}(\pi_*(-/\!/\mathbf{A}) * F,\ B)
\end{aligned}$$

since $\mathbf{A}^{\mathrm{op}}/\!/A = A/\!/\mathbf{A}$. Now, when $\mathbf{B} = \mathrm{Cat}$, one has the situation



where "$\underleftarrow{\mathrm{llim}}_{\mathbf{A}}$" $= \underleftarrow{\mathrm{llim}}_{\mathbf{A}} \circ J$ is the restriction of $\underleftarrow{\mathrm{llim}}_{\mathbf{A}}$ to $\mathrm{Cat}^{\mathbf{A}}$.

Since $l\,\mathrm{Lan}_{\mathbf{A}} \dashv J$ and $\nabla_{\mathbf{A}} \dashv \underleftarrow{\mathrm{llim}}_{\mathbf{A}}$, it follows that $l\,\mathrm{Lan}_{\mathbf{A}} \circ \nabla_{\mathbf{A}} \dashv$ "$\underleftarrow{\mathrm{llim}}_{\mathbf{A}}$". Hence "$\underleftarrow{\mathrm{llim}}_{\mathbf{A}}$" is a representable functor, being represented by

$$l\,\mathrm{Lan}_{\mathbf{A}}(\nabla_{\mathbf{A}}(1)) = \pi_*(\mathbf{A}/\!\!/-) \times 1 = \pi_*(\mathbf{A}/\!\!/-)$$

by 3.4.1. Thus

$$\underleftarrow{\mathrm{llim}}_{\mathbf{A}}(F) = \mathrm{Cat}^{\mathbf{A}}(\pi_*(\mathbf{A}/\!\!/-),\, F) = \{\pi_*(\mathbf{A}/\!\!/-),\, F\}_{\mathrm{Cat}}$$

by the comment following 2.4.1.

3.3.3. Remarks. Two other proofs of this can be found in Street [19]. It follows from 2.4.1 and 2.5.1 that $\underleftarrow{\mathrm{llim}}_{\mathbf{A}}(-) = \{\pi_*(\mathbf{A}/\!\!/\bullet), -\}_{\mathbf{B}}$ has a left adjoint given by $\pi_*(\mathbf{A}/\!\!/\bullet)\bar{\otimes}-$ and $\underrightarrow{\mathrm{llim}}_{\mathbf{A}}(-) = \pi_*(\bullet/\!\!/\mathbf{A}) * (-)$ has a right adjoint given by $\pi_*(\bullet/\!\!/\mathbf{A}) \pitchfork (-)$. In case $\mathbf{B} = \mathrm{Cat}$, then $\underleftarrow{\mathrm{llim}}_{\mathbf{A}}(-) = \mathrm{Cat}^{\mathbf{A}}(\pi_*(\mathbf{A}/\!\!/\bullet), -)$ with left adjoint $\pi_*(\mathbf{A}/\!\!/\bullet) \times (-)$ and $\underrightarrow{\mathrm{llim}}_{\mathbf{A}}(-)$ is Kan extension along Yoneda of $\pi_*(\bullet/\!\!/\mathbf{A})$ with right adjoint $(-)^{\pi_*(\bullet/\!\!/\mathbf{A})}$. Following Street [19], one can identify $\pi_*(\mathbf{A}/\!\!/-)$ in another way since, by the Yoneda lemma, $\mathrm{Cat}^{\mathbf{A}}(Y^{\mathbf{A}}(-), F) = F(-)$, one has $\underleftarrow{\mathrm{llim}}_{\mathbf{A}} F = \underleftarrow{\mathrm{llim}}_{\mathbf{A}} \mathrm{Cat}^{\mathbf{A}}(Y^{\mathbf{A}}(-), F) = \mathrm{Cat}^{\mathbf{A}}(\underrightarrow{\mathrm{llim}}_{\mathbf{A}^{\mathrm{op}}} Y^{\mathbf{A}}(-), F)$. Hence $\pi_*(\mathbf{A}/\!\!/-) = \underrightarrow{\mathrm{llim}}_{\mathbf{A}^{\mathrm{op}}} Y^{\mathbf{A}}$.

4. Homotopy limits

4.1. Set valued functor categories

Let $\mathbb{I}$ be a small category. The functor category $\mathrm{Sets}^{\mathbb{I}}$ is a cartesian closed category; i.e., $\otimes = \times$ and the internal hom, written exponentially, is given by the usual formula

$$G^F(i) = \mathrm{Sets}_0^{\mathbb{I}}(F \times \mathbb{I}(i,-),\, G).$$

The underlying set functor is $U = \underleftarrow{\lim}_{\mathbb{I}} : \mathrm{Sets}^{\mathbb{I}} \to \mathrm{Sets}$ and its left adjoint is $\check{U} = \nabla_{\mathbb{I}}$, since

$$\begin{aligned}\underleftarrow{\lim}_{\mathbb{I}} G^F &= \mathrm{Sets}(1, \underleftarrow{\lim}\, G^F) = \mathrm{Sets}_0^{\mathbb{I}}(\nabla_{\mathbb{I}} 1,\, G^F) \\ &= \mathrm{Sets}_0^{\mathbb{I}}(\nabla_{\mathbb{I}} 1 \times F,\, G) = \mathrm{Sets}_0^{\mathbb{I}}(F,\, G).\end{aligned}$$

We omit the subscript $\mathbb{I}$ from $\nabla_{\mathbb{I}}$ when no confusion is possible. Note that ∇ also has a left adjoint, $\underrightarrow{\lim}_{\mathbb{I}}$, but this is $\mathrm{Sets}^{\mathbb{I}}$-left adjoint to ∇ only if $\mathbb{I}$ is directed. (Cf., 2.2.2.) As a Sets-category, $\mathrm{Sets}^{\mathbb{I}}$ is tensored and cotensored by $X \bar{\otimes} F = \nabla X \times F$ and $X \pitchfork F = F^{\nabla X}$. (Cf., 2.3.1.)

