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Higher homotopies of natural constructions

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1. Introduction and main results

In this note I would like to draw attention to a precise formulation of the following intuitively obvious claim "Any functorial change introduced in lower dimensions of a space must have a similar influence via 'higher homotopies' on higher dimensions." We will consider homotopy properties of functors $\boldsymbol{F}$ from spaces to spaces. Although we restrict attention to idempotent functors (with $\boldsymbol{F} \sim \boldsymbol{F} \circ \boldsymbol{F}$) this is by no means a crucial assumption as explained in the last sections (3.3). It simply makes the formulation neater. We have in mind localization functors [7] but also the James construction, Dold–Kan functor $X \to RX$, Bousfield–Kan functor $X \to R_\infty X$, etc.

Here is an example of a result in the above-mentioned direction. A functor $\boldsymbol{F}: \{\text{spaces}\} \to \{\text{spaces}\}$ is called continuous if the function between mapping spaces map $(X, Y) \to$ map $(\boldsymbol{F}X, \boldsymbol{F}Y)$ is continuous.

1.1. Proposition. *Let $\boldsymbol{F}$ be a continuous, co-augmented, homotopy idempotent functor from spaces to spaces. If $\boldsymbol{F}\{0, 1\}$ is a connected space then $\boldsymbol{F}X$ is a contractible space for all X.*

Proof. For a simple, "one diagram", proof see 1.10.

Remark. Thus if a natural coaugmented construction: $X \to \boldsymbol{F}X$ kills $\pi_0 X$ then it kills $\pi_n X$ for all $n \geq 0$. Similarly in the pointed category one cannot kill $\pi_j X$ in a natural and idempotent fashion for any $j \geq 1$ without killing at the same time all higher homotopy groups $\pi_l X$ for $l \geq j$.

We now give a general "higher homotopies" theorem of which the above is a special case and from which some other interesting cases follow directly.

We consider functors $\boldsymbol{F}: \{\text{spaces}\} \to \{\text{spaces}\}$ that are continuous, coaugmented and idempotent up to homotopy:

Continuity means that the functorial correspondence of spaces of maps: $\text{map}(X, Y) \to \text{map}(\boldsymbol{F}X, \boldsymbol{F}Y)$, where $f \mapsto \boldsymbol{F}f$ is a continuous function or simplicial one

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if one works simplicially. It is equivalent by adjunction to the condition that $\boldsymbol{F}$ comes with a natural transformation

$$K \times \boldsymbol{F}X \to \boldsymbol{F}(K \times X)$$

between these two-variable functors, that commutes with the obvious maps from $K \times X$ compare [5].

Idempotency for coaugmented $\boldsymbol{F}$ means that if $j: X \to \boldsymbol{F}X$ is the coaugmentation then both $j\boldsymbol{F}$ and $\boldsymbol{F}j: \boldsymbol{F}X \to \boldsymbol{FF}X$ are homotopy equivalences. Examples of such functors are localization functors $X \to X_{(p)}$ and taking a Postnikov section, $X \to P_n X$ (Although in the usual presentation they are not continuous, they can be made such).

Pointed, unpointed: We will state the results for the category $\mathscr{S}$ of (unpointed) spaces but all the major results in this paper are true without change in the pointed category. We are given a function complex construction in our category of spaces that we denote either by $\mathrm{map}(X, Y)$ or by Y^X. The pointed case allows us to consider higher homotopy groups. For example the pointed version of 1.2 when applied to the map $\varphi: * \to S^n$ says that if a functor $\boldsymbol{F}$ "kills" $\pi_n X$ it "kills" $\Omega^n X$, i.e., then it kills also all higher homotopy groups (above n) of any space X.

1.2 Theorem. *Let $\boldsymbol{F}: \{spaces\} \to \{spaces\}$ be a continuous, coaugmented and idempotent functor and let $\varphi: A \to B$ be any cofibration. If the map induced by φ, $\boldsymbol{F}$:*

$$map(\varphi, \boldsymbol{F}X): map(B, \boldsymbol{F}X) \to map(A, \boldsymbol{F}X)$$

is

(1) *a one-to-one map on π_0 for $X = B$,*

(2) *a surjective map on π_0 for $X = A$;*

then it is a homotopy equivalence for every space X.

The point of 1.2 is that we assume a π_0-surjection and π_0-injection for certain spaces (A, B) and get homotopy equivalence for all spaces. A proof appears in 2.1.2.

