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A-m-Isometric operators in semi-Hilbertian spaces

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ABSTRACT

In this work, the concept of m-isometry on a Hilbert space are generalized when an additional semi-inner product is considered. This new concept is described by means of oblique projections.

1. Introduction and terminologies

The concept of partial isometries, quasi-isometries and contractions in semi-Hilbertian spaces was introduced by Arias et al. [7] and Suciu [9,10].

The class of m-isometric and in particular 2-isometric operators on a complex Hilbert space has been the object of some intensive study, especially by Agler and Stankus [1], Patel [11] but also by Richter, Shimorin and other authors (see [2,6]).

Our goal in this paper is to study the class of m-isometric with respect to a semi-norm $\|.\|_A$. An operator in this class will be called A-m isometry. We show that many results from [1,11] remain true if we consider an additional semi-inner product defined by a positive semi-definite operator A.

The contents of the paper are the following. In Section 1, we set up notation and terminology. In Section 2, we introduce the concept of A-m isometries. Several properties are proved which result in a

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generalization of well known assertions of m-isometries . In Section 3, we shall specialize to the case $m = 2$. We explore some properties of A-2-isometry.

Several spectral properties of A-m-isometries are obtained in Section 4, concerning the point spectrum, the approximate spectrum and the spectrum. The spectral radius of T will be denoted by $r(T)$.

Throughout this paper $\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{K}$ stands for a complex Hilbert space with inner product $\langle\ |\ \rangle$ and $\mathcal{L}(\mathbb{K})$ is the Banach algebra of all bounded linear operators on $\mathbb{K}$, $I = I_{\mathbb{K}}$ being the identity operator. $\mathcal{L}(\mathbb{K})^+$ is the cone of positive (semi-definite) operators, i.e.,

$$\mathcal{L}(\mathbb{K})^+ = \{A \in \mathcal{L}(\mathbb{K}) : \langle Au,\ |\ u\rangle \geqslant 0,\ \forall\, u \in \mathbb{K}\ \}.$$

For every $T \in \mathcal{L}(\mathbb{K})$ its range is denoted by $R(T)$, its null space by $N(T)$ and its adjoint by T^*. If $\mathbb{M} \subset \mathbb{K}$ is a closed subspace, $P_{\mathbb{M}}$ is the orthogonal projection onto $\mathbb{M}$. The subspace $\mathbb{M}$ is invariant for T if $T\mathbb{M} \subset \mathbb{M}$, and $\mathbb{M}$ reduces T it is invariant for T and T^*. We write $\rho(T), \sigma(T), \sigma_p(T)$ and $\sigma_{ap}(T)$, respectively, for the resolvent set, the spectrum, the point spectrum and the approximate point spectrum of the operator T. Any $A \in \mathcal{L}(\mathbb{K})^+$ defines a positive semi-definite sesquilinear form:

$$\langle\ |\ \rangle_A : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{C},\ \langle u \mid v\rangle_A = \langle Au \mid v\rangle.$$

By $\|.\|_A$ we denote the semi-norm induced by $\langle\ |\ \rangle_A$, i.e.,$\|u\|_A = \langle u \mid u\rangle_A^{\frac{1}{2}}$. Observe that $\|u\|_A = 0$ if and only if $u \in N(A)$. Then $\|.\|_A$ is a norm if and only if A is an injective operator, and the semi-normed space $(\mathcal{L}(\mathbb{K}), \|.\|_A)$ is complete if and only if $R(A)$ is closed. Moreover $\langle\ |\ \rangle_A$ induced a semi-norm on a certain subspace of $\mathcal{L}(\mathbb{K})$, namely, on the subset of all $T \in \mathcal{L}(\mathbb{K})$ for which there exists a constant $c > 0$ such that $\|Tu\|_A \leqslant c\|u\|_A$ for every $u \in \mathcal{L}(\mathbb{K})$. For these operators it holds

$$\|T\|_A = \sup_{u \in \overline{R(A)}, u \neq 0} \frac{\|Tu\|_A}{\|u\|_A} < \infty.$$

It is straightforward that

$$\|T\|_A = \sup\{|\langle Tu \mid v\rangle_A| : u, v \in \mathbb{K} \text{ and } \|u\|_A \leqslant 1, \|v\|_A \leqslant 1\ \}.$$