4.2. Simplicial sets

Let Δ denote the category whose objects are the sets $[n] = \{0, \ldots, n\}$ and whose morphisms are non-decreasing functions. The category $\mathscr{K} = \mathrm{Sets}^{\Gamma^{\mathrm{op}}}$ is called the category of *simplicial sets*. It is cartesian closed with the structures described in 4.1. Standard terminology for $F \in \mathscr{S}$ is F_n instead of $F([n])$. Also, the representable functor $\nabla(-, [n])$ is denoted by $\nabla[n]$, so exponentiation is given by

$$(G^F)_n = \mathscr{S}_0(F \times \nabla[n], G),$$

the elements of which are called higher homotopies. The underlying set functor $\varprojlim_{\nabla^{op}}$ is evaluation at $[0]$ since $[0]$ is terminal in ∇.

4.3. *Simplicial-set-valued functor categories*

Let $\mathbb{I}$ be a small category. There are several ways to view the functor category $\mathscr{S}^{\mathbb{I}}$.

(i) $\mathscr{S}^{\mathbb{I}} = (\mathrm{Sets}^{\nabla^{op}})^{\mathbb{I}} = \mathrm{Sets}^{\nabla^{op} \times \mathbb{I}}$ which, as a Set-valued functor category, is cartesian closed with the structures described in 4.1. Hereafter $\mathscr{S}^{\mathbb{I}}$ will denote this closed category with the internal hom written exponentially; viz., Q^P for $P, Q \in \mathscr{S}^{\mathbb{I}}$.

(ii) $\mathscr{S}_0^{\mathbb{I}}$ is the underlying category of the complete and cocomplete $\mathscr{S}$-category $[(\nabla)_* \mathbb{I}, \mathscr{S}]$, by 2.3.1, in which the $\mathscr{S}$-valued hom functor is given by

$$[(\nabla)_* \mathbb{I}, \mathscr{S}](P, Q) = \textstyle\int_i \mathscr{S}(P(i), Q(i)) = \int_i Q(i)^{P(i)}.$$

Here ∇ refers to $\nabla_{\Delta^{op}}$.

(iii) There is another way to make $\mathscr{S}^{\mathbb{I}}$ into a $\mathscr{S}$-category using the adjoint pair

$$\nabla_{\mathbb{I}} \dashv \varprojlim_{\mathbb{I}} : \mathscr{S}^{\mathbb{I}} \to \mathscr{S}$$

in which $\mathscr{S}$ plays essentially the same role as Sets in 4.1. Since $\varprojlim_{\mathbb{I}}$ preserves products, it is a closed functor with respect to the cartesian closed structures and, since $\nabla_{\mathbb{I}}$ preserves products, it is a $\mathscr{S}$-left adjoint to $\varprojlim_{\mathbb{I}}$ (cf. 2.2.2). We can form the $\mathscr{S}$-category $(\varprojlim_{\mathbb{I}})_* \mathscr{S}^{\mathbb{I}}$ in which the K-valued hom between P and Q in $\mathscr{S}^{\mathbb{I}}$ is given by $\varprojlim_{\mathbb{I}}(Q^P)$. These two structures on $\mathscr{S}_0^{\mathbb{I}}$ coincide; i.e., we have the following result.

4.3.1. Proposition. $[(\nabla)_* \mathbb{I}, \mathscr{S}] = (\varprojlim_{\mathbb{I}})_* \mathscr{S}^{\mathbb{I}}$.

Proof. The underlying categories are the same and there are two possible $\mathscr{S}$-valued hom functors. To show they are the same it is sufficient to show the corresponding tensor products are the same. Let $F \in \mathscr{S}$, $P \in \mathscr{S}_0^{\mathbb{I}}$. The tensor product for $[(\nabla)_* \mathbb{I}, \mathscr{S}]$ is given by

$$(F \bar{\otimes} P)(i) = F \otimes P(i) = F \times P(i)$$

by 2.3.1, while the tensor product for $(\varprojlim_{\mathbb{I}})_* \mathscr{S}^{\mathbb{I}}$ is given by $F \bar{\otimes} P = \nabla_{\mathbb{I}} F \times P$, by 2.2.2. These are clearly the same.

4.4. *The nerve functor*

Let $J : \Delta \to \mathrm{Cat}$ be the "inclusion" functor given by $J([n]) = \mathbf{n} \oplus \mathbf{1}$. Here $\mathbf{n}$ denotes the ordered category with n objects and $\oplus$ means ordinal sum, so $\mathbf{n} \oplus \mathbf{1} = \mathbf{m}$ where

$m = n+1$. J is clearly a full embedding. Let $\check{N} : \mathscr{S} \to \mathrm{Cat}$ be the left Kan extension of J along Y_Δ as in 2.6, where Cat is treated as a Sets-category. As such, Cat is tensored by $X \tilde{\otimes} \mathbb{C} = DX \times \mathbb{C}$ (cf. Section 3), so if $F \in \mathscr{S}$, one has

$$\check{N}(F) = F * J = \textstyle\int^n DF_n \times (\mathbf{n} \oplus \mathbf{1}).$$

By 2.7.6, $\check{N}$ has a right adjoint $N : \mathrm{Cat} \to \mathscr{S}$ given by $N(\mathbb{C}) = \mathrm{Cat}_0(J(-), \mathbb{C})$. The functor N is called the *nerve* functor (cf., [9, 13], etc.) and $\check{N}$ (written this way to emphasize that it is the left adjoint to N) is called the *categorization* functor. It is important to know that $\check{N}$ has an *explicit description* from [9]; namely, $\check{N}(F)$ is the category whose set of objects is F_0 and whose maps are generated by the elements of F_1 subject to the relation that for each $\sigma \in F_2$, $(d_0\sigma)(d_2\sigma) = d_1\sigma$.