The main tool for proving 1.2 is theorem 2.1.

Other readable formulations of 1.2 are the following weaker versions.

1.3 Corollary. *Let $\boldsymbol{F}$, $\varphi: A \to B$ as in 1.2. If for every space X the map of function spaces: $map(\varphi, \boldsymbol{F}X)$ induces an isomorphism on the sets of path components, i.e. on π_0, then, for every space X, it is a homotopy equivalence.*

Remark. If we take φ in 1.2 to be the map from the two points space to the one point space we get 1.1 as a special case of 1.2.

1.4 Corollary. *If $\varphi: A \to A$ is a self map and map $(\varphi, \boldsymbol{F}X)$ induces an isomorphism on π_0 for $X = A$ then it is a homotopy equivalence for all X.*

Proof. See 2.1.1.

Two familiar examples: As an illustration we offer two examples of application of the principle encoded in 1.2 and 1.4. The first says that if a functor $H\mathbb{Z}$-localizes in the sense of Bousfield [2] the homotopy groups in a given dimension then it does the same to homotopy groups in all higher dimensions. The second concerns turning the homotopy groups and modules into p-divisible ones and is of a similar nature:

1.5 Theorem. *Let $\boldsymbol{F}:\{spaces\} \to \{spaces\}$ be continuous, coaugmented, homotopy, idempotent functor. Assume that $\pi_1\boldsymbol{F}(\bigvee_\infty S^1)$ is an $H\mathbb{Z}$-local group where $\bigvee_\infty S^1$ is a countable wedge of circles. Then $\pi_n\boldsymbol{F}X$ are $H\mathbb{Z}$-local (as groups and as $\pi_1\boldsymbol{F}X$ modules) for all $n \geq 1$ and for every space X; therefore $\boldsymbol{F}X$ is $H\mathbb{Z}$-local space for every space X.*

1.6 Corollary. *Let $\boldsymbol{F}$ be as above. If $\pi_j\boldsymbol{F}X$ is $H\mathbb{Z}$-local for some given $j \geq 1$ then the modules $\pi_i\boldsymbol{F}X$ are $H\mathbb{Z}$-local for all $i \geq j$.*

Proofs. See 2.5 and 2.6.

A related result claims that when forming $H\mathbb{Z}$-localization [2] it is sufficient to "localize with respect to all self maps of $\bigvee_\infty S^1$ inducing an identity on $H_1(:,\mathbb{Z})$."

1.7 Proposition. *Let $\Psi_\alpha:\bigvee_\infty S^1 \to \bigvee_\infty S^1$ be the set of all self-maps with $H_1(\Psi_\alpha,\mathbb{Z})$ the identity map. The homotopy localization with respect to Ψ_α (see [3, 6, 7, 1.A]) is naturally equivalent to the $H\mathbb{Z}$-localization in the sense of Bousfield.*

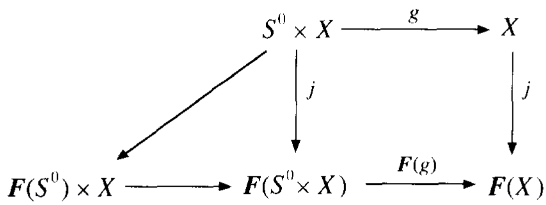
Proof. See Section 3.

The second example of this type: (for a proof see 2.7).

1.8 Theorem. *Let $\boldsymbol{F}$ be a continuous, co-augmented, homotopy idempotent functor from pointed spaces to pointed spaces. Assume that $\pi_1\boldsymbol{F}S^1$ is uniquely p-divisible for some prime $p \geq 2$. Then for all spaces and all $n \geq 1$ the group $\pi_n\boldsymbol{F}X$ is uniquely p-divisible and so is the semi-direct product $\pi_n\boldsymbol{F}X\tilde{X}\pi_1\boldsymbol{F}X$. In fact the map $\Omega\boldsymbol{F}X \to \Omega\boldsymbol{F}X$ that takes any loop to its pth power is a homotopy equivalence.*

1.9 Remark on localizations. One evident consequence of the above is that even if one attempts to define φ-localization in terms of homotopy classes of map (i.e. Y is $\varphi:A \to B$-local if $[B,Y] \to [A,Y]$ is an isomorphism of sets) it always entails a homotopy equivalence of whole function spaces (see 2.1). In particular it implies that certain universal examples do not exist. Namely those defined simply by homotopy classes of maps for *too small* families of maps. For example, there is no universal map among all maps of $\mathbb{R}P^2$ to 1-connected spaces (Guido Mislin).

