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Oscillation or nonoscillation property for semilinear wave equations

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Abstract

We consider

$$P_{\pm}u = \partial_t^2 u - \Delta u \pm F(u) = 0,$$

where $F(u) = au + f(u)$ with $a \geqslant 0$ and $f(0) = 0$. For the Cauchy problem of P_+ if $a > 0$, every nontrivial solution oscillates for any initial data. On the other hand for the Cauchy problem of P_- if $a \geqslant 0$, the solution does not change its sign for some initial data, namely it has nonoscillation property.

1. Introduction

We treat the following PDEs in this paper

$$P_{\pm}u = \partial_t^2 u - \Delta u \pm F(u) = 0.$$

The semilinear term F is defined by

$$F(u) = a(x,t)u + f(u),$$

where $a(x,t) \geqslant 0$, and $f \in C^1$ and $f(0) = 0$.

Example. $F(u) = au + |u|^{p-1}u$ with $p > 1$.

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We consider the following Cauchy problem:

$$\mathbf{P}(\pm) = \begin{cases} P_{\pm}u = 0 \text{ in } \mathbb{R}^n \times I, \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x) \text{ in } \mathbb{R}^n, \end{cases}$$

where $I = [0, T)$ with $T \leqslant \infty$.

We shall discuss the change of the signs$(+,-)$ of a solution $u(x,t)$, namely oscillation property. Our aim is to show the following two facts:

1. Let $a > 0$. For any initial data every nontrivial strong solution u of **P(+)** is oscillatory in some sense for the space dimension $n \geqslant 1$.
2. Let the space dimension n be $n = 1, 2, 3$. For some initial data a solution u of $\mathbf{P}(-)$ with $a \geqslant 0$ is nonoscillatory.

Up to this time the following three subjects have been investigated in Oscillation Problem for the Cauchy problem or the initial-boundary value problem.

1. Global oscillation property (GOP). Let t_0 be any fixed positive number. There is such a finite time interval $\mathbf{J}$ of which left-end point is t_0 and its length $|\mathbf{J}|$ depends on a_0 defined below that $meas.\{(x,t) \in \Omega \times \mathbf{J}/u(x,t) > 0\} > 0$ and $meas.\{(x,t) \in \Omega \times \mathbf{J}/u(x,t) < 0\} > 0$,
2. Pointwise oscillation property (POP). For any fixed $x_0 \in \Omega$, there exists $\{t_n\} \subset \mathbb{R}$ such that $t_n \to \infty$ as $n \to \infty$ and $u(x_0, t_n)u(x_0, t_{n+1}) < 0$,
3. Nonoscillation property (NOP). There exists some constant $t_0 \in \mathbb{R}$ such that $u(x,t)$ does not change its sign for any $t > t_0$ and any $x \in \Omega$.

Refer to [2,4,11,12] for GOP, and to [4,12–14] for POP. In those papers oscillation problems have been considered in a bounded Ω. For NOP refer to [15].

In this paper we shall discuss only GOP and NOP to the Cauchy problem $\mathbf{P}(\pm)$.

2. Global oscillation property

In this section we consider

$$\mathbf{P}(+) = \begin{cases} P_{+}u = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x) \text{ in } \mathbb{R}^n. \end{cases}$$

We shall show only GOP for **P(+)** with $a(x,t) > 0$. For that purpose we set some assumptions.

Assumption 2.1. The Cauchy problem **P(+)** in this section has a strong solution and the solution exists globally in time.

On existence of strong and global solutions of **P(+)** see [4,6,7] and [9].

Assumption 2.2. (1) $F(u) = a(x,t)u + f(u)$, $a(x,t) \geqslant \exists a_0(\text{const.}) > 0$.
(2) $f(s)s \geqslant 0$ and $f(0) = 0$ for $\forall s \in \mathbb{R}$.

Assumption 2.3. Let u be a solution of **P(+)**. Let t_0 be any fixed nonnegative number. If $((u(x,t_0), \partial_t(u(x,t_0)) \equiv 0$, then $u(x,t) \equiv 0$.

We apply the function $\phi(x)$ introduced in [2] to show GOP. They used $\phi(x)$ to show GOP for the Cauchy problem of a linear wave equation. We use it for the semilinear problem.

First the function $\psi(x)$ is defined by

$$\psi(x) := (1+|x|^2)^{-\rho}, \ \rho > \frac{n}{2}, \quad x \in \mathbb{R}^n.$$

ψ has the following properties and these are proved by elementary calculation.

1. $\psi \in H^2 \cap C^\infty$,
2. $\Delta\psi(x) = \frac{1}{1+r^2}\{2\rho(2\rho+2-n) - \frac{4\rho(\rho+1)}{1+r^2}\}\psi(x), \ r = |x|$,
3. $\exists M > 0$ such that $|\Delta\psi(x)| \leqslant M\psi(x)$.