Given a subspace $\mathbb{M}$ of $\mathbb{K}$ its A-orthogonal subspace is the subspace

$$\mathbb{M}^{\perp_A} = \{\ v \in \mathbb{K};\ \langle v \mid u\rangle_A = 0,\quad \forall\, u \in \mathbb{M}\ \ \}.$$

It easy to check that

$$\mathbb{M}^{\perp_A} = (A\mathbb{M})^{\perp} = A^{-1}(\mathbb{M}^{\perp}).$$

Moreover, since $A(A^{-1}(\mathbb{M}) = \mathbb{M} \cap R(A)$, then

$$(\mathbb{M}^{\perp_A})^{\perp_A} = \left(\mathbb{M}^{\perp} \cap R(A)\right)^{\perp}.$$

Theorem 1.1 (Douglas [4])**.** *Let $A, B \in \mathcal{L}(\mathbb{K})$. The following conditions are equivalents.*

1. *$R(B) \subset R(A)$.*
2. *There exists a positive number λ such that $BB^* \leqslant \lambda AA^*$.*
3. *There exists $C \in \mathcal{L}(\mathbb{K})$ such that $AC = B$.*

Definition 1.1. Let $T \in \mathcal{L}(\mathbb{K})$, an operator $W \in \mathcal{L}(\mathbb{K})$ is called an A-adjoint of T if

$$\langle Tu \mid v\rangle_A = \langle u \mid Wv\rangle_A \text{ for every } u, v \in \mathbb{K},$$

i.e.,

$$AW = T^*A;$$

T is called A-selfadjoint if

$$AT = T^*A.$$

The existence of an A-adjoint operator is not guaranteed. Observe that a given $T \in \mathcal{L}(\mathbb{K})$ may admit none, one or many A-adjoints: in fact, if W is an A-adjoint of T and $AZ = 0$for some $Z \in \mathcal{L}(\mathbb{K})$ then $W + Z$ is also an A-adjoint of T.

By Douglas theorem, T admits an A-adjoint if and only if $R(T^*A) \subset R(A)$. From now on, $\mathcal{L}_A(\mathbb{K})$ denotes the set of all $T \in \mathcal{L}(\mathbb{K})$ which admit an A-adjoint, it is

$$\mathcal{L}_A(\mathbb{K}) = \{ \ T \in \mathcal{L}(\mathbb{K}) : \ R(T^*A) \subset R(A) \ \}.$$

$\mathcal{L}_A(\mathbb{K})$ is a subalgebra of $\mathcal{L}(\mathbb{K})$ which is neither closed nor dense in $\mathcal{L}(\mathbb{K})$.

On the other hand if

$$\|u\|_A = \langle Au \mid u\rangle^{\frac{1}{2}} = \|A^{\frac{1}{2}}u\|,$$

the set of all $\|.\|_A$-bounded operators in $\mathcal{L}(\mathbb{K})$is

$$\mathcal{L}_{A^{\frac{1}{2}}}(\mathbb{K}) = \{ \ T \in \mathcal{L}(\mathbb{K}) : \ T^*R(A^{\frac{1}{2}}) \subset R(A^{\frac{1}{2}}) \ \} = \{ \ T \in \mathcal{L}(\mathbb{K}) : \ R(A^{\frac{1}{2}}T^*A^{\frac{1}{2}}) \subset R(A) \ \}.$$

Note that $\mathcal{L}_A(\mathbb{K}) \subset \mathcal{L}_{A^{\frac{1}{2}}}(\mathbb{K})$, which shows that if T admits an A-adjoint then it is A-bounded.

1.1. The A-adjoint operator $T^\sharp$

If $T \in \mathcal{L}(\mathbb{K})$admits an A-adjoint operator, i.e. if $R(T^*A) \subset R(A)$, then there exists a distinguished A-adjoint operator of T, namely, the reduced solution of the equation $AX = T^*A$, i.e. $T^\sharp = A^\dagger T^*A$, where $T^\dagger$ is the Moore–Penrose inverse of T and the A-adjoint operator $T^\sharp$ verifies

$$AT^\sharp = T^*A, \ R(T^\sharp) \subseteq \overline{R(A)} \text{ and } \ N(T^\sharp) = N(T^*A).$$

In the next we add without proof some properties of $T^\sharp$, (for more details we refer the reader to [8]).