1.10 A simple example of the results above is of course 1.1. In a simpler version it says that if $S^0 \to \boldsymbol{F}(S^0)$ is null homotopic then so is $j:X \to \boldsymbol{F}X$ for all X, for any continuous coaugmented $\boldsymbol{F}$. This can be seen directly by the following argument due to M. Hopkins: Consider the map $g: X \amalg X = S^0 \times X \to X$ which is null on the first summand and identity on the second. To prove the assertion we must show that $j \circ g$ extends from $X \amalg X$ to $X \times I$. But since $S^0 \to \boldsymbol{F}(S^0)$ is a null it is sufficient to show that the composition $j \circ g: X \times S^0 \to X \to FX$ factors through $X \times F(S^0)$. Now due to continuity one can construct a commutative diagram



So we get the desired factorization.

2. Higher homotopies of idempotent constructions, proofs of 1.2–1.8

In this section we prove some properties of idempotent functors which makes them look very close to localizations functors. We also make some statements about the relation between higher and lower homotopy groups of φ-localizations. These general properties are used to prove the main results.

Recall. For any given map $\varphi: A \to B$ one defines a φ-local space Y as a (fibrant) space for which the map of function spaces

$$\mathrm{map}(\varphi, Y): \mathrm{map}(B, Y) \to \mathrm{map}(A, Y)$$

is a weak homotopy equivalence. For any such map there exist a continuous, coaugmented, homotopy idempotent functor L_φ such that $j_X: X \to L_\varphi X$ is universal among all maps of X to φ-local spaces. The same goes for any set of maps $\{\varphi_i\}$. [3, 7].

Note two properties of the localization $X \to L_\varphi X$: first for any local space Y and any space Z the function space $\mathrm{map}(Z, Y)$ is again φ-local as can be checked directly by adjunction. Secondly, notice that L_φ preserves, up to homotopy, finite products: This again follows from the exponential law and continuity of L_φ.

The main property of continuous, coaugmented idempotent functors that makes them "look-like localizations" is the following useful truism (compare [7, 1.A]:

2.1 Theorem. *Let $\boldsymbol{F}: \mathscr{S} \to \mathscr{S}$ be a continuous coaugmented functor. Then the following three conditions are equivalent:*

(1) *F is homotopy idempotent.*

(2) *For all $W, X \in \mathscr{S}$ the map induced by $j_W: W \to \boldsymbol{F}W$ on homotopy classes:*

$$[\boldsymbol{F}W, \boldsymbol{F}X] \to [W, \boldsymbol{F}X]$$

is an isomorphism of sets.

(3) *For all* $W, X \in S$ *the map induced by* $j_W: W \to FW$ *on function complexes*

$$\mathrm{map}(FW, FX) \to \mathrm{map}(W, FX)$$

is a homotopy equivalence.

Remarks. The equivalence of (2) and (3) is in some way the essence of the claim that "functorial properties of lower homotopy imply similar properties of higher homotopies".

Notice that (1), (2) each implies that if F is idempotent then $j_{FX} \sim F(j_X)$ are homotopic (see 2.1.4). By definition we only assume that each is a homotopy equivalence.

An immediate consequence is:

2.1.1 Corollary. *If* $\varphi: A \to B$ *is any map and* F *a continuous, coaugmented and idempotent functor as in* 2.1. *Then* $F(\varphi)$ *an equivalence* $F(A) \simeq F(B)$ *if and only if for all* $X \in \mathcal{S}$ *the map* φ *induces an equivalence*

$$\mathrm{map}(B, FX) \xrightarrow{\simeq} \mathrm{map}(A, FX).$$

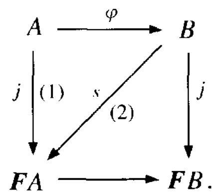
Proof. Since F is assumed to be idempotent we get from (1) $\Leftrightarrow$ (3) in 2.1 that $\mathrm{map}(FW, FX) \xrightarrow{\simeq} \mathrm{map}(W, FX)$ is a homotopy equivalence for both $W = A$ and $W = B$. But we assume $F_\varphi: FA \simeq FB$ hence $\mathrm{map}(B, FX) \simeq \mathrm{map}(A, FX)$ via the map φ. The other direction follows similarly.