Then we set

$$\phi(x) = \psi(\sqrt{\varepsilon/M}x). \tag{2.1}$$

Thus we have

Lemma 2.1. *Let ε be any positive number. There exists $\phi \in H^2 \cap C^\infty$ and $\phi > 0$ such that $-\Delta\phi \geqslant -\varepsilon\phi$ in $\mathbb{R}^n$.*

Theorem 2.1. *Let **J** be any interval with $|\mathbf{J}| \geqslant \pi/\sqrt{a_0}$. Then for any nontrivial solution u* of **P(+)** *GOP holds.*

Proof. The proof on the linear case was written in [2].

We take ε as $a_0 > \varepsilon > 0$. $U(t)$ is defined by $U(t) = \int_{\mathbb{R}^n} u(x,t)\phi(x)\,\mathrm{d}x$. Then by applying the Green formula

$$\begin{aligned} U''(t) &= \int_{\mathbb{R}^n} \partial_t^2 u(x,t)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \{\Delta u - au - f(u)\}\phi\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \{u\Delta\phi - au\phi - f(u)\phi\}\,\mathrm{d}x. \end{aligned} \tag{2.2}$$

If $u \geqslant 0$, from Lemma 2.1 $-\int_{\mathbb{R}^n} u\Delta\phi\,\mathrm{d}x \geqslant -\varepsilon\int_{\mathbb{R}^n} u\phi\,\mathrm{d}x = -\varepsilon U$. We take some finite interval $K = [t_0, T]$. Without loss of generality we can put $t_0 = 0$. Therefore $K = [0, T]$. Assume $u \geqslant 0$ in $\mathbb{R}^n \times K$ and $u(x,0) \geqslant 0$ in $\mathbb{R}^n$. Then $U(t) \geqslant 0$ in K and $\int_{\mathbb{R}^n} f(u)\phi\,\mathrm{d}x \geqslant 0$. Thus from $\int_{\mathbb{R}^n} P_+ u\phi\,\mathrm{d}x = 0$ and (2.2) we have

$$0 \geqslant U'' + (a_0 - \varepsilon)U. \tag{2.3}$$

We put $\omega = \sqrt{a_0 - \varepsilon}$ and $T = \pi/\omega$. By elementary calculation we have

$$0 \geqslant \int_0^T (U'' + (a_0 - \varepsilon)U) \sin \omega t \, \mathrm{d}t = \omega(U(T) + U(0)). \tag{2.4}$$

Because of $U(0) \geqslant 0$, $U(T) \leqslant 0$. This is contradiction. Hence there is a $T_0 \leqslant T$ such that $U(T_0) \leqslant 0$. Then the following two cases are considered:

1. **P(+)** has GOP in $[0, T_0]$,
   or
2. $u(x, T_0) \equiv 0$ and $u(x,t) > 0$ in the neighborhood of T_0.

Let the case 2 come up. $u(x,t)$ takes a minimum value 0 at $t = T_0$ and so $\partial_t u(x, T_0) = 0$. Hence $u(x,t) \equiv 0$ and u is a trivial solution because of Assumption 2.3. Hence we have the case 1 only. Because ε is arbitrary, we can conclude the statement of the theorem.

When $u < 0$, we can repeat the same argument and get the theorem. □

3. Nonoscillation property

We shall show the positivity of the nontrivial solution $u(x,t)$ of $\mathbf{P}(-)$ with initial data imposed on some conditions. Then we can maintain that NOP certainly holds. NOP of the Cauchy problem has been proved for one space dimensional general semilinear wave equations including first order derivatives in [15]. Here we show NOP for two or three space dimension case, mainly for two dimension.

Now we treat P_- in $\mathbb{R}^n \times I$, where $I = [0, T)$ with $T \leqslant \infty$ and consider the problem

$$\mathbf{P}(-) = \begin{cases} P_- u = 0 \ \text{ in } \ \mathbb{R}^n \times I, \\ u(x,0) = u_0(x), \ \ \partial_t u(x,0) = u_1(x) \ \text{ in } \ \mathbb{R}^n. \end{cases} \tag{3.1}$$

Assumption 3.1. Solutions in this section are sufficiently smooth, i.e. C^3-solutions. They may be local solutions or global solutions. Initial data (u_0, u_1) fulfill the requested regularity to assure the regularity of the solutions.

Regarding existence of local or global solutions for **P(-)**, refer to [1,3,5–7] and [9,10].

Assumption 3.2. The coefficient a in $F(u) = au + f(u)$ depends only on $x \in \mathbb{R}^n$ and $a(x) \geqslant 0$ for $\forall x \in \mathbb{R}^n$,

Assumption 3.3. Let $n = 2$.

