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Cohomology of Unipotent and Prounipotent Groups

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A *pro-affine algebraic group* G, over the algebraically closed field k, is an inverse limit of affine algebraic groups over k [5, p. 1127]. This paper studies the cohomology, in the category of rational modules, of prounipotent groups when k has characteristic zero (a finite-dimensional rational module V for a pro-affine group G is an abstract G-module such that the corresponding homomorphism $G \to GL(V)$ is a morphism of pro-affine groups and in general a *rational* G-*module* is a direct limit of finite-dimensional ones). This theory closely parallels that of the cohomology of pro-p groups [12, Chap. I]: the free prounipotent groups turn out to be precisely those of cohomological dimension ones, and the dimension of the first and second cohomology groups give numbers of generators and relations. In addition, one-relator groups turn out to have cohomological dimension two, parallel to the situation for discrete groups [9]. Finally, we apply our theory to the universal pro-affine hull A of a group Γ to conclude that if Γ has a free subgroup of finite index, then the prounipotent radical of A is a free prounipotent group.

The paper is divided into five sections. The first contains preliminary material on: the pro-variety structure of prounipotent groups, the existence and description of injective rational modules and cohomology, and various technical results used in the rest of the paper. In the second section we define free prounipotent groups and characterize them as groups of cohomological dimension one. This allows us to show that pro-affine subgroups of free prounipotent groups are free prounipotent. In Section 3 we interpret the low-dimensional cohomology of a free prounipotent group with coefficients in the

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trivial one-dimensional module in terms of generators and relations and show the one-relator groups have cohomological dimension two. Section 4 contains the application noted above to pro-affine hulls of groups. In Section 5, which is essentially independent of the rest of the paper, we show that any non-zero module endomorphism of the coordinate ring of a prounipotent group is onto. This result is used in our theorem in Section 3 that one-relator groups have cohomological dimension two.

We adopt the following notation and conventions: k is a fixed algebrically closed field of characteristic zero. If G is a pro-affine algebraic group over k, $k[G]$ is its coordinate Hopf algebra [5, p. 1127] and G^* is the Zariski closure of its commutator subgroup (G, G). If $f \in k[G]$ and $x \in G$, $f \cdot x$ and $x \cdot f$ in $k[G]$ are defined by $(f \cdot x)(y) = f(xy)$ and $(x \cdot f)(y) = f(yx)$. We use k for the one-dimensional trivial module and G_a for the one-dimensional unipotent group. If Γ is an abstract group, $A(\Gamma)$ is the *universal pro-affine hull* of Γ; that is, $k[A(\Gamma)]$ is the Hopf algebra of all representative functions on Γ [10, p. 3]. For ease of exposition, we often drop the adjectives "pro-affine" and "rational" when dealing with subgroups, homomorphisms, and modules. When distinctions are necessary we add the adjective "abstract" to refer to non-pro-affine groups or non-rational modules. We depart from this convention and call abstract groups "discrete groups" when this is the customary usage. $(\ )^I$ and $(\ )^{(I)}$ denote direct product and direct sum indexed by I.

1. Preliminaries

In this section we introduce the basic properties of prounipotent groups and their cohomology. Many of the results recorded in this section are easy extensions of facts about unipotent groups to prounipotents, and so we may only supply a reference for the affine case. For the same reason, results in this section are only numbered and not labeled as theorems, propositions, etc.

Let G be a pro-affine algebraic group over k, let $\{A_i \mid i \in I\}$ be the set of all finitely generated Hopf subalgebras of $k[G]$ and for each i, let G_i be the affine algebraic group with $k[G_i] = A_i$. Since $k[G] = \operatorname{dir\,lim} A_i$, $G = \operatorname{proj\,lim}(G_i)$ and we refer to this description of G as the *standard limit* for G. Let $L_i = \operatorname{Lie}(G_i)$; then $L = \operatorname{proj\,lim}(L_i)$ is the *Lie algebra* of G, denoted $\operatorname{Lie}(G)$ [11, p. 221]. As an inverse limit of finite-dimensional vector spaces, $\operatorname{Lie}(G)$ may be regarded as a linearly compact vector space in the sense of [9, Definition 27.1, p. 78], and $\operatorname{Lie}(\ )$ is a functor from pro-affine groups to linearly compact vector spaces. The exponential maps $\exp : \operatorname{Lie}(G_i) \to G_i$ give a map $\exp : \operatorname{Lie}(G) \to G$.

(1.1) *If G is a prounipotent group,* $\exp : \mathrm{Lie}(G) \to G$ *is an isomorphism of pro-affine varieties* [11, *p.* 221].

If G in (1.1) is abelian, exp is an isomorphism of prounipotent groups. Hence we have:

(1.2) Lie( ) *is an equivalence between the category of abelian prounipotent groups and the category of linearly compact k-vector spaces. Since every linearly compact k-vector space is of the form k^I for suitable I* [8, 32.1, p. 83], *every abelian prounipotent group is a suitable product of copies of G_a.*

Exact sequences of linearly compact vector spaces split, so it follows from (1.1) that:

(1.3) *Let $1 \to K \to G \to H \to 1$ be an exact sequence of prounipotent groups. Then there is a morphism $H \to G$ of pro-affine varieties whose composite with $G \to H$ is the identity* [11, p. 222].

(1.4) *Let G be a prounipotent group and H a subgroup (not necessarily normal). Then there is a Zariski-closed subset X of G such that $G = H \times X$ as pro-affine varieties. In particular, $k[G]$ is faithfully flat over $k[H]$.*

Proof. The inclusion $\mathrm{Lie}(H) \to \mathrm{Lie}(G)$ splits (in linearly compact k-spaces) so $\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus Y$, and $X = \exp(Y)$.

We now establish some inverse limit results.

(1.5) *Let G be a pro-affine group and $\mathscr{S}$ a family of normal subgroups of G such that:*

(a) *If $H \in \mathscr{S}$, G/H is affine.*

(b) *If $H, H' \in \mathscr{S}$, there is $H'' \in \mathscr{S}$ such that $H'' \subseteq H \cap H'$.*

Let $X = \bigcap \{H \mid H \in \mathscr{S}\}$. Then G/X is isomorphic to $\text{proj lim}(\{G/H \mid H \in \mathscr{S}\})$.

Proof. Let G' be the projective limit. Then $k[G'] = \text{dir lim}(k[G/H]) = \text{dir lim}(k[G]^H)$. We have a homomorphism $f\colon G/X \to G'$ which is injective by definition of X. Let $\bar{G}$ be the image of f. Then $k[G'] \to k[\bar{G}]$ is onto, and $k[\bar{G}] = k[G/X] = k[G]^X$. Since $k[G]^H \to k[G]^X$ is into for each H, $k[G'] = k[\bar{G}]$ and f is an isomorphism.

(1.6) *Let $G =$* $\text{proj lim}(G_i)$ *be the standard limit for the pro-affine group G, let K_i be the kernel of $G \to G_i$ and let X be a closed subgroup of G.*

(a) $X = \bigcap (XK_i)$.

(b) *If X is normal and G/X is affine, X contains some K_i.*

Proof. (a) X is contained in $\bigcap (XK_i)$ and both are Zariski closed. If $f \in k[G]$ and $f(X) = 0$, $f \in k[G]^{K_i}$ for some i, so $f(XK_i) = 0$, and hence $f(\bigcap (XK_i)) = 0$. It follows that $X = \bigcap (XK_i)$.