From 2.1.1 we get 1.2 rather easily:

2.1.2 Proof of 1.2. We need to show that for any $\varphi: A \to B$ in $\mathcal{S}$ if $[B, FA] \to [A, FA]$ is surjective and $[B, FB] \to [A, FB]$ is injective, then for all $X \in \mathcal{S}$ the map $\mathrm{map}(\varphi, FX)$ is an equivalence.

We prove 1.2 by reducing it to Corollary 2.1.1 showing that 1.2, (1), (2) directly imply that $FA \xrightarrow{\simeq} FB$ is an equivalence.

Consider the diagram in $ho\mathcal{S}$, the homotopy category:



We have chosen the arrow s by surjectivity of $[B, FA] \to [A, FA]$ thus the triangle (1) commutes. Now triangle (2) commutes after composing back to A and therefore by the injectivity assumption triangle (2) commutes. So the chosen s renders the diagram homotopy cummutative. Apply F to the diagram. Since $\boldsymbol{F}(j)$ is an equivalence we get that $\boldsymbol{F}(s)$ is the desired homotopy inverse to $\boldsymbol{F}_{\varphi}$.

Proof of 2.1. We proceed with the proof of 2.1 by first giving a characterization of idempotent coaugmented functors: this proves the equivalence (1) $\Leftrightarrow$ (2).

2.1.3 Lemma. *A coaugment functor $\boldsymbol{F}:\mathcal{S} \to \mathcal{S}$ is idempotent if and only if for all W, $X \in \mathcal{S}$ the coaugmentation $j: W \to \boldsymbol{F}W$ induces an isomorphism of sets:*

$$j_W^*:[\boldsymbol{F}W,\boldsymbol{F}X] \xrightarrow{\simeq} [W,\boldsymbol{F}X].$$

Proof. Here we work in $ho\mathcal{S}$ the homotopy category: Assume first that j_W^* is an isomorphism. To find a homotopy inverse $j_{FX} = j:\boldsymbol{F}X \to \boldsymbol{FF}X$ we consider the homotopy classes of maps of this j into $\boldsymbol{F}X$, in particular there is a map $i:\boldsymbol{FF}X \to \boldsymbol{F}X$ that corresponds to the identity $\boldsymbol{F}X \to \boldsymbol{F}X$. Thus $i \circ j = id_{FX}$. To show that $j \circ i \sim id_{FFX}$ we use the injectivity of the map of homotopy classes: $[j_{FX}, \boldsymbol{FF}X]$: Since $j \circ i \circ j \sim j_{FX}$ and also $id_{FFX} \circ j_{FX} \sim j_{FX}$ we get $j \circ i \sim id_{FFX}$ as needed. So i is a homotopy inverse to $j_{FX} = j$.

But now consider the (homotopy) commutative square:

$$\begin{array}{ccc} X & \xrightarrow{j_X} & \boldsymbol{F}X \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_{FX}} \\ \boldsymbol{F}X & \xrightarrow{F(j_X)} & \boldsymbol{FF}X \end{array} \qquad (2.1.4)$$

Since both j_{FX} and $\boldsymbol{F}(j_X)$ compose with j_X to the same map $X \to \boldsymbol{FF}X$ by our condition on j_W^* for $W = X$ they are homotopic. Since j_{FX} is an equivalence so is $\boldsymbol{F}(j_X)$ and thus $\boldsymbol{F}X$ is idempotent by definition.

Now assume that $\boldsymbol{F}$ is idempotent thus both j_{FX} and $\boldsymbol{F}(j_X)$ are equivalences. To prove that j_W^* is an isomorphism of sets we consider the following square for any given $f: W \to \boldsymbol{F}X$.

$$\begin{array}{ccc} W & \xrightarrow{f} & \boldsymbol{F}X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \wr\ j_{FX}} \\ \boldsymbol{F}W & \xrightarrow{\boldsymbol{F}f} & \boldsymbol{FF}X \end{array} \qquad (2.1.5)$$

It follows from j_{FX} being an equivalence that j_W^* is surjective, since we can lift $\boldsymbol{F}(f)$ across $j_F X$. On the other hand to prove injectivity of j_W^* we assume that two maps f, g

in the following diagram:

$$W \xrightarrow{j_W} FW \underset{g}{\overset{f}{\rightrightarrows}} FX$$

pull back to homotopic maps: $f \circ j_W \sim g \circ j_W$. Now use the assumption that $F(j_W)$ is an equivalence to get $F(f) \sim F(g)$. But since F is idempotent the class of $F(g)$ uniquely determines the class of g. Thus $f \sim g$ as needed. This proves 2.1.3, i.e. (1) $\Leftrightarrow$ (2).