There are a number of simple properties of this situation.

(i) Since Y_Δ is fully faithful, $\check{N}Y = J$, so $\check{N}(\Delta[n]) = \mathbf{n} \oplus \mathbf{1}$.

(ii) Since J is fully faithful, $N(\mathbf{n} \oplus \mathbf{1}) = \Delta[n]$.

(iii) By the explicit description, the adjunction map $\varepsilon : \check{N}N \to \mathrm{id}$ is an isomorphism, so N is fully faithful and $\check{N}$ is dense.

(iv) Since N has a left adjoint, it preserves products and hence N is a normal closed functor with respect to the cartesian closed structures on Cat and $\mathscr{S}$.

(v) By the explicit construction, $\check{N}$ preserves products, so by 2.2.3, $\check{N}$ is $\mathscr{S}$-left adjoint to $\bar{N} : N_*\mathrm{Cat} \to \mathscr{S}$ i.e., if $\mathbb{C} \in \mathrm{Cat}$ and $F \in \mathscr{S}$, then

$$N(\mathbb{C}^{\check{N}F}) = (N\mathbb{C})^F.$$

Furthermore, $\check{N}$ is a normal closed functor.

(vi) By (iii) and (v), N preserves exponentiation; i.e.,

$$(N\mathbb{C})^{(N\mathbb{D})} \cong N(\mathbb{C}^{\check{N}N\mathbb{D}}) \cong N(\mathbb{C}^{\mathbb{D}}).$$

Hence $\bar{N} : N_*\mathrm{Cat} \to \mathscr{S}$ is $\mathscr{S}$-fully faithful.

4.5. *Homotopy limits*

We are interested in the simplicial version of homotopy limits as in [3] rather than the topological version as in [21]. There is apparently no convenient simplicial notion of a homotopy natural transformation between (homotopy) $\mathscr{S}$-functors (although there probably is a cubical version of this), so we cannot mimic the development of lax limits in Section 3. Instead, following [3], we shall use an analogue of 3.4.2 as a definition. In order to do so, given a $\mathscr{S}$-category $\mathbf{A}$, we need a $\mathscr{S}$-functor $Z_{\mathbf{A}} : \mathbf{A} \to \mathscr{S}$ which is analogous to $\pi_*(\mathbf{A}/\!/-)$ for a 2-category and which satisfies suitable properties. To describe this let $\pi_*(\check{N}_*\mathbf{A}/\!/-) : \check{N}_*\mathbf{A} \to \mathrm{Cat}$ be the 2-functor described in 3.1 and let $\bar{N} : N_*\mathrm{Cat} \to \mathscr{S}$ be the $\mathscr{S}$-functor associated with the normal closed functor $N : \mathrm{Cat} \to \mathscr{S}$ as in 2.2. By 2.3.3, $\check{N}_* \dashv N_*$ so $\pi_*(\check{N}_*\mathbf{A}/\!/-)$ corresponds to a $\mathscr{S}$-functor

$$\pi_*(\check{N}_*\mathbf{A}/\!/-)^b : \mathbf{A} \to N_*\mathrm{Cat}.$$

4.5.1. Definition. $Z_{\mathbf{A}} = \bar{N} \circ \pi_*(\check{N}_* \mathbf{A} /\!/ -)^b : \mathbf{A} \to \mathscr{K}$.

4.5.2. Proposition. (i) $\check{N}^d_{\mathbf{A}}(Z_{\mathbf{A}}) = \pi_*(\check{N}_* \mathbf{A} /\!/ -)$.
(ii) *If* $\mathbb{I}$ *is a small category, then*

$$Z_{\Gamma_* \mathbb{I}} = N(\mathbb{I}/-)^{\#}.$$

Proof. (i) By 2.3.6,

$$\check{N}^d_{\mathbf{A}}(Z_{\mathbf{A}}) = (\check{N} \circ Z_{\mathbf{A}})^{\#} = (\check{N} \circ \bar{N} \circ \pi_*(\check{N}_* \mathbf{A} /\!/ -)^b)^{\#}$$
$$\cong \pi_*(\check{N}_* \mathbf{A} /\!/ -)^{b\#} = \pi_*(\check{N}_* \mathbf{A} /\!/ -)$$

since $\check{N} \circ \bar{N} \simeq \mathrm{id}$, by 4.4(iii).
(ii) The composition

$$\mathbb{I} \xrightarrow{\mathbb{I}/-} \mathrm{Cat}_0 \xrightarrow{N} \mathscr{K}_0$$

determines $N(\mathbb{I}/-)^{\#} : \Gamma_* \mathbb{I} \to \mathscr{K}$ by adjointness. On the other hand, $\mathbb{I}/- : \mathbb{I} \to \mathrm{Cat}_0$ corresponds by adjointness to $\pi_*(D_* \mathbb{I} /\!/ -) : D_* \mathbb{I} \to \mathrm{Cat}$ by the comment at the end of 3.1. But $D_* \mathbb{I} = \check{N}_* \Gamma_* \mathbb{I}$ and, by naturality, $\bar{N} \circ \pi_*(D_* \mathbb{I} /\!/ -)^b = N(\mathbb{I}/-)^{\#}$ since $\bar{N}_0 = N$.