(b) By (a), $X = \bigcap (XK_i)$. Choose K_i such that XK_i/X has minimal dimension and minimal number of components. If K_i contains K_j, then XK_j must equal XK_i, so $XK_i = X$ and $K_i \subseteq X$.

If G is a pro-affine group and H any subgroup of G, then $k[G]^H = \{f \in k[G] \mid f \cdot x = f \text{ for all } x \in H\}$. Let $H' = \{x \in G \mid f \cdot x = f$ for all $f \in k[G]^H$ and call H *observable* in G if the inclusion $H \subseteq H'$ is equality.

(1.7) *A subgroup H of a prounipotent group G is observable.*

Proof. Let $\{K_i\}$ be as in (1.6). Since $H = \bigcap (HK_i)$ by (1.6)(a), $H' = \bigcap (HK_i)'$. Thus we may assume $H = HK_i$ contains some K_i. By [2, Lemma 4.2, p. 9], H/K_i is observable in G/K_i. Now $k[G]^H = k[G/K_i]^{H/K_i}$, so x in G is in H' if and only if xK_i is in $(H/K_i)' = K/K_i$. Since $K_i \subseteq H'$, this means $H = H'$.

We say a subgroup H of prounipotent group G is of *codimension* n if $k[G]^H$ is affine of dimension n.

(1.8) *A subgroup H of a prounipotent group G is of finite codimension if and only if H contains a normal subgroup K of G with G/K affine. If H is of finite codimension, the coset space $H \backslash G$ maps bijectively to the affine variety with coordinatte ring $k[G]^H$.*

Proof. If $k[G]^H$ is affine, $k[G]^H \subseteq k[G]^{K_i}$ for some i, where $\{K_i\}$ is as in (1.6). Then $K_i = K_i' \subseteq H' = H$ by (1.7). Conversely, if $K \subseteq H$ is normal in G with G/K affine, by (1.6)(b) $K_i \subseteq H$. Thus $k[G]^H = k[G/K_i]^{H/K_i}$, and the latter is the coordinate ring of the affine variety $(H/K_i) \backslash (G/K_i) = H \backslash G$.

Our next task is to define rational cohomology for pro-affine groups. We follow the procedure of [6]: let G be pro-affine group, let M be a G-module, and let E' be the injective hull of the (abstract) G-module M in the category of abstract G-modules. The largest (rational) submodule E of E' is then an injective (rational) G-module containing M, so the category of (rational) G-modules has enough injectives.

(1.9) *Let G be a pro-affine group and let M be a G-module. Then $H^i(G, M)$ denotes the* ith *right derived functor of* $(\ )^G$ *evaluated at M.*

(1.10) *Let $1 \to K \to G \to H \to 1$ be an exact sequence of pro-affine groups, and let M be a G-module. Then there is a spectral sequence $H^p(H, H^q(K, N))$ with abutment $H^n(G, M)$.*

Proof. If I is an injective G-module, I^K is an injective H-module. The functor $(\ )^G$ can be regarded as the composite $((\ )^K)^H$, so there is a spectral sequence of the composite functor by [3, Theorem 2.4.1, p. 148] and we use definition (1.9) to describe the right derived functors which occur. We must see that an injective G-module I is an injective K-module. It is sufficient to consider the case $I = k[G]$. Let W be a finite-dimensional K-module, V a K-submodule and $f: V \to I$ a K-module homomorphism. There is a normal subgroup H_1 of K of finite codimension such that H_1 is trivial on W, a normal subgroup H_2 of G of finite codimension such that $H_2 \cap K \subseteq H_1$ and a normal subgroup H_3 of G of finite codimension such that $f(V)$ is contained in the H_3 invariants. Let $L = H_2 \cap H_3$. L is normal in G of finite codimension, $L \cap K$ is trivial on W and $f(V) \subseteq k[G]^L$. Let $\bar{G} = G/L$ and $K = L \cap K$. Then W is a $\bar{K}$-module and $f(V) \subseteq k[\bar{G}]$. It follows from [2, Section 2] that $k[\bar{G}]$ is $\bar{K}$-injective, so f extends to a $\bar{K}$-map $W \to k[\bar{G}]$. Since $k[\bar{G}] \subseteq k[G]$, this is an extension of f to W, and it follows that I is K-injective.

We can be more precise about the nature of injectives with a prounipotent group G.

(1.11) *Let G be a prounipotent group and let V be a G-module. Then $k[G] \otimes V^G$ is an injective G-module containing V. Any injective G-module containing V contains a copy of $k[G] \otimes V^G$.*

Proof. We first note that $k[G]$ is an injective G-module, as in [2, p. 4]. Next we observe that a G-module I is injective if for any monomorphism of G-modules $M \to N$ with N finitely generated (hence finite dimensional), the map $\operatorname{Hom}_G(N, I) \to \operatorname{Hom}_G(M, I)$ is surjective, as in the usual arguments. Thus arbitrary direct sums of injectives are injective, so $k[G] \otimes V^G$ is injective. The inclusion $V^G \to k[G] \otimes V^G$ by $x \to 1 \otimes x$ is a G-module homomorphism so it extends to a G-module homomorphism $f: V \to k[G] \otimes V^G$ by the injectivity of $k[G] \otimes V^G$. Since $\operatorname{Ker}(f)^G = 0$, $\operatorname{Ker}(f) = 0$. Any inclusion $V \to E$ with E injective extends to a homomorphism $g: k[G] \otimes V^G \to E$. Then $\operatorname{Ker}(g)^G$ is contained in $(k[G] \otimes V^G)^G = V^G$; so $\operatorname{Ker}(g)^G = 0$ and $\operatorname{Ker}(g) = 0$.

Because of (1.11), we define $E_0(V) = k[G] \otimes V^G$ to be the *injective hull* of the G-module V. We let $E_{-1}(V) = V$ and let $d_{-1}: E_{-1}(V) \to E_0(V)$ be inclusion.

(1.12) *Let G be a prounipotent group and let V be a G-module. Define $E_i(V)$ and $d_i: E_i(V) \to E_{i+1}(V)$ inductively by $E_{i+1}(V) = E_0(E_i(V)/d_{i-1}(E_{i-1}(V)))$ and $d_i = E_i(V) \to E_i(V)/d_{i-1}(E_{i-1}(V)) \to E_{i+1}$. Then $\{E_i(V), d_i\}$ is an injective resolution of V with $H^i(G, V) = E_i(V)^G$.*

Proof. By construction, $\{E_i(V), d_i\}$ is an injective resolution of V, and $E_i(V) = E_0(\operatorname{Ker}(d_i)) = k[G] \otimes \operatorname{Ker}(d_i)^G$, the last equality by (1.11). So

$E_i(V)^G = \mathrm{Ker}(d_i)^G$ and the complex $\{E_i(V)^G\}$ has zero differentiation, so $H^i(G, V) = E_i(V)^G$.

We call $\{E_i(V), d_i\}$ the minimal injective resolution of V [1, Section 2, p. 10].

We say the pro-affine group G has *cohomological dimension* n, and write $\mathrm{cd}(G) = n$, if for every G-module V and every $i > n$, $H^i(G, V) = 0$ and $H^n(G, V) \neq 0$. If G is prounipotent, $\mathrm{cd}(G) \leqslant n$ if and only if $H^{n+1}(G, k) = 0$, since k is the unique simple G-module.