Since clearly (3) $\Rightarrow$ (2) we complete the proof of 2.1 by showing that (1) $\Rightarrow$ (3). For this end one consider the composition (where we use the power notation for function complexes):

$$FX^W \xrightarrow{F(-)} FFX^{FW} \xrightarrow{i_*} FX^{FW},$$

Notation. To decipher the power notation uniquely note that the functor F is never applied here to function complexes so that FX^Y always means $(FX)^Y$.

Here $F(-)$ is given by continuity of F, i_* is composition where $i{:}FFX \to FX$ is a homotopy inverse to $j{:}FX \to FFX$. We show that the above composition is a homotopy inverse (left and right) to the map induced by coagmentation: $j_W^*{:}(FX)^{FW} \to (FX)^W$.

First we show that it is a right homotopy inverse: For this it is enough to show that the composition

$$F(-) \circ j_W^*{:}FX^{FW} \to FX^W \xrightarrow{F(-)} (FFX)^{FW}$$

is homotopic to map$(FW, j_{FX}){:}FW^{FW} \to (FFX)^{FW}$, since $j \circ i \sim id$.

By definition the composition $F(-) \circ j_W^*$ takes a map $f{:}FW \to FX$ to a map $F(f) \circ F(j)$ in the following ladder:

$$\begin{array}{ccccc} W & \xrightarrow{j} & FW & \xrightarrow{f} & FX \\ \downarrow & & \downarrow & & \downarrow \\ FW & \xrightarrow{F(j)} & FFW & \xrightarrow{F(f)} & FFX \end{array}$$

Therefore the composition $F(-) \circ j_W^*$ is equal to the composition (where $F^2 \equiv FF$):

$$(FX)^{FW} \xrightarrow{(F(-)} (F^2X)^{F^2W} \xrightarrow{(Fj)^*} (F^2X)^{FW}.$$

(here we just used functoriality of F: $F(f \circ j) = F(f) \circ F(j)$). But we saw in (2.1.4) that 2.1(2) implies that the map $Fj{:}FW \to F^2W$ is homotopic to the coaugmentation map $j_{FW} := j_F{:}FW \to F^2W$.

Therefore our composition can be rewritten as

$$(FX)^{FW} \xrightarrow{F(-)} (F^2X)^{F^2W} \xrightarrow{j_F} (F^2X)^{FW}.$$

Now the composition $\mathrm{map}(\boldsymbol{F}W, j):(\boldsymbol{F}X)^{\boldsymbol{F}W} \to (\boldsymbol{FF}X)^{\boldsymbol{F}W}$ is homotopic, by commutativity of a version of (2.1.5) to $j_F \circ \boldsymbol{F}(-)$ as above. (Takes f to $\boldsymbol{F}(f)$ composed with $j_{FW} = j:\boldsymbol{F}W \to \boldsymbol{FF}W$.) Therefore, our original composition above is just $\mathrm{map}(\boldsymbol{F}W, j_{FX})$ as needed.

Secondly, we show that the same composition $i_* \circ \boldsymbol{F}(-)$ is a left inverse to the induced map j_W^*. For this end we use the diagram:

$$\begin{array}{ccccc} W & \xrightarrow{f} & \boldsymbol{F}X & & \\ {\scriptstyle j_W}\downarrow & & \downarrow{\scriptstyle j_{FX}} & & \\ \boldsymbol{F}W & \xrightarrow{\boldsymbol{F}f} & \boldsymbol{FF}X & \xrightarrow{i} & \boldsymbol{F}X \end{array}$$

where f is an arbitrary map. In terms of function complexes the commutativity of the square is equivalent to the commutativity of the triangle

$$\begin{array}{ccc} & (\boldsymbol{F}X)^W & \\ {\scriptstyle \boldsymbol{F}(-)}\swarrow & & \searrow{\scriptstyle (j_{FX})_*} \\ (\boldsymbol{FF}X)^{\boldsymbol{F}W} & \xrightarrow{j_W^*} & (\boldsymbol{FF}X)^W \end{array}$$