1. $u_0(x) \geqslant 0$,
2. there exist a constant $M > 0$ and a positive integer $k > 2$ such that $\Delta u_0(x) + F(u_0(x)) \geqslant M/(1 + |x|)^k$ for $\forall x \in \mathbb{R}^2$,
3. $u_1(x) = 0$,
4. $f(s)$ is C^1-function and $F'(s) = a + f'(s) \geqslant 0$ for $\forall s \in \mathbb{R}$.

Theorem 3.1. *Let* $n=2$. *Assume* Assumption 3.1–3.3. *Let* u *be a solution of* **P(-)**. *Then* $u(x,t)\geqslant 0$ *and*

1. $\partial_t u(x,t)\geqslant \frac{M}{(1+|x|)^k}\frac{t}{(1+t)^k}$,
2. $u(x,t)\geqslant u_0(x)+\frac{M}{(1+|x|)^k}\{\frac{1}{(k-2)}(1-\frac{1}{(1+t)^{k-2}})-\frac{1}{(k-1)}(1-\frac{1}{(1+t)^{k-1}})\}$.

Proof. Let $x=(x_1,x_2)$, $\xi=(\xi_1,\xi_2)$ and $|x|=\sqrt{x_1^2+x_2^2}$. $D_t(x)$ is the set $\{x\in\mathbb{R}^2/|x|<t\}$ in two dimensional space.

Applying the well-known representation formula of a solution of the Cauchy problem for the inhomogeneous wave equation, the sufficiently smooth solution of **P(-)** satisfies

$$u(x,t)=\frac{\partial}{\partial t}\frac{1}{2\pi}\iint_{D_t(\xi)}\frac{u_0(x+\xi)}{\sqrt{t^2-|\xi|^2}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2+\frac{1}{2\pi}\iint_{D_t(\xi)}\frac{u_1(x+\xi)}{\sqrt{t^2-|\xi|^2}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$
$$+\frac{1}{2\pi}\int_0^t\mathrm{d}\tau\iint_{D_{t-\tau}(\xi)}\frac{F(u(x+\xi,\tau))}{\sqrt{(t-\tau)^2-|\xi|^2}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \tag{3.2}$$

Differentiating **P(-)** by t, the solution u satisfies

$$\begin{aligned}&\Box\partial_t u=F'(u)\partial_t u\quad\text{in }\mathbb{R}^2\times I,\\&\partial_t u(x,0)=0,\ \ \partial_t^2 u(x,0)=u_2(x)\quad\text{in }\mathbb{R}^2,\end{aligned}\tag{3.3}$$

where $u_2(x)=\Delta u_0(x)+F(u_0(x))$. Noting $u_1(x)=0$, $\partial_t u$ is given by

$$\partial_t u(x,t)=I_2(x,t)+I_3(x,t), \tag{3.4}$$

where

$$I_2(x,t)=\frac{1}{2\pi}\iint_{D_t(\xi)}\frac{u_2(x+\xi)}{\sqrt{t^2-|\xi|^2}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

and

$$I_3(x,t)=\frac{1}{2\pi}\int_0^t\mathrm{d}\tau\iint_{D_{t-\tau}(\xi)}\frac{F'(u(x+\xi,\tau))\partial_t u(x+\xi,\tau)}{\sqrt{(t-\tau)^2-|\xi|^2}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

Applying the polar coordinate $(\xi_1,\ \xi_2)=r\mathbf{n}$ with $\mathbf{n}=(\cos\theta,\ \sin\theta)$ to I_2 and moreover putting $t^2-r^2=s^2$ with $s\geqslant 0$, we get

$$I_2(x,t)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\int_0^t u_2(x+\sqrt{t^2-s^2}\mathbf{n})\,\mathrm{d}s. \tag{3.5}$$

By Assumption 3.3 we have

$$\int_0^t u_2\,\mathrm{d}s\geqslant\int_0^t\frac{M}{(1+|x+\sqrt{t^2-s^2}\mathbf{n}|)^k}\,\mathrm{d}s.$$

Applying the elementary inequality

$$\frac{1}{(1+|x+y|)^k} \geqslant \frac{1}{(1+|x|)^k}\,\frac{1}{(1+|y|)^k}$$

to the above integral, we get

$$\begin{aligned}
&\int_0^t u_2(x+\sqrt{t^2-s^2}\mathbf{n})\,\mathrm{d}s \\
&\quad\geqslant \frac{M}{(1+|x|)^k}\int_0^t \frac{\mathrm{d}s}{(1+\sqrt{t^2-s^2})^k} = \frac{M}{(1+|x|)^k}\int_0^{\pi/2}\frac{t\cos\theta}{(1+t\cos\theta)^k}\,\mathrm{d}\theta \\
&\quad\geqslant \frac{M}{(1+|x|)^k}\int_0^{\pi/2}\frac{t\cos\theta}{(1+t)^k}\,\mathrm{d}\theta = \frac{M}{(1+|x|)^k}\,\frac{t}{(1+t)^k}.
\end{aligned}$$