We are next going to show that if H is a subgroup of the prounipotent group G, then $\mathrm{cd}(H) \leqslant \mathrm{cd}(G)$, following the proof of [2, Theorem 4.3, p. 9]. A necessary technical step is supplied by the following identity.

(1.13) *Let G be a prounipotent group, let H be a subgroup and let $B = k[G]^H$. Then there is an isomorphism $k[G] \otimes_B k[G] \to k[H] \otimes k[G]$.*

Proof. Let $\{K_i\}$ be as in (1.6), let $H_i = HK_i$ and let $B_i = k[G]^{H_i}$. Then $k[G] = \mathrm{dir\,lim}(k[G]^{K_i})$ and $B = \mathrm{dir\,lim}(B_i)$. Also, $k[G]^{K_i} = k[G/K_i]$ and $B_i = k[G/K_i]^{H_i/K_i}$; so by [2, (**), p. 9], $k[G/K_i] \otimes_B k[G/K_i]$ is naturally isomorphic to $k[H_i/K_i] \otimes k[G/K_i]$. By taking direct limits, we have $k[G] \otimes_B k[G]$ isomorphic to $C \otimes k[G]$, where $C = \mathrm{dir\,lim}(k[H_i/K_i])$. Now $H_i/K_i = H/H \cap K_i$, and the set $\mathscr{S} = \{H \cap K_i\}$ of subgroups of H satisfies (1.5). Since $\bigcap \{H \cap K_i\} = \{e\}$, by (1.5) $H = \mathrm{proj\,lim}(H/H \cap K_i)$; so $C = k[H]$.

Next we define induced modules as in [2, p. 2]. Let G be a pro-affine group, let H be a subgroup and let V be an H-module. Then $\mathrm{Map}_H(G, V)$ denotes the set of functions $f\colon G \to V$ such that $f(G)$ spans a finite-dimensional subspace W of V with $f\colon G \to W$ a morphism of pro-affine varieties and such that $f(hg) = hf(g)$ for all $h \in H$ and $g \in G$. Then $\mathrm{Map}_H(G, V)$ is a G-module and there is a natural isomorphism $\mathrm{Hom}_G(X, \mathrm{Map}_H(G, V)) \to \mathrm{Hom}_H(X, V)$ for G-modules X [2, Proposition 1.4, p. 3]. We call $\mathrm{Map}_H(G, V)$ the *G-module induced from H by V.*

(1.14) *Let G be a prounipotent group and H a subgroup. For any H-module V, $H^i(G, \mathrm{Map}_H(G, V)) = H^i(H, V)$ for all i. In particular* $\mathrm{cd}(H) \leqslant \mathrm{cd}(G)$.

Proof. Use the argument of [2, Theorem 4.3, p. 9], (a) $\Rightarrow$ (b), with (1.13) replacing the identity (**).

Finally, we make some elementary cohomological calculations.

(1.15) *Let U be a unipotent group of dimension n. Then $\mathrm{cd}(U) = n$ and $H^n(U, k)$ is one dimensional.*

Proof. We use induction on n. For the case $n = 1$, we have $U = G_a$.

Choose t so that $k[G_a]$ is the polynomial ring $k[t]$, and let $d\colon k[t] \to d[t]$ be derivative with respect to t. Then $1 \to k \to k[G_a] \to^d k[G_a] \to 1$ is an exact sequence of G_a-modules. By (1.11), this is an injective resolution of k as G_a-module, and taking cohomology we see that $H^1(G_a, k) = k$ and $\mathrm{cd}(G_a) = 1$. Now assume the result for $n-1$. Then U has a normal subgroup U_0 of codimension one, and the spectral sequence (1.10) associated to the exact sequence $1 \to U_0 \to U \to G_a \to 1$ is $H^p(G_a, H^q(U_0, k)) \Rightarrow H^{p+q}(U, k)$. Since $\mathrm{cd}(G_a) = 1$ and $\mathrm{cd}(U_0) = n-1$, we have $\mathrm{cd}(U) \leqslant n$, and that $H^1(G_a, H^{n-1}(U_0, k)) = H^n(U, k)$. Now $H^{n-1}(U_0, k) = k$ by induction and $H^1(G_a, k) = k$ by the above; so $H^n(U, k) = k$ and $\mathrm{cd}(U) = n$.

(1.16) *Let G be a prounipotent group. Then* $H^1(G, k) = \mathrm{Hom}(G, G_a)$.

Proof. By (1.12), $H^1(G, k) = E_1(k)^G = (E_0(k)/E_{-1}(k))^G = (k[G]/k)^G$. Suppose $a \in k[G]$ is invariant modulo k. We can normalize a so that $a(e) = 0$. Define $f\colon G \to k$ by $f(g) = g \cdot a - a$. It follows that f is actually a homomorphism from G to G_a. Conversely, given a homomorphism $h\colon G \to G_a$, we can regard h as an element of $k[G]$ and then the class of h modulo k is invariant. This correspondence gives the desired equality.

(1.17) *Let G be a prounipotent group. Then isomorphism classes of extensions $1 \to G_a \to \bar{X} \to G \to 1$ correspond to elements of $H^2(G, k)$ with split extensions corresponding to zero.*

Proof. Because of (1.3), the proof of [6, Theorem 6.1, p. 519] applies to the prounipotent case to establish a one–one correspondence between extensions and elements of $H^2(G, k)$ with the split extensions corresponding to zero.

We restate (1.17) in a slightly different form we will use below.

(1.18) *Let G be a prounipotent group. Then $H^2(G, k) = 0$ if and only if for every exact sequence $1 \to G_a \to E \to^g F \to 1$ of unipotent groups and every homomorphism $f\colon G \to F$ there is a homomorphism $h\colon G \to E$ with $gh = f$.*

Proof. Let $X = \{(a, b) \in E \times G \mid g(a) = f(b)\}$. The projection $E \times G \to G$ induces a surjective homomorphism $X \to G$ with kernel G_a, hence an extension $1 \to G_a \to X \to G \to 1$. If there is an h with $gh = f$, $b \to (h(b), b)$ splits this extension, and conversely if $h_0\colon G \to X$ splits the extension, h_0 composed with projection of X to E gives a map h with $gh = f$. The result now follows from (1.17).

2. Free Prounipotent Groups

In this section we define free prounipotent groups and show that they are precisely the prounipotent groups of cohomological dimension one.

Let I be a set. We are going to construct a prounipotent group $U(I)$ and an injection $\phi\colon I \to U(I)$; $U(I)$ will be called the *free prounipotent group on I.* The justification for the term "free" will be explained by Theorem 2.4. Let $L(I)$ be the (discrete) free group on I and let $A = A(L(I))$. Let R_u be the prounipotent radical of A and let P be a pro-reductive subgroup of A such that $A = R_u \cdot P$. Let N be the minimal normal subgroup of A containing P (since the choices for P are conjugates, N depends only on A). Let $U_0(I) = A/N$. $U_0(I)$ is prounipotent, and we have a map $\psi\colon I \to U_0(I)$ given by the composition of the canonical maps $I \to L(I) \to A \to U_0(I)$. The pair $(U_0(I), \psi)$ satisfies a universal mapping property: suppose B is a prounipotent group and $f\colon I \to B$ is a function. Then f induces a unique abstract group homomorphism $L(I) \to B$, which induces a unique homomorphism $A \to B$ which vanishes on N and hence induces a unique homomorphism $\bar{f}\colon U_0(I) \to B$ characterized by the property that $\bar{f}\psi(x) = f(x)$ for all $x \in I$.