In which the maps are the obvious induced maps. Notice that associativity of composition $i \circ (\boldsymbol{F}f \circ j_W) = (i \circ \boldsymbol{F}f) \circ j_W$ in expressed on function space level as commutativity of the square:

$$\begin{array}{ccc} (\boldsymbol{F}X)^W & \xrightarrow{i_* \circ \boldsymbol{F}(-)} & (\boldsymbol{F}X)^{\boldsymbol{F}W} \\ {\scriptstyle j_W^* \circ \boldsymbol{F}(-)}\downarrow & & \downarrow{\scriptstyle j_W^*} \\ (\boldsymbol{FF}X)^W & \xrightarrow{i_*} & (\boldsymbol{F}X)^W . \end{array}$$

Using both diagrams we can compute: $i_* \circ \boldsymbol{F}(-) \circ j_W^* = i_* \circ j_W^* \circ \boldsymbol{F}(-) = i_* \circ (j_{FX})_* \sim id$, since i is a homotopy inverse to j_{FX} by assumption. This completes the proof of 2.1.

The following is a special case of 2.1.1 and 2.1:

2.1.6 Corollary. *Let $\varphi: A \to A$ be a self-map of a space A, $\boldsymbol{F}$ a continuous coaugmented, idempotent functor. Assume that $\boldsymbol{F}A$ is, up to homotopy, uniquely φ-divisible, (i.e. $[\varphi, \boldsymbol{F}A]$ is an isomorphism of sets) then*

(1) *$\boldsymbol{F}_\varphi$ is an equivalence*

(2) *$map(\varphi, \boldsymbol{F}X): map(A, \boldsymbol{F}X) \to map(A, \boldsymbol{F}X)$ is a homotopy equivalence for all X.*

We now proceed to show that if one can identify a specific property of the fundamental group $\pi_1 L$ for any φ-local space where $\varphi: A \to B$ a any map, this immediately entails an analogous property of the higher homotopy groups as π_1-modules.

2.2 Proposition. *Let $\varphi: A \to B$ be any map. Assume that C_φ is a class of groups such that for every φ-local space X one has $\pi_1 X \in C_\varphi$. Then for every φ-local space X the semi-direct product $\pi_n X \tilde{X} \pi_1 X$, with respect to the natural action of $\pi_1 X$ on $\pi_n X$, is in C_φ.*

In order to prove it we need

2.3 Lemma. *For any connected pointed space X $\pi_1 \mathrm{map}_\circ^{\mathrm{free}}(S^{n-1}, X)$ is the semi-direct product of $\pi_n X$ with $\pi_1 X$ with the natural action where $\mathrm{map}_\circ^{\mathrm{free}}$ denotes the null component of the space of free (unpointed) maps.*

Proof. (Cf. [10]) There is a fibration sequence with a section

$$(\Omega^{n-1} X)_\circ \to \mathrm{map}_\circ^{\mathrm{free}}(S^{n-1}, X) \underset{\mathrm{ev}}{\leftrightarrows} X$$

from which the above lemma follows upon taking fundamental groups: The fundamental group of the total space is naturally isomorphic to the group of components of $\mathrm{map}^{\mathrm{free}}(S^{n-1}, \Omega X)$ which is evidently the semi-direct product: $(x\alpha)\cdot(y\beta) = xy(y^{-1}\alpha y)\beta$ where α, β, $y^{-1}\alpha y$ are in the null component of ΩX.

2.4 Proof of 2.2. Given a φ-local space X, it is evident from the definition that for any space Y each component of the free function space $\mathrm{map}^{\mathrm{free}}(Y, X)$ is φ-local (recall above). Therefore $\mathrm{map}_\circ^{\mathrm{free}}(S^{n-1}, X)$ is φ-local and its fundamental group is by assumption in C_φ. But this group is the desired semi-direct product by 2.3.

2.5 Proofs of 1.5 and 1.8. We first make two claims.

2.5.1 Claim. *Let G be an $H\mathbb{Z}$-local group then a G module M is $H\mathbb{Z}$-local (in the sense of Bousfield) as a G-module if and only if the semi-direct product $M\tilde{X}G$ is again an $H\mathbb{Z}$-local group. This follows directly from* [2, 4] *and* 2.2.