Proposition 1.1. *Let $A \in \mathcal{L}(\mathbb{K})^+$ and $T \in \mathcal{L}_A(\mathbb{K})$. Then*

1. *$T^\sharp T$ and $TT^\sharp$are A-selfadjoint.*
2. $\|T\|_A = \|T^\sharp\|_A = \|T^\sharp T\|_A^{\frac{1}{2}}$.
3. $\|W\|_A = \|T^\sharp\|_A$*for every $W \in \mathcal{L}(\mathbb{K})$ which is an A-adjoint of T.*
4. *If $W \in \mathcal{L}_A(\mathbb{K})$ then $\|TW\|_A = \|WT\|_A$.*
5. $\|T^\sharp\| \leqslant \|W\|$*for every $W \in \mathcal{L}(\mathbb{K})$ which is an A-adjoint of T. Nevertheless, $T^\sharp$ is not in general the unique A-adjoint of T that realizes the minimal norm.*

We work with the next A-operators classes.

Definition 1.2. Let $T \in \mathcal{L}(\mathbb{K})$.

1. T is an A-isometry if $\|Tu\|_A = \|u\|_A$ for every $u \in \mathbb{K}$.
2. $T \in \mathcal{L}_A(\mathbb{K})$is an A-normal operator if $T^\sharp T = TT^\sharp$.

2. The class of *A-m*-isometric operators

Recall that for $m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathbb{K})$ is called an m-isometry if,

$$T^{*m}T^m - \binom{m}{1} T^{*m-1}T^{m-1} + \binom{m}{2} T^{*m-2}T^{m-2} + \ldots + (-1)^{m-1} \binom{m}{m-1} T^*T + (-1)^m I = 0.$$

Obviously, every m-isometry is injective. Moreover, T is an m-isometry if and only if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^{m-k}u\|^2 = 0, \quad \forall \ u \in \mathbb{K}.$$

In this section we will introduce and prove some basic properties of an A-m-isometry for which we give the following definition.

Definition 2.1. For $m \in \mathbb{N}$ and $A \in \mathcal{L}(\mathbb{K})^+$, an operator $T \in \mathcal{L}(\mathbb{K})$ is called an A-m-isometry if,

$$T^{*m}AT^m - \binom{m}{1} T^{*m-1}AT^{m-1} + \binom{m}{2} T^{*m-2}AT^{m-2} + \ldots + (-1)^{m-1}\binom{m}{m-1} T^*AT + (-1)^m A = 0.$$

or equivalently

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\|T^{m-k}u\|_A^2 = 0,\ \forall\, u \in \mathbb{K},$$

that is, T is an m-isometry relative to the semi-norm on $\mathbb{K}$ induced by A.

Remark 2.1

1. Let $T \in \mathcal{L}_A(\mathbb{K})$, a simple computation shows that the following statements are equivalent:
   (a) T is an A-m-isometry.
   (b) $\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{\sharp m-k}T^{m-k} = 0$.
2. An A-1-isometry is an A-isometry and if $A = I$ then an A-m-isometry is an m-isometry.
3. If T is an A-m-isometry then $N(T) \subset N(A)$. In particular if A is injective, then T is injective.

Example 2.1. The following examples of A-m-isometries can be easily proved.

1. The identity operator and the orthogonal projection on $\overline{R(A)}$ are A-m-isometries for all $m \geqslant 1$. Moreover, if T is an A-m-isometry then $\{T + S,\ R(S) \subset N(A)\ \}$ is a set of A-m-isometries.
2. Let $\mathbb{K} = \mathbb{C}^2$ be equipped with the norm $\|(x,y)\|^2 = |x|^2 + |y|^2$, and consider the operators $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in \mathcal{L}(\mathbb{K})^+$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{L}(\mathbb{K})$. It is easy to check that T admits A-adjoint operators and by direct computation, we see that

   $$\|T^3(x,y)\|_A^2 - 3\|T^2(x,y)\|_A^2 + 3\|T(x,y)\|_A^2 - \|(x,y)\|_A^2 = 0,$$

   and

   $$\|Tx\|_A^2 \neq \|x\|_A^2.$$

Thus, T is an A-3-isometry but is not an A-isometry. In general an A-m-isometry is not an m-isometry and vice versa, for example if $\mathbb{K} = \mathbb{C}^3$ equipped with the canonical norm, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, then T is a 2-isometry but it is not an A-isometry. On the other hand if $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then S is a B-isometry but it is not a 3-isometry.

Under some additional conditions, we obtain the following result.