Now let $\mathscr{S}$ be the set of all normal subgroups H of $U_0(I)$ of finite codimension such that $\{x \in I \mid \psi(x) \notin H\}$ is finite. If $H, H' \in \mathscr{S}$, then $H \cap H' \in \mathscr{S}$. Let $K = \bigcap \{H \mid H \in \mathscr{S}\}$, and let $U(I) = U_0(I)/K$. By (1.5), $U(I) = \operatorname{proj\,lim}\{U_0(I)/H \mid H \in \mathscr{S}\}$. Let $\phi\colon I \to U(I)$ be the composite of ψ with the canonical map $U_0(I) \to U(I)$.

We want to observe that ϕ is an injection: let $x \in I$ and let $e_x\colon I \to G_a$ be given by $e_x(y) = 0$ if $y \neq x$ and $e_x(x) = 1$. Then e_x induces a unique homomorphism $\bar{e}_x\colon U_0(I) \to G_a$ such that $\bar{e}_x\psi = e_x$. The kernel of $\bar{e}_x$ is in $\mathscr{S}$, so $\bar{e}_x$ induces a homomorphism $E_x\colon U(I) \to G_a$ such that $E_x\phi = e_x$. So if $\phi(x) = \phi(y)$, we have $x = y$.

We shall use the notation introduced in the preceding three paragraphs for the rest of the paper.

Definition 2.1. Let I be a set. The group $U(I)$ is called the *free prounipotent group on I.* I is regarded as a subset of $U(I)$ via ϕ.

We will see below (2.8) that the group $U_0(I)$ is also free (in general, not on I). The following proposition gives a first explanation for the term "free" in (2.1).

Proposition 2.2. *Let I be a set and let U be a unipotent group. Then there is a bijection between homomorphisms $U(I) \to U$, and sets of elements of U $\{x_i \mid i \in I\}$ where $\{i \mid x_i \neq e\}$ is finite, such that the homomorphism f corresponds to the set $\{f(i) \mid i \in I\}$.*

Proof. If $f: U(I) \to U$ is a homomorphism, then since $U(I) = \operatorname{proj\,lim}(\{U_0(I)/H \mid H \in \mathscr{S}\})$, it follows from (1.6) that $\operatorname{Ker}(f)$ contains an element of $\mathscr{S}$, so that $\{f(i) \mid i \in I\}$ has the desired property. Conversely, given a set of elements $\{x_i \mid i \in I\}$ there is a homomorphism $h : U_0(I) \to U$ with $h\psi(i) = x_i$, and since $\{i \mid x_i \neq e\}$ is finite, $\operatorname{Ker}(h) \in \mathscr{S}$; so we get an induced homomorphism $g : U(I) \to U$ with $g(i) = x_i$.

We will need the following property of free prounipotent groups in our characterization of free groups.

LEMMA 2.3. *Let I be a set. Then the k-vector space* $\operatorname{Hom}(U(I), G_a)$ *has* $\{E_x \mid x \in I\}$ *as basis.*

Proof. By Proposition 2.2, $\operatorname{Hom}(U(I), G_a)$ corresponds to sets of elements $\{x_i \mid i \in I\}$ of $G_a = k$ with all but finitely many $x_i = 0$. These sets of elements are the same as the elements of the k-vector space $k^{(I)}$ and $\{E_x \mid x \in I\}$ corresponds to the standard basis.

It is a consequence of Proposition 2.2 that a free group $U(I)$ has the following lifting property: if $1 \to G_a \to E \to^g U \to 1$ is an exact sequence of unipotent groups and $f: U(I) \to U$ is a homomorphism, then there is a homomorphism $h : U(I) \to E$ such that $gh = f$. (We get a set of elements of E by choosing preimages of the elements $f(i)$ $i \in I$, choosing e as a preimage if $f(i) = e$.) This property turns out to be characteristic for free groups. To see this, we will need to see that the property is equivalent to an apparently stronger one. This result is based closely on the proof of [4, Proposition 1, p. 157].

THEOREM 2.4. *Let G be a prounipotent group. The following conditions are equivalent*:

(a) *If $1 \to K \to E \to^g F \to 1$ is any exact sequence of prounipotent groups and $f: G \to F$ is a homomorphism, then there is a homomorphism $h : G \to E$ such that $gh = f$.*

(b) *If $1 \to G_a \to E \to^g F \to 1$ is any exact sequence of unipotent groups and $f: G \to F$ is a homomorphism, then there is a homomorphism $h : G \to E$ such that $gh = f$.*

Proof. To establish the non-trivial implication (b) $\Rightarrow$ (a) we begin by assuming E is unipotent and use induction on the dimension of K, the case where $\dim K = 1$ being (b). For $\dim K > 1$, we observe that the center Z of K is a module for the unipotent group F acting via conjugation, and hence Z contains a one-dimensional subgroup L normalized by F, hence normal in E. Then we apply induction to the sequence $1 \to K/L \to E/L \to F \to 1$ and the homomorphism $G \to F$ to obtain a homomorphism $G \to E/L$, and then use

(b) with this homomorphism and the sequence $1 \to L \to E \to E/L \to 1$. This establishes (a) for E unipotent.

Next we want to establish (a) with K unipotent (E can now be prounipotent). We claim that there is a normal subgroup H of E of finite codimension with $H \cap K = \{e\}$: we can choose a collection $\{H_a \mid a \in \mathscr{A}\}$ of normal subgroups of E of finite codimension with $\bigcap \{H_a \mid a \in \mathscr{A}\} = \{e\}$. Then $\{H_a \cap K \mid a \in \mathscr{A}\}$ is a collection of closed connected subgroups of K whose intersection is $\{e\}$; since K is finite dimensional some finite collection of these subgroups of K must intersect in $\{e\}$. Now we can apply (a) to the sequence $1 \to K \to E/H \to F' \to 1$ (where $F' = F/g(H)$) and the homomorphism $G \to F \to F'$ since E/H is unipotent. This produces a homomorphism $G \to E/H$ which combines with the original homomorphism $G \to F$ to produce a homomorphism from G to the fibre product $E/H \times_{F'} F = E$.

Finally, we establish (a) in general. We let $\mathscr{P}$ be the set of pairs (P, s), where P is a normal subgroup of E contained in K and $s : G \to E/P$ solves (a) for the sequence $1 \to K/P \to E/P \to F \to 1$. $\mathscr{P}$ is partially ordered by the relation $(P_1, s_1) \leqslant (P_2, s_2)$ if P_1 contains P_2 and s_1 is the composite of s_2 and the map $E/P_2 \to E/P_1$. Chains in $\mathscr{P}$ have upper bounds so $\mathscr{P}$ has a maximal element (P, s). If $P = \{e\}$, (a) holds. If not, we can choose a normal subgroup H of E of finite codimension with $H \cap P \neq P$: this is possible since otherwise P is in every normal subgroup of E of finite codimension and these intersect to $\{e\}$. Let $P' = P \cap H$. Then P/P' is unipotent and (a) applied to the sequence $1 \to P/P' \to E/P' \to E/P \to 1$ and the homomorphism $s : G \to E/P$ gives $s' : G \to E/P'$ with $(P', s') > (P, s)$. This contradiction shows that $P = \{e\}$ and that (a) holds.

We shall temporarily refer to groups satisfying the conditions of Theorem 2.4 as groups with the *lifting property*. (We see below Proposition 2.8 that they are actually free.) The remarks prior to Theorem 2.4 show that free groups have the lifting property. The universal mapping property of $U_0(I)$ shows that it also has the lifting property.