4.5.3. Definition. Let $\mathbf{A}$ be a small $\mathscr{K}$-category and $M : \mathbf{A} \to \mathbf{B}$ a $\mathscr{K}$-functor. Then

$$\underset{\overleftarrow{\mathbf{A}}}{\mathrm{holim}}\, M = \{Z_{\mathbf{A}}, M\}_{\mathbf{B}}, \qquad \underset{\overrightarrow{\mathbf{A}}}{\mathrm{holim}}\, M = Z_{\mathbf{A}^{\mathrm{op}}} * M.$$

Remarks. If $\mathbf{A} = \Gamma_* \mathbb{I}$ where $\mathbb{I}$ is a small category, then by 4.5.2(ii),

$$\underset{\overleftarrow{\mathbb{I}}}{\mathrm{holim}}\, M = \{Z_{\Gamma_* \mathbb{I}}, M\}_{\mathbf{B}} = \textstyle\int_i N(\mathbb{I}/i) \pitchfork M(i),$$
$$\underset{\overrightarrow{\mathbb{I}}}{\mathrm{holim}}\, M = Z_{\Gamma_* \mathbb{I}^{\mathrm{op}}} * M = \textstyle\int^i N(i/\mathbb{I}) \bar{\otimes} M(i).$$

If $\mathbf{B} = \mathscr{K}$, then these formulas agree with the definitions in [3].
One can derive a general description of $\underset{\overleftarrow{\mathbf{A}}}{\mathrm{holim}}\, M$ from 2.4.1, since

$$\mathbf{B}(B, \underset{\overleftarrow{\mathbf{A}}}{\mathrm{holim}}\, M) = \mathbf{B}(B, \{Z_{\mathbf{A}}, M\})$$
$$\simeq [\mathbf{A}, \mathscr{K}](Z_{\mathbf{A}}(\cdot), \mathbf{B}(B, M(\cdot))).$$

If $B = \underset{\overleftarrow{\mathbf{A}}}{\mathrm{holim}}\, M$, then the identity map of B on the left hand side corresponds to a $\mathscr{K}$-natural transformation η with components

$$\eta_A : Z_{\mathbf{A}}(A) = N(\pi_*(\check{N}_* \mathbf{A} /\!/ A)) \to \mathbf{B}(\underset{\overleftarrow{\mathbf{A}}}{\mathrm{holim}}\, M, M(A))$$

η_A is a map in $\mathscr{K}$ so it has components for each n,

$$(\eta_A)_n : N(\pi_*(\check{N}_* \mathbf{A} /\!/ A))_n \to \mathbf{B}(\underset{\overleftarrow{\mathbf{A}}}{\mathrm{holim}}\, M, M(A))_n$$

E.g., if $\mathbf{B} = \mathscr{K}$, then the right hand side here is

$$\mathscr{K}(\underset{\overleftarrow{\mathbf{A}}}{\mathrm{holim}}\, M, M(A))_n = \mathscr{K}_0(\underset{\overleftarrow{\mathbf{A}}}{\mathrm{holim}}\, M \times \Delta[n], M(A)),$$

so that if $\sigma_n \in N(\pi_*(\check{N}_*\mathbf{A}/\!\!/A))_n$, then $(\eta_A)_n(\sigma_n)$ has components

$$((\eta_A)_n(\sigma_n))_p : (\underset{\longleftarrow\mathbf{A}}{\mathrm{holim}}\, M)_p \times \Delta(p,\ n) \rightarrow M(A)_p$$

satisfying a number of equations. (See [3].) The $\mathscr{K}$-natural transformation η is universal in the sense that given any $\mathscr{K}$-natural transformation, $\varphi : Z_{\mathbf{A}} \rightarrow \mathbf{B}(B,\ M)$, then there is a unique map $f : B \rightarrow \underset{\longleftarrow\mathbf{A}}{\mathrm{holim}}\, M$ such that $\varphi = \mathbf{B}(f,\ M) \circ \eta$; e.g., if $\mathbf{B} = \mathscr{K}$ as above, then

$$(\varphi_n(\sigma_n))_p = (f_p \times \Delta(p,\ n)) \circ ((\eta_A)_n(\sigma_n))_p.$$

4.5.4. Proposition. *Let $F : \mathbf{B} \rightarrow \mathbf{B}'$ be a $\mathscr{K}$-functor with a $\mathscr{K}$-left adjoint $\check{F}$. Then*

$$F(\underset{\longleftarrow\mathbf{A}}{\mathrm{holim}}\, M) = \underset{\longleftarrow\mathbf{A}}{\mathrm{holim}}\, FM, \qquad \check{F}(\underset{\longrightarrow\mathbf{A}}{\mathrm{holim}}\, M) = \underset{\longrightarrow\mathbf{A}}{\mathrm{holim}}\, \check{F}M.$$

Proof. Immediate from 2.4.2 and 2.5.2.

4.5.5. Proposition. (i) $\underset{\longleftarrow\mathbf{A}}{\mathrm{holim}} : [\mathbf{A},\ \mathbf{B}] \rightarrow \mathbf{B}$ *has a $\mathscr{K}$-left adjoint given by $L(B) = Z_{\mathbf{A}}(-)\bar{\otimes} B$.*

(ii) $\underset{\longrightarrow\mathbf{A}}{\mathrm{holim}} : [\mathbf{A},\ \mathbf{B}] \rightarrow \mathbf{B}$ *has a $\mathscr{K}$-right adjoint given by $R(B) = Z_{\mathbf{A}^{\mathrm{op}}}(-) \pitchfork B$.*

Proof. Immediate from 2.4.1 and 2.5.1.