Proposition 2.1. *Let A be a positive, injective bounded operator on $\mathbb{K}$ and $T \in \mathcal{L}(\mathbb{K})$. Then the following statements hold*:

1. *If $TA = AT$, then $T^\sharp = T^*$. In particular T is an A-m-isometry if and only if T is an m-isometry*.
2. *If T is A-normal, then T is an A-m-isometry if and only if $T^\sharp$ is an A-m-isometry*.

Proof

1. Note first that the conditions imposed on A and on T imply that $\overline{R(A)} = \mathbb{K}$ and that $T^*R(A) \subset R(A)$. So $T^\sharp$ exists. Moreover

   $$T^\sharp = A^\dagger T^* A = A^\dagger A T^* = P_{\overline{R(A)}} T^* = T^*.$$

   Thus the assertion follows.
2. Follows from the definition of an A-m-isometry and the Lemma 2.1, [7].
   More generally, as it happens with A-isometries, A-m-isometries can be characterized in terms of m-isometries. □

Proposition 2.2. *Let $T \in \mathcal{L}(\mathbb{K})$, then $T \in \mathcal{L}_{A^{\frac{1}{2}}}(\mathbb{K})$ if and only if there exists $S \in \mathcal{L}(\mathbb{K})$ such that $A^{\frac{1}{2}}S = T^*A^{\frac{1}{2}}$. In this case T is an A-m-isometry if and only if $S^*|_{\overline{R(A^{\frac{1}{2}})}}$ is an m-isometry.*

Proof. The first statement follows by applying Dauglas theorem. Now

$$A^{\frac{1}{2}}S = T^*A^{\frac{1}{2}} \iff S^*A^{\frac{1}{2}} = A^{\frac{1}{2}}T.$$

This implies that $\overline{R(A^{\frac{1}{2}})}$ is invariant for S^*. Moreover we have $S^{*k}A^{\frac{1}{2}} = A^{\frac{1}{2}}T^k$, for all positive integer k. Thus for all $u \in \mathbb{K}$,

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\|T^{m-k}u\|_A^2 = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\|A^{\frac{1}{2}}T^{m-k}u\|^2 = \sum_{k=0}^{m}(-1)^m\binom{m}{k}\|S^{*m-k}A^{\frac{1}{2}}u\|^2$$

from which we can conclude. Not that if in Proposition 2.2 the positive operator A is injective then S^* is an m-isometry. □

Proposition 2.3. *Let $T \in \mathcal{L}(\mathbb{K})$ be an A-isometry and $S \in \mathcal{L}(\mathbb{K})$ with $ST = TS$, then ST is an A-m-isometry if and only if S is also an A-m-isometry.*

Proof. Let $u \in \mathbb{K}$. The assumption T is A-isometry implies:

$$\|T^jS^ju\|_A = \|S^ju\|_A, \quad j = 0, 1, \ldots, m.$$

Using this and $ST = TS$, we obtain

$$\begin{aligned}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|(TS)^{m-j}u\|_A^2 &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\|T^{m-j}S^{m-j}u\|_A^2 \\ &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\|S^{m-j}u\|_A^2,\end{aligned}$$

which gives the desired result. □

Proposition 2.4. *Let $T \in \mathcal{L}(\mathbb{K})$ be an A-m-isometry, then for all $k \geqslant 0$, T is an A-$(m+k)$-isometry.*

Proof. It is enough to prove the result for $k = 1$; so we have

$$\begin{aligned}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\|T^{m+1-j}u\|_A^2 &= \|T^{m+1}u\|_A^2 + \sum_{j=1}^{m}(-1)^j\binom{m+1}{j}\|T^{m+1-j}u\|_A^2 - (-1)^m\|u\|_A^2 \\ &= \|T^{m+1}u\|_A^2 + \sum_{j=1}^{m}(-1)^j\left(\binom{m}{j} + \binom{m}{j-1}\right)\|T^{m+1-j}u\|_A^2 - (-1)^m\|u\|_A^2 \\ &= \|T^m(Tu)\|_A^2 + \sum_{j=1}^{m}(-1)^j\binom{m}{j}\|T^{m-j}Tu\|_A^2\end{aligned}$$

$$+\sum_{j=1}^{m}(-1)^j\binom{m}{j-1}\|T^{m+1-j}u\|_A^2-(-1)^m\|u\|_A^2$$
$$=0-\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|T^{m-j}u\|_A^2=0.$$

Hence T is an A-$(m+1)$-isometry. □

Proposition 2.5. *Let $T\in\mathcal{L}(\mathbb{K})$ be an invertible A-m-isometry, then T^{-1} is also an A-m-isometry.*