We recall that if G is a prounipotent group G^* denotes the closure of the commutator subgroup (G, G) of G. The next lemma shows that G^* behaves like the Frattini subgroup of a finite group.

LEMMA 2.5. *Let $f: G \to H$ be a homomorphism of prounipotent groups such that the induced homomorphism $G/G^* \to H/H^*$ is onto. Then f is onto.*

Proof. We can replace G by $f(G)$ and assume that G is a subgroup of H and f is inclusion. If $G \neq H$, by (1.6) there is a finite-dimensional quotient H_0 of H such that the image G_0 of G in H_0 is distinct from H_0. Since H/H^* maps onto $H_0/(H_0, H_0)$, we can replace G and H by G_0 and H_0. By [7, Proposition 12.4, p. 112] there is a subgroup of H_0 of codimension one containing G_0. But this subgroup must contain (H_0, H_0) and $G_0(H_0, H_0) = H_0$ by assumption.

COROLLARY 2.6. *Let $f: G \to H$ be a homomorphism of prounipotent groups such that the induced homomorphism $G/G^* \to H/H^*$ is an isomorphism. Assume H has the lifting property. Then f is an isomorphism.*

Proof. By Lemma 2.5, f is onto. We can apply the lifting property to the sequence $1 \to \mathrm{Ker}(f) \to G \to H \to 1$ and the identity homomorphism $H \to H$ to obtain a homomorphism $g: H \to G$ such that $fg = id$. Then g is one–one, and the map $H/H^* \to G/G^*$ induced from g is onto; so by Lemma 2.5, g is onto and hence an isomorphism.

To apply Lemma 2.5 and Corollary 2.6, we need criteria to tell when homomorphisms $G/G^* \to H/H^*$ are onto or isoorphisms. These are provided by the equivalence (1.2) between abelian prounipotent groups and linear compact k-vector spaces.

LEMMA 2.7. *Let $f: G \to H$ be a homomorphism of prounipotent groups and let $f_*: G/G^* \to H/H^*$ and $f^*: \mathrm{Hom}(H, G_a) \to \mathrm{Hom}(G, G_a)$ be the induced maps. Then f_* is an isomorphism if and only if f^* is an isomorphism.*

Proof. We may assume G and H are abelian so $f = f_*$. By (1.2), G and H correspond to the linearly compact vector spaces $\mathrm{Lie}(G)$ and $\mathrm{Lie}(H)$, and $\mathrm{Hom}(\mathrm{G}, G_a)$ and $\mathrm{Hom}(H, G_a)$ are the continuous duals of these spaces, so the result follows from the duality between linearly compact and discrete vector spaces [8, Theorem 29.1, p. 81].

PROPOSITION 2.8. *Let G be a prounipotent group. Then there is a free prounipotent group $U(I)$ and a surjection $f: U(I) \to G$. I and f can be chosen so that* $\mathrm{Hom}(G, G_a)$ *has dimension equal to the cardinarlity of I. If J is any set and $g: U(J) \to G$ is onto, the cardinality of I is less than or equal to that of J. If G has the lifting property, f is an isomorphism.*

Proof. Choose a basis $\{f_i \mid i \in I\}$ of $\mathrm{Hom}(G, G_a)$. Let $U = U(I)$ and let $\phi: \mathrm{Hom}(G, G_a) \to \mathrm{Hom}(U, G_a)$ be the linear isomorphism with $\phi(f_i) = E_i$, Lemma 2.3. By duality for linearly compact vector spaces, ϕ comes from an isomorphism $F: U/U^* \to G/G^*$. Apply the lifting property, Theorem 2.4(a), to the sequence $1 \to G^* \to G \to G/G^* \to 1$ and the homomorphism $U \to G/G^*$ factoring through F to obtain a homomorphism $f: U \to G$ such that, in the notation of Lemma 2.7, $f_* = F$ and $f^* = \phi$. By Lemma 2.7, f_* is an isomorphism; so by Lemma 2.5, f is onto. If G has the lifting property, f is an isomorphism by Corollary 2.6. If $g: U(J) \to G$ is onto, $\mathrm{Hom}(G, G_a) \to \mathrm{Hom}(U(J), G_a)$ is one–one. By Lemma 2.3, $\dim(\mathrm{Hom}(G, G_a)) =$

$\dim(\mathrm{Hom}(U(I), G_a)) = \mathrm{card}(I)$ and $\dim(\mathrm{Hom}(U(J), G_a)) = \mathrm{card}(J)$; so $\mathrm{card}(I) \leqslant \mathrm{card}(J)$.

We can now characterize free groups cohomologically.

THEOREM 2.9. *Let G be a prounipotent group. Then G is free if and only if* $\mathrm{cd}(G) \leqslant 1$.

Proof. If G is free, G satisfies Theorem 2.4(b); so $H^2(G, k) = 0$ and $\mathrm{cd}(G) \leqslant 1$. If $\mathrm{cd}(G) \leqslant 1$, then G satisfies Theorem 2.4(b) and hence Theorem 2.4(a); so G has the lifting property and thus G is free by Proposition 2.8. (We are using 1.18).

By Proposition 2.8, groups with the lifting property are free (and conversely, by Theorem 2.4.). In particular, the groups $U_0(I)$ are free.

The cohomological description of free groups gives us the following important result on subgroups.

COROLLARY 2.10. *Let G be a free prounipotent group and let H be a subgroup of G. Then H is free.*

Proof. By (1.14), $\mathrm{cd}(H) \leqslant \mathrm{cd}(G)$; so the corollary follows from Proposition 2.8.

We also note that Theorem 2.9 shows that $U(\{1\}) = G_a$: for G_a has cohomological dimension one (1.15); so $G_a = U(I)$ for some I, and $H^1(G_a, k) = 1$ (1.15), thus $\mathrm{card}(I) = 1$.

We can also describe in part the free prounipotent groups on a finite set. We will use the prounipotent analogue of the lower central series.

DEFINITION 2.11. Let G be a prounipotent group. Then $G^{*1} = G^*$ and for $i \geqslant 1$, $G^{*(i+1)}$ is the Zariski closure of the abstract subgroup (G, G^{*i}).

We write $U(n)$ for $U(\{1,..., n\})$ and $U(n, r)$ for $U(n)/U(n)^{*r}$. Any unipotent homomorphic image of $U(n, r)$ is generated by n elements and is nilpotent of class r, and conversely any such unipotent group is a homomorphic image of $U(n, r)$. We can describe $U(n, r)$ in terms of free Lie algebras [13, Definition 2.1, p. LA4.4]: let $L(n)$ denote the free k-Lie algebra on $\{1,..., n\}$, let $L(n)^m$ denote the subspace spanned by products of length m, let $L(n)_m = \bigcup \{L(n)^i \mid i \geqslant m\}$ and let $L(n, r) = L(n)/L(n)_{r+1}$. Then $L(n, r)$ is finite dimensional [13, p. LA4.5] and if $U'(n, r)$ is the unipotent group with Lie algebra $L(n, r)$, every n-generated unipotent group which is nilpotent of class r is a homomorphic image of $U'(n, r)$ and conversely. It follows that $U(n, r) = U'(n, r)$ and, in particular, $U(n, r)$ is finite dimensional of dimension equal to that of $L(n, r)$ (the latter is given by [13, p. LA4.5]). Thus the exact sequence $1 \to U(n)^{*r}/U(n)^{*(r+1)} \to U(n, r+1) \to U(n, r) \to 1$

shows that $U(u)U^r/U(n)^{*(r+1)}$ has dimension equal to that of $L(n)^{r+2}$, namely, $l_n(r+2)=(r+2)^{-1}\sum_{m|(r+2)}\mu(m)\,n^{m/r+2}$, μ the Moëbius function.