We now turn to the replacement schemes of [3] for calculating $\overleftarrow{\mathrm{holim}}$ and $\overrightarrow{\mathrm{holim}}$. We consider only the case in which $\mathbf{A} = \nabla_*\mathbb{I}$, where $\mathbb{I}$ is an ordinary small category. These results are direct consequences of 2.7.5. In the situation described before 2.7.3, we take $\mathbf{V} = \mathrm{Sets}$, and $\mathbf{A} = \Delta^{\mathrm{op}}$ so $[\mathbf{A},\ \mathbf{V}] = \mathrm{Sets}^{\Delta^{\mathrm{op}}}$ and $[[\mathbf{A},\ \mathbf{V}]] = \mathscr{K}$. The functor Φ is the underlying set functor, $\underleftarrow{\lim}_{\Delta^{\mathrm{op}}}$, and clearly $\Phi_*\mathscr{K} = \mathscr{K}_0 = \mathrm{Sets}^{\Delta^{\mathrm{op}}}$. The Yoneda functor $Y^{\Delta^{\mathrm{op}}} : (\Delta^{\mathrm{op}})^{\mathrm{op}} \rightarrow \mathrm{Sets}^{\Delta^{\mathrm{op}}}$ is the same as $Y_\Delta : \Delta \rightarrow \mathrm{Sets}^{\Delta^{\mathrm{op}}}$ and it corresponds to the $\mathscr{K}$-functor $Y_\Delta^{\#} : \nabla_*\Delta \rightarrow \mathscr{K}$.

The total object and the diagonal object introduced in 2.7.3 in this case are described as follows. Let $F : \nabla_*\Delta \rightarrow \mathbf{C}$ be a $\mathscr{K}$-functor. Then

$$\mathrm{Tot}\ F = \{Y_\Delta^{\#},\ F\}_{\mathbf{C}} = \textstyle\int_n Y_\Delta^{\#}(n) \pitchfork F(n);$$

e.g., if $\mathbf{C} = \mathscr{K}$, then

$$\mathrm{Tot}\ F = \textstyle\int_n F(n)^{Y_\Delta^{\#}(n)}$$

so in degree p,

$$\begin{aligned}(\mathrm{Tot}\ F)_p &= (\textstyle\int_n F(n)^{Y_\Delta^{\#}(n)})_p = \textstyle\int_n (F(n)^{Y_\Delta^{\#}(n)})_p \\ &= \textstyle\int_n \mathrm{Sets}^{\Delta^{\mathrm{op}}}(Y_\Delta^{\#}(n) \times \Delta[p],\ F(n)) \\ &= \textstyle\int_n \mathrm{Sets}^{\Delta^{\mathrm{op}}}(\Delta(-,\ n) \times \Delta(-,\ p),\ F(n)(-))\end{aligned}$$

On the other hand, if $H : \nabla_*\Delta^{\mathrm{op}} \rightarrow \mathbf{C}$ is a $\mathscr{K}$-functor, then

$$\mathrm{Diag}\ H = (Y_\Delta^{\#} * H)_{\mathbf{C}} = \textstyle\int^n Y_\Delta^{\#}(n) \bar{\otimes} H(n);$$

e.g., if $\mathbf{C}=\mathscr{K}$, then

$$\text{Diag } H=\int^n Y^{\#}_{\Delta}(n)\times H(n)$$

so in degree p,

$$(\text{Diag } H)_p=(\int^n Y^{\#}_{\Delta}(n)\times H(n))_p=\int^n (Y^{\#}_{\Delta}(n)\times H(n))_p$$
$$=\int^n \Delta(p,\ n)\times H(n)_p=H(p)_p.$$

The "replacement" functors introduced in 2.7.4 have the following description in this case. The functor $N(\mathbb{I}/-):\mathbb{I}\to\text{Sets}^{\Delta^{\text{op}}}$ corresponds by adjointness to $Q=N(\mathbb{I}/-)^t:\Delta^{\text{op}}\times\mathbb{I}\to\text{Sets}$. If $R:\nabla_*\mathbb{I}\to\mathbf{C}$ is a $\mathscr{K}$-functor, then (following the terminology of [3]) we set

$$\prod{}^* R=\tilde{Q}(R)=\{N(\mathbb{I}/\bullet)_{(-)},\ R^b(\bullet)\}^{\#}_{\mathbf{C}_0}$$
$$=(\int_i N(\mathbb{I}/i)_{(-)}\pitchfork R(i))^{\#}\ :\ \nabla_*\Delta\to\mathbf{C}.$$

The cotensor here is over Sets, so by the remark preceding 2.2.1, in degree n,

$$\prod{}^* R(n)=\int_i \prod_{N(\mathbb{I}/i)_n} R(i).$$

Similarly, the functor $N(-/\mathbb{I}):\mathbb{I}^{\text{op}}\to\text{Sets}^{\Delta^{\text{op}}}$ corresponds to $Q=N(-/\mathbb{I})^b:\Delta^{\text{op}}\times\mathbb{I}^{\text{op}}\to\text{Sets}$. If $R:\nabla_*\mathbb{I}\to\mathbf{C}$ is a $\mathscr{K}$-functor, then we set

$$\coprod{}_*\mathbf{R}=\underset{\sim}{Q}(R)=(N(\bullet/\mathbb{I})_{(-)}*R^b(\bullet))_{\mathbf{C}}$$
$$=(\int^i N(i/\mathbb{I})_{(-)}\tilde{\otimes}R(i))^{\#}\ :\ \nabla_*\Delta^{\text{op}}\to\mathbf{C},$$

so in degree n,

$$\coprod{}_*R(n)=\int^i \coprod_{N(i/\mathbb{I})_n} R(i).$$

4.5.6. Theorem (The replacement schemes). *Let $R:\nabla_*\mathbb{I}\to\mathbf{C}$ be a $\mathscr{K}$-functor. Then*

$$\underset{\longleftarrow\nabla_*\mathbb{I}}{\text{holim}}\, R=\text{Tot}\ \prod{}^*R,\qquad \underset{\longrightarrow\nabla_*\mathbb{I}}{\text{holim}}\, R=\text{Diag}\ \coprod{}_*R.$$

Proof. Immediate from 2.7.5.

Using the preceding descriptions, one can describe these homotopy limits as follows, in case $\mathbf{C}=\mathscr{K}$.