Proof. Since $\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|T^{m-j}u\|_A^2=0,\ \forall\ u\in\mathbb{K}$. By using the identity $\binom{m}{j}=\binom{m}{m-j},\ j=0,1,\ldots,m$, and replacing u by $(T^{-1})^m u$, we deduce that

$$0=\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|T^{m-j}((T^{-1})^m u)\|_A^2=\sum_{j=0}^{m}(-1)^j\binom{m}{m-j}\|T^{-j}u\|_A^2$$
$$=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|(T^{-1})^{m-k}u\|_A^2$$
$$=(-1)^m\sum_{k=0}^{m}(-1)^k\binom{m}{k}\|(T^{-1})^{m-k}u\|_A^2.$$

Hence, the result.

For $n,\ k=0,1,2,\ldots$, we denote

$$n^{(k)}=\begin{cases}1, & \text{if, } n=0 \text{ or } k=0\\ n(n-1)\ldots(n-k+1), & \text{otherwise}\end{cases}$$

For $T\in\mathcal{L}(\mathbb{K})$ and $k=0,1,2\ldots$, we consider the operator

$$\beta_k(T)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}T^{*j}AT^j.$$

Observe that if T is an A-m-isometry, then $\beta_k(T)=0$ for every $k\geqslant m$. The symbol $S_T(n)=T^{*n}AT^n$ of T, can be written

$$S_T(n)=\sum_{k=0}^{\infty}n^{(k)}\beta_k(T).$$

Hence if T is an A-m-isometry then

$$S_T(n)=\sum_{k=0}^{m-1}n^{(k)}\beta_k(T)$$

and consequently

$$\|T^n u\|_A^2=\sum_{k=0}^{m-1}n^{(k)}\langle\beta_k(T)u\mid u\rangle,\ \text{for all } u\in\mathbb{K}.$$

Moreover, the A-covariance operator Δ_T is defined by

$$\Delta_T:=\beta_{m-1}(T).$$

Note that the operator Δ_T depends on T, m and also on A. The next results have been proved in the special case $A=I$ by Agler and Stankus [1]. Here we present a generalization in more general context. □

Theorem 2.1. *Let $T \in \mathcal{L}(\mathbb{K})$. If T is an A-m-isometry, then the following properties hold.*

1. Δ_T *is positive and for all* $u \in \mathbb{K}$,

$$\langle \Delta_T u \mid u \rangle = \sum_{k=0}^{m-1} (-1)^{m-k-1} \frac{1}{k!(m-k-1)!} \|T^k u\|_A^2.$$

2. *The null space $N(\Delta_T)$ of Δ_T is an invariant subspace for T. Moreover, if $N(\Delta_T)$ is invariant for A and $A_0 = A|_{N(\Delta_T)}$, then the restriction operator $T|_{N(\Delta_T)}$ is an A_0-$(m-1)$-isometry.*
3. *If $\mathbb{M} \subset \mathbb{K}$ is an invariant subspace for T and A such that $T|_{\mathbb{M}}$ is an $A|_{\mathbb{M}}$-$(m-1)$-isometry, then $\mathbb{M} \subset N(\Delta_T)$.*

Proof

1. We have

$$\langle \Delta_T u \mid u \rangle = \lim_{n\to\infty} \frac{1}{n^{(m-1)}} \langle S_T(n)u \mid u \rangle = \lim_{n\to\infty} \frac{1}{n^{(m-1)}} \|T^n u\|_A^2 \geqslant 0.$$

2. A simple computation shows that if T is an A-m-isometry, then

$T^* \Delta_T T - \Delta_T = 0.$

Let $u \in N(\Delta_T)$,

$\langle \Delta_T T u \mid Tu \rangle = \langle T^* \Delta_T T u \mid u \rangle = \langle \Delta_T u \mid u \rangle = 0.$

The positivity of Δ_T implies that $\Delta_T Tu = 0$, and so Δ_T is invariant for T.
Furthermore, from 1., it follows that $T|_{N(\Delta_T)}$ is an A_0-$(m-1)$-isometry.
3. Let $u \in \mathbb{M}$, since $T|_{\mathbb{M}}$ is an $A|_{\mathbb{M}}$-$(m-1)$-isometry, then for $u \in \mathbb{M}$, the statement 1. implies that $\langle \Delta_T u \mid u \rangle_A = 0$. Hence one conclude that $u \in N(\Delta_T)$.