2.5.2 Claim. *Let $\{\varphi^1: \bigvee_\infty S^1 \to \bigvee_\infty S^1\}$ be the set of all self-maps of a countable copies of the circle which induce an identity on the first homology with $\mathbb{Z}$ coefficients $H_1(-, \mathbb{Z})$. Then a space X has $H\mathbb{Z}$-local fundamental group if and only if the map on classes $[\bigvee_\infty S^1, X] \xrightarrow{\varphi^1} [\bigvee_\infty S^1, X]$ is an isomorphism for every φ^1 as above. A proof of* 2.5.2 *is given in* 2.8.

2.6 Proof of 1.5. Since $\pi_1 \boldsymbol{F} \bigvee_\infty S^1$ is $H\mathbb{Z}$-local by assumption, then by 2.5.2 the map $[\varphi^1, \boldsymbol{F} \to \bigvee_\infty S^1]$ induced by any map $\varphi^1: \bigvee_\infty S^1 \to \bigvee_\infty S^1$ with $H_1(\varphi_1, \mathbb{Z}) = \text{identity}$, is a set isomorphism. Therefore by 2.1 $[\varphi_1, \boldsymbol{F}X]$ is a set isomorphism for every space X and we can apply 1.2 for the set of map $\{\varphi^1\}$ as above. Therefore for every such a map φ^1 the space $\boldsymbol{F}X$ is φ^1-local – namely map $(\varphi^1, \boldsymbol{F}X)$ is a weak homotopy equivalence. But we know again by 2.5.2 that for every space which is φ^1-local for every φ^1 with $H_1(\varphi^1, \mathbb{Z})$ the identity, the fundamental group is $H\mathbb{Z}$-local. Therefore by 2.2 the semi-direct products $\pi_n \boldsymbol{F}X \tilde{X} \pi_1 \boldsymbol{F}X$ are $H\mathbb{Z}$-local groups and by 2.5.1 $\pi_n \boldsymbol{F}X$ is $H\mathbb{Z}$-local as $\pi_1 \boldsymbol{F}X$ module as needed. It follows that $\boldsymbol{F}X$ is an $H\mathbb{Z}$-local space for every space X.

2.7 Proof of 1.8. This follows easily from 2.1 again as in 2.6, consider though the remark below above p-divisible spaces:

Remark about uniquely p-divisible spaces: (Cf. [10]) It was noted by Baumslag [1] that the semi-direct product of a uniquely p-divisible group G with a uniquely p-divisible module M over G is *not*, in general, a uniquely p-divisible *gp*. To that end one needs to require that the module in M is uniquely p-divisible by all elements in the group ring of the form $(1 + \xi + \xi^2 + \xi^3 + \cdots + \xi^{p-1})$. In our context this means as noted by [10] that $\mathrm{map}(p, X): \mathrm{map}(S^1, X) \xrightarrow{p} \mathrm{map}(S^1, X)$, induced by $p: S^1 \to S^1$ the degree p map, is a homotopy equivalence if and only if $\pi_1 X$ is uniquely p-divisible and the higher homotopy groups are divisible by the above mentioned elements.

Therefore we get the "algebraic" conclusion that $\pi_1 \boldsymbol{F}S^1$ is uniquely p-divisible for a functor $\boldsymbol{F}$ as above iff $\pi_n \boldsymbol{F}X$ is uniquely divisible by $(1 + \xi + \xi^2 + \cdots + \xi^{p-1})$ for all X and all $\xi \in \pi_1 X$ and all $n \geq 1$.

2.8. $H\mathbb{Z}$ local groups as uniquely φ^1-divisible ones. We would like to express the property that $\pi_1 X$ is $H\mathbb{Z}$-local by saying that for certain collections of maps $\varphi: A \to B$ the induced map π_0 map (φ, X) is an isomorphism. It turns out that one may take a set of maps with $A = B = S^1$ for all these maps where $\bigvee_\infty$ is a countable wedge of circles. To this end we recall [8] that being $H\mathbb{Z}$-local can be expressed in terms of uniqueness and existence of solutions to certain family of equations.

Proposition. *Let $\{\varphi^1 : (\bigvee_\infty S^1) \to (\bigvee_\infty S^1)\}$ be the set of all self maps $\bigvee_\infty S^1 \xrightarrow{\varphi^1} \bigvee_\infty S^1$ of countable wedge of circle which induce an identity on $H_1(\mathbb{Z})$. Then $\pi_1 X$ is $H\mathbb{Z}$-local in the sense of Bousfield if and only if π_0 map (φ^1, X) is on isomorphism of sets. Furthermore a space X is Bousfield $H\mathbb{Z}$-local iff map (φ^1, X) is a weak equivalence.*

Proof. Recall [8] that a group is $H\mathbb{Z}$-local in if and only if every system of countable equations of the form $X_\alpha^\alpha = g_\alpha C_\alpha(X_{\beta 1} \cdots X_{\beta n})$ in a countable number of unknowns $\{X_j\}$ and where C_α is a word in X_j with the total degree of each X_j is 0, has a unique solution. We can assume with no loss of generality that C_α is a word in X_j with no group elements since for each group element one can add one more equation $X_g = g$.