(i) $(\underset{\longleftarrow\nabla_*\mathbb{I}}{\text{holim}}\, R)_p=(\text{Tot}\ \prod{}^*R)_p$

$$=\int_n \text{Sets}^{\Delta^{\text{op}}}(\Delta(-,\ n)\times\Delta(-,\ p),\ \int_i \prod_{N(\mathbb{I}/i)_n} R(i)(-))$$
$$=\int_i\int_n \prod_{N(\mathbb{I}/i)_n} \text{Sets}^{\Delta^{\text{op}}}(\Delta(-,\ n)\times\Delta(-,\ p),\ R(i)(-)).$$

Thus a p-simplex of $\underset{\longleftarrow\nabla_*\mathbb{I}}{\text{holim}}\, R$ is a choice for each element of $N(\mathbb{I}/i)_n$ of a map (in $\mathscr{X}$) of $\Delta[n]\times\Delta[p]$ into $R(i)$ in such a way that the choice is "natural" in n and i.

(ii) $$(\underrightarrow{\mathrm{holim}}_{\Gamma.i}\, R)_p = (\mathrm{Diag}\ \coprod{}_* R)_p$$
$$= (\coprod{}_* R(p))_p = \int_i \coprod_{N(i/-)_p} R(i)_p$$

4.6. Homotopy limits in Cat

In this section, the relation between homotopy limits and lax limits will be discussed. Let **B** be a 2-category. In the corresponding $\mathscr{X}$-category $N_*\mathbf{B}$, we can consider homotopy limits indexed by a $\mathscr{X}$-category $\mathbf{A}'$.

4.6.1. Proposition. *Let* $F:\mathbf{A}'\to N_*\mathbf{B}$ *be a* $\mathscr{X}$*-functor, with* $F^\#:\check{N}_*\mathbf{A}'\to\mathbf{B}$ *the corresponding* 2*-functor. Then*

$$\underleftarrow{\mathrm{holim}}_{\mathbf{A}'}\, F \simeq \underleftarrow{\mathrm{llim}}_{\check{N}_*\mathbf{A}'}\, F^\#, \qquad \underrightarrow{\mathrm{holim}}_{\mathbf{A}'}\, F \simeq \underrightarrow{\mathrm{llim}}_{\check{N}_*\mathbf{A}'}\, F^\#.$$

Proof.

$$\underleftarrow{\mathrm{holim}}_{\mathbf{A}'}\, F = \{Z_{\mathbf{A}'},\ F\}_{N_*\mathbf{B}} \cong \{\check{N}^d_{\mathbf{A}'}(Z_{\mathbf{A}'}),\ F^\#\}_{\mathbf{B}}$$
$$\cong \{\pi_*(\check{N}_*\mathbf{A}'/\!/-),\ F^\#\}_{\mathbf{B}} = \underleftarrow{\mathrm{llim}}_{\check{N}_*\mathbf{A}'}\, F^\#,$$

the first equality by Definition 4.5.3, the second by 2.4.3, the third by 4.5.2 and the last by 3.4.2. The second formula follows similarly, replacing 2.4.3 by 2.5.3.

4.6.2. Corollary. *Let* $G:\mathbf{A}\to\mathbf{B}$ *be a* 2*-functor. Then*

$$\underleftarrow{\mathrm{holim}}_{N_*\mathbf{A}}\, N_*G \cong \underleftarrow{\mathrm{llim}}_{\mathbf{A}}\, G, \qquad \underrightarrow{\mathrm{holim}}_{N_*\mathbf{A}}\, N_*G \cong \underrightarrow{\mathrm{llim}}_{\mathbf{A}}\, G.$$

Proof. This is immediate from 4.6.1 since $\check{N}_*N_*\mathbf{A}\simeq\mathbf{A}$ and $(N_*G)^\#\simeq G$.

4.6.3. Proposition. (i) *Let* $G:\mathbf{A}\to\mathrm{Cat}$ *be a* 2*-functor. Then*

$$\bar{N}(\underleftarrow{\mathrm{llim}}_{\mathbf{A}}\, G) \cong \underleftarrow{\mathrm{holim}}_{N_*\mathbf{A}}\, N^d_{\mathbf{A}}(G).$$

(ii) *Let* $F:\mathbf{A}'\to\mathscr{X}$ *be a* $\mathscr{X}$*-functor. Then*

$$\check{N}(\underrightarrow{\mathrm{holim}}_{\mathbf{A}'}\, F) \cong \underrightarrow{\mathrm{llim}}_{\check{N}_*\mathbf{A}'}\, \check{N}^d_{\mathbf{A}'}(F).$$

Proof.

$$\bar{N}(\underleftarrow{\mathrm{llim}}_{\mathbf{A}}\, G) \cong \bar{N}(\underleftarrow{\mathrm{holim}}_{N_*\mathbf{A}}\, N_*G)$$
$$\cong \underleftarrow{\mathrm{holim}}_{N_*\mathbf{A}}\, (\bar{N}\circ N_*G) \cong \underleftarrow{\mathrm{holim}}_{N_*\mathbf{A}}\, N^d_{\mathbf{A}}(G),$$

the first equality by 4.6.1, the second by 4.5.4 and the third by the definition of $N^d_{\mathbf{A}}$ in 2.3.5(iii). The second formula follows similarly, replacing 2.3.5 by 2.3.6.