Then we have

$$\partial_t u(x,t) \geqslant \frac{M}{(1+|x|)^k}\,\frac{t}{(1+t)^k} + I_3(x,t). \tag{3.6}$$

We use the similar argument to that in [8]. Let x be any fixed point in $\mathbb{R}^2$. We set

$$Z(x) = \{t\in(0,\infty)/\partial_t u(x,t)=0\}.$$

We assume that $Z(x)\neq\emptyset$. Let $t_0=\inf Z(x)$. Then $t_0>0$ and $\partial_t u(x,t)>0$ for $0\leqslant t<t_0$. Becasue if $t_0=0$, there exists a sequence $\{t_n\}\searrow 0$ such that $\partial_t u(x,t_n)=0$. Hence there exists a sequence $\{t'_n\}\searrow 0$ such that $\partial_t^2 u(x,t'_n)=\Delta u(x,t'_n)+F(u(x,t'_n))=0$. Thus $u_2(x)=0$, which contradicts 2 of Assumption 3.3. $C(x,t_0)$ denotes the characteristic cone of which vertex is (x,t_0) and its bottom lies in $t=0$. Then there exists $(x',t')\in C(x,t_0)$ such that $\partial_t u(x',t')=0$ and $\partial_t u(x,t)>0$ for $(x,t)\in C(x',t'), t<t'$. We have the same expression for $\partial_t u(x',t')$ as (3.4) and therefore

$$0=\partial_t u(x',t') \geqslant \frac{M}{(1+|x'|)^k}\,\frac{t'}{(1+t')^k} + I_3(x',t'), \tag{3.7}$$

where

$$I_3(x',t') = \frac{1}{2\pi}\int_0^{t'}\mathrm{d}\tau\iint_{D_{t'-\tau}(\xi)}\frac{F'(u(x'+\xi,\tau))\partial_t u(x'+\xi,\tau)}{\sqrt{(t'-\tau)^2-|\xi|^2}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

Noting that $\partial_t u(x'+\xi,\tau)>0$ for $(x'+\xi,\tau)\in C(x',t'), \tau<t'$, we can lead a contradiction from (3.7). Thus we can insist that there is no such (x',t') in $C(x',t')$. By the same argument we can show that $Z(x)=\emptyset$ for any $x\in\mathbb{R}^2$, whence $I_3(x,t)>0$. Thus we get for any $(x,t)\in\mathbb{R}^2\times I$

$$\partial_t u(x,t) \geqslant \frac{M}{(1+|x|)^k}\,\frac{t}{(1+t)^k}. \tag{3.8}$$

Integrating (3.8) from 0 to t, we can get $u(x,t)>0$ and the desired results of the theorem. □

Assumption 3.4. (1) $F(u) > 0$ if $u > 0$,
(2) $u_0 \equiv 0$,
(3) $u_1 \geqslant 0$ and $u_1 \not\equiv 0$.

Theorem 3.2. *Let* $n = 2$. *Assume* Assumptions 3.1 *and* 3.4. *Then the solution* $u(x,t)$ *of* **P(-)** *satisfies* $u(x,t) \geqslant 0$.

Proof. We use the expression (3.2) and have

$$u(x,t) = \frac{1}{2\pi} \iint_{D_t(\xi)} \frac{u_1(x+\xi)}{\sqrt{t^2 - |\xi|^2}} \, d\xi_1 \, d\xi_2 + \frac{1}{2\pi} \int_0^t d\tau \iint_{D_{t-\tau}(\xi)} \frac{F(u(x+\xi,\tau))}{\sqrt{(t-\tau)^2 - |\xi|^2}} \, d\xi_1 \, d\xi_2. \tag{3.9}$$

The way of the proof is almost the same as in Theorem 3.1 and therefore we omit rest of the proof. □

In three space dimension we can show almost the same results as the above with the similar argument to two dimensional cases. We show only the result and omit the proof.

Assumption 3.5. Let $n = 3$.

1. $u_1 \geqslant 0$,
2. There exists some positive constants M_1, M_2 such that $|\nabla u_1(x)| \leqslant M_1$ and $u_2(x) = \Delta u_0 + F(u_0) \geqslant M_2$, and $M_2 - M_1 \geqslant 0$,
3. $F'(u) \geqslant 0$.

Theorem 3.3. *Let* $n = 3$. *Assume* Assumptions 3.1–3.2 *and* 3.5. *Then any nontrivial solution* u *satisfies*

1. $\partial_t u(x,t) \geqslant (M_2 - M_1)t$,
2. $u(x,t) \geqslant u_0(x) + \frac{1}{2}(M_2 - M_1)t^2$.

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