Each system of equations $x_\alpha = g_\alpha C_\alpha$ can also be written as $g_\alpha = X_\alpha \cdot C_\alpha$. Thus a group G is $H\mathbb{Z}$-local if for each such system e, the map $f_e:\prod_\infty G \to \prod_\infty G$ taking the sequence $(X_\alpha)_\alpha$ to $(X_\alpha C_\alpha)_\alpha$ is an isomorphism of sets. Now $\prod_\infty G = \text{map}(\boldsymbol{F}(\infty), G)$ where $\boldsymbol{F}(\infty)$ is the free group on a countable number of generators $\{y_j\}$. And the above map f_e is induced by a map $g_e:\boldsymbol{F}(\infty) \to \boldsymbol{F}(\infty)$ sending the generator y_α to $y_\alpha C_\alpha(y_{\beta 1} \cdots y_{\beta n})$. Notice however that this map $\boldsymbol{F}(\infty) \to \boldsymbol{F}(\infty)$ is the most general map that induces an identity on the abelianization of $\boldsymbol{F}(\infty)$, since the abelianization of $y_\alpha C_\alpha$ is $[y_\alpha]$.

Therefore, we get the first part of the conclusion. The rest follows immediately from 1.2, 2.2 and 2.1.

3. Further examples and questions

We begin with a proof of 1.7.

3.1 Theorem. *Let $\{\varphi:\bigvee_\infty S^1 \to VS^1\}$ be the set of all self-maps of countable wedge of circles that induce an identity on the $\mathbb{Z}$-homology group H_1, (,$\mathbb{Z}$). The homotopy localization functor with respect to $\{\varphi\}$ is the Bousfield $H\mathbb{Z}$-localization functor for all spaces.*

Proof. Given a space X, let $\Phi: X \to L_{\{\varphi\}}X$ be the homotopy localization with respect to $\{\varphi\}$ clearly $H_*(\Phi, \mathbb{Z})$ is an isomorphism by construction. But we saw above (1.5) that $L_{\{\varphi\}}X$ is $H\mathbb{Z}$-local space since all its homotopy groups are $H\mathbb{Z}$-local. It follows from [4] that Φ is equivalent to the $H\mathbb{Z}$-localization.

3.2 Question. One suspects that for other interesting homological localizations with respect to h_* in the sense of Bousfield there exist a relatively small sets of map $\Psi: A \to B$ of "generating h_*-isomorphism" with $L_{\{\Psi\}} = L_{h^*}$. For example it might be true that if Ψ is the collection of Adams maps between Moore spaces we get K-theory localization as $L_{\{\Psi\}}$.

3.3 Non-idempotent functors. Many functors are known in homotopy theory that are continuous and coaugmented but are not idempotent. For example JX, $\mathbb{Z}_s X$, $\mathbb{Z}_\infty X$, $\Omega^n S^n X$, QX, etc. A careful reading of the proof of 1.2 shows that the assumption of idempotency can be dropped if a slight weakening of the conclusion is allowed. In fact, for an appropriate class C of groups and π-modules the following can be proved by similar arguments as above.

3.3 Proposition. Let $\boldsymbol{F}$ be a continuous, co-augmented functor. If the map $\pi_1 X \to \pi_1 \boldsymbol{F}X$ factors through a group in $C: \pi_1 X \to G \to \pi_1 \boldsymbol{F}X$ for $G \in C$, then for all $n \geq 1$ $\pi_n X \to \pi_n \boldsymbol{F}X$ factor through a π-module in C.

3.4 Examples.

(1) $C_0 = \{0\}$ the class that contain only the trivial group.

(2) C_N = nilpotent group and nilpotent π-modules

(3) C_Z = Bousfield groups and modules.

(4) C_r = groups G and G-modules M with $\Gamma_r G = 0 (\Gamma_r M = 0)$ for a fixed integer $r \geq 1$.

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