4.6.4. Corollary. *Let* $G:\mathbf{A}\to\mathrm{Cat}$. *Then*

$$\underleftarrow{\mathrm{llim}}_{\mathbf{A}}\, G = \check{N}\ \underleftarrow{\mathrm{holim}}_{N_*\mathbf{A}}\, N^d_{\mathbf{A}}(G), \qquad \underrightarrow{\mathrm{llim}}_{\mathbf{A}}\, G = \check{N}\ \underrightarrow{\mathrm{holim}}_{N_*\mathbf{A}}\, N^d_{\mathbf{A}}(G).$$

Proof. The first formula follows by applying $\check{N}$ to the first formula in 4.6.3 since $\check{N}\bar{N} = \text{id}$. For the second, observe that under the isomorphism $\text{Cat} \simeq \check{N}_* N_* \text{Cat}$, one has $\check{N}^d_{N.\text{Cat}}(\bar{N}) \cong \text{id}$. Hence $G \simeq \check{N}_* N_* G \simeq \check{N}^d_{N.\mathbf{A}}(\bar{N} \circ N_* G) = \check{N}^d_{N.\mathbf{A}}(N^d_{\mathbf{A}}(G))$, so

$$\underrightarrow{\text{llim}}_{\mathbf{A}}\, G \simeq \underrightarrow{\text{llim}}_{\check{N}.N.\mathbf{A}}\, \check{N}^d_{N.\mathbf{A}}(N^d_{\mathbf{A}}(G)) \simeq \check{N}(\underrightarrow{\text{holim}}_{N.\mathbf{A}}\, N^d_{A}(G)).$$

If the index category over which these limits are taken is an ordinary category $\mathbb{I}$, then much of the notation can be suppressed by ignoring the various closed category structures and just treating everything as ordinary categories; i.e., $N : \text{Cat}_0 \to \mathscr{K}_0$ with an ordinary left adjoint $\check{N}$.

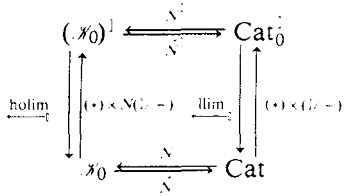
4.6.5. Proposition. *Let* $F : \mathbb{I} \to \mathscr{K}_0$ *and* $G : \mathbb{I} \to \text{Cat}$ *be functors. Then*

$$N(\underleftarrow{\text{llim}}_{\mathbb{I}}\, G) = \underleftarrow{\text{holim}}_{\mathbb{I}}\, NG, \qquad \underrightarrow{\text{llim}}_{\mathbb{I}}\, \check{N}F = \check{N}\, \underrightarrow{\text{holim}}_{\mathbb{I}}\, F$$

and

$$\underleftarrow{\text{llim}}_{\mathbb{I}}\, G = \check{N}\, \underleftarrow{\text{holim}}_{\mathbb{I}}\, NG, \qquad \underrightarrow{\text{llim}}_{\mathbb{I}}\, G = \check{N}\, \underrightarrow{\text{holim}}_{\mathbb{I}}\, NG.$$

Proof. These follow from the preceding discussion or from the following direct argument. Consider the diagram



Then $\check{N} \dashv N$, $\check{N}^{\mathbb{I}} \dashv N^{\mathbb{I}}$, $\underleftarrow{\text{holim}}_{\mathbb{I}} \dashv (\bullet) \times N(\mathbb{I}/-)$ by 4.5.5 and $\underleftarrow{\text{llim}}_{\mathbb{I}} \dashv (\bullet) \times (\mathbb{I}/-)$ by 3.4.3. But by 4.4(v), $\check{N}$ preserves products and, since $\check{N}N \cong \text{id}$, one has

$$\check{N}^{\mathbb{I}}((\bullet) \times N(\mathbb{I}/-)) = \check{N}(\bullet) \times \check{N}N(\mathbb{I}/-) = \check{N}(\bullet) \times (\mathbb{I}/-)$$

so the diagram of left adjoints commutes. Hence so does the diagram of right adjoints, i.e.,

$$N\, \underleftarrow{\text{llim}}_{\mathbb{I}} = \underleftarrow{\text{holim}}_{\mathbb{I}}\, N^{\mathbb{I}}.$$

The formula for colimits follows similarly, using the fact that $\check{N}$ preserves exponentiation.

4.6.6. Remark. In [20], it is shown that $N(\underleftarrow{\text{llim}}_{\mathbb{I}}\, G)$ and $\underleftarrow{\text{holim}}_{\mathbb{I}}\, NG$ are homotopy equivalent. The only unknown relation is that between $\check{N}(\underleftarrow{\text{holim}}_{\mathbb{I}}\, F)$ and $\underleftarrow{\text{llim}}_{\mathbb{I}}\, \check{N}F$. If $F = NG$, these are isomorphic by the above, but we have not tried to determine the answer in general.

4.6.7. Example. It follows from 4.6.4 or 4.6.5 that any lax limit in Cat can be calculated from a corresponding homotopy limit in $\mathscr{K}$. These in turn can be constructed from the replacement schemes in 4.5.6. For instance, if $G : \mathbb{I} \to \mathrm{Cat}$, then by the description following 4.5.6,

$$\begin{aligned}(\underset{\longleftarrow\mathbb{I}}{\mathrm{holim}}\, NG)_p &= \textstyle\int_i \int_n \prod_{N(\mathbb{I}/i)_n} \mathrm{Sets}^{\Delta^{\mathrm{op}}}(\Delta(-, n) \times \Delta(-, p), NG(i)(-)) \\ &= \textstyle\int_i \int_n \prod_{N(\mathbb{I}/i)_n} \mathrm{Cat}_0((\mathbf{n} \oplus \mathbf{1}) \times (\mathbf{p} \oplus \mathbf{1}), G(i)) \\ &= \textstyle\int_n \mathrm{Cat}_0((\mathbf{n} \oplus \mathbf{1}) \times (\mathbf{p} \oplus \mathbf{1}), \int_i \prod_{N(\mathbb{I}/i)_n} G(i))\end{aligned}$$

since $\check{N} \dashv N$, $\check{N}$ preserves products, and $\check{N}(\Delta(-, n)) = \mathbf{n} \oplus \mathbf{1}$. Since $\underset{\longleftarrow\mathbb{I}}{\mathrm{llim}}\, G = \check{N}\, \underset{\longleftarrow\mathbb{I}}{\mathrm{holim}}\, NG$ we only need the expressions above for $p = 0$, 1, and possibly 2 if composition is not obvious.