We have surjections $U(n,r+1)\to U(n,r)$ and in fact $U(n)=$ proj $\lim(U(n,r))$.

3. Generators and Relations

In this section we interpret the low-dimensional cohomology of a prounipotent group in terms of generators and relations and show that one-relator groups have cohomological dimension two.

We know from Proposition 2.8 that every prounipotent group G is a quotient of a free prounipotent group $U(I)$ with $\mathrm{card}(I)=\dim(\mathrm{Hom}(G,G_a))$. When I is finite, this fact gives an interpretation of $H^1(G,G_a)$ in terms of minimal number of generators of G.

Definition 3.1. A prounipotent group G is *finitely generated* if there is a set of elements $\{g_1,\dots,g_n\}$ of G such that the abstract subgroup of G generated by $g_1,\dots,g_n$ is Zariski-dense in G. In this case $\{g_1,\dots,g_n\}$ is a set of generators of G. If G is finitely generated, the *rank* of G is the cardinality of a set of generators of minimal cardinality.

Theorem 3.2. *Let G be a prounipotent group. G is finitely generated if and only if $H^1(G,k)$ is finite dimensional. If G is finitely generated, the rank of G is the dimension of $H^1(G,k)$.*

Proof. As we observed in (1.16), $H^1(G,k)=\mathrm{Hom}(G,G_a)$. By Proposition 2.8, $H^1(G,k)$ finite dimensional implies that there is a surjection $g: U(I)\to G$ with I finite, so $\{g(i)\mid i\in I\}$ is a finite set of generators of G. Conversely, if $\{g_1,\dots,g_n\}$ generate G, there is an onto homomorphism $g: U(\{1,\dots,n\})\to G$ with $g(i)=g_i$ so by Proposition 2.8, $H^1(G,k)$ has dimension at most n. For G finitely generated the rank assertion follows similarly from Proposition 2.8.

By Theorem 3.2 and Lemma 2.3 we can refer to $U(\{1,2,\dots,n\})$ as the free group of rank n. We next consider subgroups of free groups of finite rank.

Lemma 3.3. *Let F be a free prounipotent group of rank e, and let F_0 be a normal subgroup of F of codimension one. Then $H^1(F_0,k)$ is isomorphic to $k[G_a]^{(e-1)}$ as a G_a-module.*

Proof. $F/F_0=G_a$, so $H^1(F_0,k)$ is a G_a-module. We apply the spectral sequence (1.10) to the extension $1\to F_0\to F\to F_a\to 1$: we have $H^p(G_a,H^q(F_0,k))\Rightarrow H^{p+q}(F,k)$. F, G_a, and F_0 all have cohomological dimension one by Theorem 2.9 and Corollary 2.10; so

$H^1(G_a, H^1(F_0, k)) = 0$. This means $H^1(F_0, k)$ is injective as a G_a-module; so by (1.11), $H^1(F_0, k)) = k[G_a]^{(I)}$. To determine card($I$), we use the exact sequence of low degree from the spectral sequence: $0 \to H^1(G_a, k) \to H^1(F, k) \to H^1(F_0, k)^{G_a} \to H^2(G_a, k) = 0$. Since $\dim(H^1(G_a, k)) = 1$ and $\dim(H^1(F, k)) = e$, Theorem 3.2, $\dim(H^1(F_0, k)^{G_a}) = e - 1$, so card$(I) = e - 1$.

We want to apply Lemma 3.3 to more general subgroups of free groups. This requires the following two technical lemmas.

LEMMA 3.4. *Let G be a prounipotent group, $G \neq \{e\}$. Then G contains a subgroup isomorphic to G_a.*

Proof. Choose $g \in G$, $g \neq e$. By Proposition 2.2 there is a homomorphism $f: G_a = U(\{1\}) \to G$ with $f(1) = g$. Thus $\operatorname{Ker}(f) \neq G_a$, so $\operatorname{Ker}(f) = 0$ and the image of f is the desired subgroup.

LEMMA 3.5. *Let G be a prounipotent group and H a subgroup of G, $H \neq G$. If H is either normal in G or of finite codimension in G, G contains a subgroup G_0 with H normal in G_0 and $G_0/H = G_a$.*

Proof. If H is normal in G, let G_0 be the inverse image in G of a subgroup of G/H isomorphic to G_a, which exists by Lemma 3.4. If H is of finite codimension in G, there is a normal subgroup H_0 of G of finite codimension contained in H, (1.8), so it will suffice to pass to the subgroup $\bar{H} = H/H_0$ of the unipotent group $\bar{G} = G/H_0$. Then a normalizer N of $\bar{H}$ in $\bar{G}$ properly contains $\bar{H}$ [7, Lemma 17.4, p. 112] so we may take G_0 to be the inverse image of a G_a in $N/\bar{H}$.

PROPOSITION 3.6. *Let F be a non-abelian free prounipotent group and let H be a proper subgroup. If H is either normal in F or of finite codimension in F, H is not finitely generated.*

Proof. By Lemma 3.5 there is a subgroup F_0 of F such that H is normal in F_0 and $F_0/H = G_a$. By Corollary 2.10, F_0 is also free and since $H \neq \{e\}$, F_0 is non-abelian. Thus we can assume $F = F_0$. If H were finitely generated of rank n, then F would be finitely generated of rank $n + 1$. Then by Lemma 3.3, $H^1(H, k) = k[G_a]^{(n)}$ is infinite dimensional, contrary to Theorem 3.3.

COROLLARY 3.7. *Let F be a non-abelian free prounipotent group and let H be a finitely generated proper subgroup. Then H is its own normalizer, and if k is any subgroup of F properly containing H, then H is of infinite codimension in K.*

Proof. Let N be the normalizer of H. N is free by Corollary 2.10. If H is a proper subgroup of N, Proposition 3.6 implies H is infinitely generated. Similarly K is free by Corollary 2.10; so if H were of finite codimension in K, Proposition 3.6 again yields a contradiction.

Now any non-abelian free prounipotent group contains finitely generated proper subgroups (for example, by Lemma 3.4); hence Corollary 3.7 actually gives examples of self-normalizing proper subgroups of prounipotent groups. Such a situation cannot occur in unipotent groups [7, Proposition 17.4, p. 112].

We can also use Lemma 3.3 to give an example of a finitely generated prounipotent group of infinite cohomological dimension.

EXAMPLE 3.8. Let F be a free prounipotent group of rank at least two and let G be a normal subgroup of F of codimension one. By Proposition 3.6, G is infinitely generated. Let $\bar{F} = F/G^*$ and let $\bar{G} = G/G^*$. It follows from Lemma 3.3 and 2.3, Corollary 2.10, 2.3, and 1.2 that $\bar{G}$ is isomorphic to G_a^I with I countable. Then $\bar{G}$ has unipotent subgroups of arbitrary large dimension, so by (1.15), $\mathrm{cd}(\bar{G}) = \infty$. Since $\bar{G}$ is a subgroup of $\bar{F}$, it follows from (1.14) that $\mathrm{cd}(\bar{F}) = \infty$. Thus $\bar{F}$ is a finitely generated prounipotent group of infinite cohomological dimension.