In simple cases this calculation is actually feasible. As an example, let $\mathbb{I} = \mathbf{2} = (0 \to 1)$. Then $\mathbb{I}/0 \cong \mathbf{1}$ and $\mathbb{I}/1 \cong \mathbf{2}$ so $N(\mathbb{I}/0) = \Delta[0]$ which has one simplex of each degree and $N(\mathbb{I}/1) = \Delta[1]$ which has $n+2$ simplices of degree n, the ith one being the map of $[n]$ to $[1]$ taking the last i entries to 1 for $0 \le i \le n+1$. A functor $G : \mathbf{2} \to \mathrm{Cat}$ determines a functor $f : \mathbb{A} \to \mathbb{B}$ in Cat, with $G(0) = \mathbb{A}$, and $G(1) = \mathbb{B}$. A simple calculation shows that

$$\textstyle\int_i \prod_{N(\mathbb{I}/i)_n} G(i) = \mathbb{A} \times \mathbb{B}^{n+1}$$

which is functorial in $n+2$ via various projections together with $f : \mathbb{A} \to \mathbb{B}$ when necessary; hence

$$\begin{aligned}(\underset{\longleftarrow\mathbf{2}}{\mathrm{holim}}\, NG)_0 &= \textstyle\int_n \mathrm{Cat}(\mathbf{n} \oplus \mathbf{1}, \mathbb{A} \times \mathbb{B}^{n+1}) \\ (\underset{\longleftarrow\mathbf{2}}{\mathrm{holim}}\, NG)_1 &= \textstyle\int_n \mathrm{Cat}(\mathbf{2} \times (\mathbf{n} \oplus \mathbf{1}), \mathbb{A} \times \mathbb{B}^{n+1}),\end{aligned}$$

so the entries in $(\underset{\longleftarrow\mathbf{2}}{\mathrm{holim}}\, NG)_1$ are natural transformations between the entries in $(\underset{\longleftarrow\mathbf{2}}{\mathrm{holim}}\, NG)_0$. To calculate these latter entries, it is sufficient to look at $n = 0$ and 1 since all higher n's are expressible (via degeneracies or colimits) in terms of $\mathbf{n} \oplus \mathbf{1} = \mathbf{2}$. For $n = 0$, one has just $\mathbb{A} \times \mathbb{B}$ and for $n = 1$, one has functors

$$\gamma : \mathbf{2} \to \mathbb{A} \times \mathbb{B} \times \mathbb{B};$$

i.e., $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ where γ_1 is a morphism in $\mathbb{A}$ and γ_2 and γ_3 are morphisms in $\mathbb{B}$. From the map $s^0 : [1] \to [0]$ it follows that $\gamma_1 = \mathrm{id}_A$ and $\gamma_3 = \mathrm{id}_B$ and from the two maps, $d^0, d^1 : [0] \to [1]$ it follows that $\gamma_2 : f(A) \to B$ in $\mathbb{B}$; i.e., $(\underset{\longleftarrow\mathbf{2}}{\mathrm{holim}}\, NG)_0$ consists of the objects of the comma category $f/\mathbb{B}$, and hence $(\underset{\longleftarrow\mathbf{2}}{\mathrm{holim}}\, NG)_1$ consists of the maps in this comma category. This yields the well-known result.

$$\underset{\longleftarrow\mathbf{2}}{\mathrm{llim}}\, G = N\, \underset{\longleftarrow\mathbf{2}}{\mathrm{holim}}\, NG = f/\mathbb{B}.$$

4.7. *Extensions*

Let $\mathbf{V}$ be an arbitrary closed category and let $M : \mathscr{X}_0 \to \mathbf{V}_0$ be any functor. Then one can more or less trivially extend the definition of homotopy limits to $\mathbf{V}$ as follows:

4.7.1. Definition. If $F : \mathbb{I} \to \mathbf{B}_0$ is a functor where $\mathbf{B}$ is a $\mathbf{V}$-category, then

$$\underleftarrow{\mathrm{holim}}_{\mathbb{I}}\, F = \{M \circ Z_{\mathbb{I}}, F\} = \int_i (MN(\mathbb{I}/i) \pitchfork F(i))$$

$$\underrightarrow{\mathrm{holim}}_{\mathbb{I}}\, F = M \circ Z_{\mathbb{I}^{op}} * F = \int^i MN(i/\mathbb{I}) \bar{\otimes} F(i).$$

If M has a right adjoint $\hat{M} : \mathbf{V} \to \mathscr{K}$ which is a normal closed functor, such that M is $\mathscr{K}$-left adjoint to $\hat{M}$, then this agrees with the notion of homotopy limits in $\hat{M}_*\mathbf{B}$ by 2.4.3 and 2.5.3. If M does not have such a right adjoint then this notion of homotopy limits may or may not be interesting. For instance, if R is a ring and if $(\mathrm{Mod}\ R)^+$ is the category of positive chain complexes of (right) R-modules, then the chain complex functor $C.(-, R) : \mathscr{K} \to (\mathrm{Mod}\ R)^+$ gives a notion of homotopy limits in $(\mathrm{Mod}\ R)^+$ whose usefulness is justified by its use in describing derived functors of limits. See, e.g., [3] or, for a much more thorough account with "classical" references, [15].

Of course, one may also generalize lax limits in the same way for any functor $M : \mathrm{Cat} \to \mathbf{V}$. Taking, for example, $N : \mathrm{Cat} \to \mathscr{K}$ shows that homotopy limits are special cases of lax limits in this generalized sense. (Note: N does not preserve coequalizes so it has no right adjoint.) Since one proves in 3.4.2 that lax limits are represented by the construction $(\mathbb{I}/-)$, this perhaps helps to understand why homotopy limits are defined in terms of $N(\mathbb{I}/-)$.

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