We will show that the second cohomology group can be used to determined numbers of relations. We need the concept of generators of a normal subgroup.

DEFINITION 3.9. Let G be a prounipotent group and let N be a normal subgroup of G. N is *finitely generated* (as a normal subgroup) if there is a set of elements $\{n_1, \ldots, n_m\}$ of N such that the abstract subgroup of N generated by all G-conjugates of the n_i is Zariski-dense. If m is minimal among all such sets, m is the minimum number of generators of N.

DEFINITION 3.10. Let G be a prounipotent group. A *proper presentation* of G is an exact sequence $1 \to R \to F \to G \to 1$ of prounipotent groups with F free such that $H^1(G, k) \to H^1(F, k)$ is an isomorphism. G is *finitely related* if there is such an exact sequence with R finitely generated as a normal subgroup of F.

THEOREM 3.11. *Let G be a prounipotent group. G is finitely related if and only if $H^2(G, k)$ is finite dimensional. If G is finitely related, and if $1 \to R \to F \to G \to 1$ is any proper presentation of G, then R is finitely generated as a normal subgroup of F and its minimal number of generators is the dimension of $H^2(G, k)$.*

Proof. Suppose G has a proper presentation $1 \to R \to F \to G \to 1$. The spectral sequence (1.10), $H^p(G, H^q(R, k)) \Rightarrow H^{p+q}(F, k)$, gives the sequence $1 \to H^1(G, k) \to H^1(F, k) \to H^1(R, k)^F \to H^2(G, k) \to H^2(F, k) = 1$ as the exact sequence of low degree. Since the presentation is proper, we conclude $H^1(R, k)^F = H^2(G, k)$. F acts on R by conjugation. If we let F act trivially on G_a, under the identification $H^1(R, k) = \mathrm{Hom}(R, G_a)$, $H^1(R, k)^F$ corresponds to the set $\mathrm{Hom}_F(R, G_a)$ of F-equivariant homomorphisms. If R is generated by $r_1, ..., r_n$ as a normal subgroup of F, then an F-equivariant homomorphism $f: R \to G_a$ is determined by $f(r_1), ..., f(r_n)$; so $\dim(H^2(G, k)) = \dim(H^1(R, k)^F) \leqslant n$. Conversely, suppose $\dim(H^2(G, k)) = n$. By Corollary 2.10, R is free, and by Proposition 2.8 we can choose a set I containing $\{1, ..., n\}$ such that $R = U(I)$ and, by Lemma 2.3, $E_1, ..., E_n$ is a basis of $\mathrm{Hom}_F(R, G_a)$. Let r_i denote the element of R corresponding to $i \in I$ for $i = 1, ..., n$. We claim that $\{r_1, ..., r_n\}$ generate R as a normal subgroup of F: for let N be the Zariski closure of the abstract subgroup of R generated by the conjugates under F of $\{r_1, ..., r_n\}$. N is normal in F. The inclusion $N \to R$ induces a homomorphism $H^1(R, k) \to H^1(N, k)$ and hence a homomorphism $H^1(R, k)^F = \mathrm{Hom}_F(R, G_a) \to \mathrm{Hom}_F(N, G_a)$. By construction, this last map is an injection. Now the kernel K of $H^1(R, k) \to H^1(N, k)$ is an F-module with $K^F = 0$; so $K = 0$. Thus $\mathrm{Hom}(R, G_a) \to \mathrm{Hom}(N, G_a)$ is an injection. If $N \neq R$, there is a non-trivial homomorphism $R/N \to G_a$, so the map could not be an injection. Thus $N = R$, and the theorem follows. (This proof is based on [12, Section 4.3].)

DEFINITION 3.12. Let G be a prounipotent group. G is a *n-relator* group if in every proper presentation $1 \to R \to F \to G \to 1$ R is finitely generated as a normal subgroup of F with minimal number of generators n.

COROLLARY 3.13. *A prounipotent group G is n-relator if and only if $H^2(G, k)$ has dimension n.*

Discrete one-relator groups essentially have cohomological dimension two [9, Corollary 11.2, p. 663]. We shall see (Corollary 3.15) that a similar statement holds for prounipotent groups. In fact, we have the following stronger result:

THEOREM 3.14. *Let G be a prounipotent group and assume that for some $n \geqslant 1$, $H^n(G, k)$ has dimension one. Then* $\mathrm{cd}(G) = n$.

Proof. Let $\{E_i \mid i = 0, 1, ...$ be the minimal injective resolution of k as a G-module, so $H^i(G, k) = E_i^G$ and $E_i = H^i(G, k) \otimes k[G]$ ((1.11) and (1.12)). In particular, $E_n = k[G]$. Since E_n is not zero, $d_{n-1}: E_{n-1} \to E_n$ is not the zero map. By Theorem (3.2), it follows that d_{n-1} is onto, so that $E_{n+1} = 0$. Thus $H^{n-1}(G, k) = 0$; so $\mathrm{cd}(G) = n$.

COROLLARY 3.15. *A* 1-*relator prounipotent group* G *has cohomological dimension two.*

Proof. Combine Theorem 3.14 and Corollary 3.13.

COROLLARY 3.16. *The only unipotent* 1-*relator group is* $G_a \times G_a$.

Proof. $G_a \times G_a$ is the only unipotent group of dimension two, so Corollary 3.15 and (1.15) supply the result.

As an illustration of the results of this section, we compute numbers of relations for the groups $U(n, r)$ introduced at the end of Section 2.

LEMMA 3.17. *Let* $1 \to R \to F \to G \to 1$ *be an exact sequence of prounipotent groups, and let* $(F, R)^*$ *denote the Zariski closure of the abstract commutator subgroup* (F, R). *Then* $\mathrm{Hom}_F(R, G_a) = \mathrm{Hom}(R/(F, R)^*, G_a)$.

Proof. If $f: R \to G_a$ is a F-homomorphism, $f(xrx^{-1}) = f(r)$ for $x \in F$ and $r \in R$; so f vanishes on $xrx^{-1}r^{-1}$ and hence on (F, R). The converse is similar.

Now in the notation of Section 2, $1 \to U(n)^{*r} \to U(n) \to U(n, r) \to 1$ is a proper presentation of $U(n, r)$. From the proof of Theorem 3.11, we know that $H^2(U(n, r), k) = H^1(U(n)^{*r}, k)^{U(n)} = \mathrm{Hom}_{U(n)}(U(N)^{*r}, G_a)$. By Corollary 3.16 and Definition 2.11 this last group is $\mathrm{Hom}(U(n)^{*r}/U(n)^{*(r+1)}, G_a)$. In Section 2, we showed that $U(n)^{*r}/U(n)^{*(r+1)}$ has dimension $l_n(r+2)$. Since it is abelian, we conclude that $H^2(U(n, r), k)$ has dimension $l_n(r+2)$. By Theorem 3.11, we know that it has this many relations.

4. APPLICATIONS TO PRO-AFFINE HULLS OF DISCRETE GROUPS

If A is any pro-affine algebraic group over k, A contains a normal prounipotent subgroup $R_u(A)$ such that the quotient $A/R_u(A)$ is an inverse limit of reductive algebraic groups [10, (2.9), p. 7]. We call $R_u(A)$ the *prounipotent radical* of A. If Γ is a discrete group and $A = A(\Gamma)$, we also write $U(\Gamma)$ for $R_u(A)$ [10, p. 7].

LEMMA 4.1. *Let* A *be a pro-affine group. Then* $\mathrm{cd}(A) = \mathrm{cd}(R_u(A))$.

Proof. Let $Q = A/R_u(A)$. Then $\mathrm{cd}(Q) = 0$, so the spectral sequence (1.10) arising from the exact sequence $1 \to R_u(A) \to A \to Q \to 1$ collapses and we get isomorphisms $H^0(Q, H^i(R_u(A), V)) \to H^i(A, V)$ for every A-module V. Thus $\mathrm{cd}(A) \leqslant \mathrm{cd}(R_u(A))$. Since by [10, (2.9), p. 7], $R_u(A)$ is a normal semi-direct factor of A, the proof of [2, Theorem 4.3, p. 9] shows that induction from $R_u(A)$-modules to A-modules is exact, so that, as in (1.14), $\mathrm{cd}(R_u(A)) \leqslant \mathrm{cd}(A)$.

PROPOSITION 4.2. *Let Γ be an abstract free group on the set X. Then* $\mathrm{cd}(A(\Gamma)) \leqslant 1$.

Proof. Let $1 \to K \to G \to^{f} A(\Gamma) \to 1$ be any exact sequence of pro-affine groups. Since Γ is free, there is an abstract homomorphism $h_0 : \Gamma \to G$ such that fh_0 is the identity on Γ. Then h_0 induces a homomorphism $h : A(\Gamma) \to G$ with fh the identity on $A(\Gamma)$. If we apply this in the case K is a finite-dimensional $A(\Gamma)$-module V, we see that $H^2(A(\Gamma), V) = 0$. Thus $\mathrm{cd}(A(\Gamma)) \leqslant 1$.

THEOREM 4.3. *Let Γ be a finitely generated discrete group and suppose Γ contains a free subgroup Γ_0 of finite index. Then $U(\Gamma)$ is a free prounipotent group.*

Proof. We may assume Γ_0 is normal. Then since $[\Gamma : \Gamma_0] < \infty$, $U(\Gamma_0) = U(\Gamma)$; so we may assume Γ is free. By Proposition 4.2, $\mathrm{cd}(A(\Gamma)) \leqslant 1$; so by Lemma 4.1 we have $\mathrm{cd}(U(\Gamma)) \leqslant 1$. Now by Corollary 2.10 we conclude $U(\Gamma)$ is free.

It follows from Theorem 4.3 that if Γ is a non-abelian free group, then $U(\Gamma)$ is infinite dimensional: for example, $A(\Gamma)$ (and hence $U(\Gamma)$) has the group $U(2, 2)$ of Section 2 as a homomorphic image, so $U(\Gamma) \neq G_a$.

In particular, we see that $U(SL_2(\mathbb{Z}))$ is infinite dimensional. This should be compared with the result [10, p. 2] that $U(SL_n(\mathbb{Z})) = 1$ for $n \geqslant 3$.

5. MODULE ENDOMORPHISMS OF THE COORDINATE RING

In this section we establish the result, used in the proof of Theorem 3.14, that any non-zero G-module endomorphism of the coordinate ring of te prounipotent group G is surjective. It turns out to be sufficient to treat the case where G is unipotent, and here the result is essentially independent of the rest of the paper.

We begin with some preliminary observations: fix a unipotent group U, and let $L = \mathrm{Lie}(U)$. We can regard L as the set of all k-algebra derivations of $k[U]$ with $D(f \cdot x) = D(f) \cdot x$ for all f in $k[U]$ and $x \in U$. If we regard $k[U]$ as a left U-module (where x in U acts on f in $k[U]$ to give $x \cdot f$), then the corresponding left L-module structure on $k[U]$ makes the (derivation) D in L act on f in $k[U]$ by $D(f)$. In particular, a U-module endomorphism T of $k[U]$ is also an L-module endomorphism, so $XT = TX$ (as k-module endomorphisms of $k[U]$) for all X in L. If $D \in L$ is a U-module endomorphism, then D commutes with all U-module endomorphisms of $k[U]$.

Now let Z be a one-dimensional normal subgroup of U. As in the proof of (1.15), we have an exact sequence $1 \to k \to k[Z] \to^d k[Z] \to 1$, where d is the derivative with respect to the parameter of Z. By [2, Theorem 4.3, p. 9] induction from Z-modules to U-modules is exact, so we get an exact sequence $1 \to \mathrm{Map}_Z(U, k) \to \mathrm{Map}_Z(U, k[Z]) \to^D \mathrm{Map}_Z(U, k[Z]) \to 1$. It is easy to check that $\mathrm{Map}_Z(U, k) = k[U/Z]$ and $\mathrm{Map}_Z(U, k[Z]) = k[U]$. If we regard $d \in \mathrm{Lie}(Z)$, then is the image of d in $\mathrm{Lie}(U)$. Thus we have an exact sequence $0 \to k[U/Z] \to k[U] \to^D k[U] \to 0$, where D is a central surjective endomorphism of $k[U]$.

THEOREM 5.1. *Let U be a unipotent group and T a non-zero U-module endomorphism of $k[U]$. Then T is onto.*

Proof. We use induction on $\dim(U)$, the case $\dim(U) = 0$ being clear. Choose a one-dimensional normal subgroup Z of U and let $E = k[U]$ and let $E_1 = k[U/Z]$. The discussion preceding the theorem shows that there is a central, surjective U-module endomorphism D of E with kernel E_1. Let $E_i = \mathrm{Ker}(D^i)$ for $i \geqslant 1$. Since D is locally nilpotent ($D \in \mathrm{Lie}(U)$), we have $E = \bigcup E_i$ and since D is onto, $DE_{i+1} = E_i$; so $D^i E_{i+1} = E_1$. Since D is central, every U-module endomorphism X of E sends E_i to E_i. Let X be a U-module endormophism of E, $X \neq 0$, and assume $X(E_i) = 0$, $X(E_{i+1}) \neq 0$. Now D^i induces a U-module isomorphism $D^i : E/E_i \to \underline{E}$ and X induces a U-module homomorphism $\bar{X}: E/E_i \to E$. If we let $Y = \bar{X}(\overline{D^i})^{-1}$, then $YD^i = X$. Now $Y(E_1) = Y(D^i E_{i+1}) = X(E_{i+1}) \neq 0$. The restriction of Y to E_1 is a U/Z-module endomorphism of $k[U/Z]$ and non-zero; so by induction $Y(E_1) = E_1$. Now the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E_1 & \to & E_{j+1} & \to & E_j & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E_1 & \to & E_{j+1} & \to & E_j & \to & 0 \end{array}$$

where the vertical maps are all Y, and $E_{j+1} \to E_j$ is D, has exact rows, so from the fact that $Y(E_1) = E_1$ we conclude inductively that $Y(E_i) = E_i$. Thus Y is surjective. Since D is surjective and $YD^i = X$, X is surjective.

COROLLARY 5.2. *Let G be a prounipotent group and T a non-zero G-module endomorphism of $k[G]$. Then T is onto.*

Proof. If H is any normal subgroup of G, then $T(k[G]^H) \subseteq k[G]^H$ and T is a G/H-module endomorphism of $k[G/H] = k[G]^H$. If $T \neq 0$, there is such an H of finite codimension with $T(k[G]^H) \neq 0$; so for every normal K of finite codimension contained in H, T is a non-zero endomorphism of $k[G]^K$, and hence onto by Theorem 5.1. Since $k[G] = \bigcup k[G]^K$ for such K, T is onto.

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