The concept of A-reduced minimum modulus of an operator is a natural generalization of the reduced minimum modulus: recall that the reduced minimum modulus of an operator $T \in \mathcal{L}(\mathbb{K})$ is defined as

$$\gamma(T) = \inf\{\|Tu\| : u \in N(T)^{\perp} \text{ and } \|u\| = 1\}. \quad \square$$

Definition 2.2. Let $T \in \mathcal{L}(\mathbb{K})$. The A-reduced minimum modulus of T is defined as

$$\gamma_A(T) = \inf\{\|Tu\|_A : u \in N(A^{\frac{1}{2}}T)^{\perp_A} \text{ and } \|u\|_A = 1\}$$

Note that

$$\|Tu\|_A \geqslant \gamma_A(T) d_A\left(u, N(A^{\frac{1}{2}}T)\right), u \in \mathbb{K}$$

where $d_A(u, V) = \inf\{\|u - v\|_A : v \in V\}, \quad V \subset \mathbb{K}$,
and if $T \in \mathcal{L}_A(\mathbb{K})$, then

$$\gamma_A(T) = \inf\{\|Tu\|_A : u \in \overline{R(T^\sharp T)} \text{ and } \|u\|_A = 1\}.$$

Proposition 2.6. *Let $T \in \mathcal{L}_A(\mathbb{K})$. Then*

$$\gamma_A(T^\sharp T) \geqslant \gamma_A(T)^2.$$

Proof. Let $u \in \mathbb{K}$, we have,

$$\|T^\sharp Tu\|_A \|u\|_A \geqslant \langle T^\sharp Tu \mid u\rangle_A = \|Tu\|_A^2 \geqslant \gamma_A(T)^2 \|u\|_A^2,$$

Hence,

$$\|T^\sharp Tu\|_A \geqslant \gamma_A(T)^2 \|u\|_A.$$

Since $T^\sharp T \in \mathcal{L}_A(\mathbb{K})$ and $R((T^\sharp T)^\sharp (T^\sharp T)) \subset R(T^\sharp T)$, we deduce that

$$\gamma_A(T^\sharp T) \geqslant \gamma_A(T)^2. \qquad \square$$

Proposition 2.7. *Let $T \in \mathcal{L}_A(\mathbb{K})$ be an A-m-isometry. Then*

1. *T is A-bounded below (i.e., $\|Tu\|_A \geqslant c\|u\|_A$, for some $c > 0$).*
2. $\|T\|_A \geqslant \gamma_A(T) \geqslant \dfrac{1}{\sqrt{m}\left(1+\|T\|_A^2\right)^{\frac{m-1}{2}}} > 0.$

Proof. Note first that if $T \in \mathcal{L}_A(\mathbb{K})$, then $N(A) \subset N(AT)$ and we have

$$\|Tu\|_A \leqslant \|T\|_A \|u\|_A, \forall u \in \mathbb{K}.$$

Since T is an A-m-isometry, then for $u \in \mathbb{K}$, it follows

1. $$\begin{aligned} \|u\|_A^2 &\leqslant \left(\|T^{m-1}\|_A^2 + \tbinom{m}{1}\|T^{m-2}\|_A^2 + \ldots + \tbinom{m}{m-1}\right)\|Tu\|_A^2 \\ &= C(m,T)\|Tu\|_A^2. \end{aligned}$$

2. $$\begin{aligned} \|u\|_A^2 &\leqslant \|T^m u\|_A^2 + \tbinom{m}{1}\|T^{m-1}u\|_A^2 + \ldots + \tbinom{m}{m-1}\|Tu\|_A^2 \\ &\leqslant \|T^{m-1}\|_A^2\|Tu\|_A^2 + \tbinom{m}{1}\|T^{m-2}\|_A^2\|Tu\|_A^2 + \ldots. + \tbinom{m}{m-1}\|T\|_A^2 \\ &\leqslant \left((\|T\|_A^2)^{m-1} + \tbinom{m}{1}(\|T\|_A^2)^{m-2} + \ldots + \tbinom{m}{m-1}\right)\|Tu\|_A^2 \\ &\leqslant m\sum_{j=0}^{m-1}\tbinom{m-1}{j}\left(\|T\|_A^2\right)^{m-1-j}\|Tu\|_A^2 \\ &\leqslant m\left(1+\|T\|_A^2\right)^{m-1}\|Tu\|_A^2. \end{aligned}$$

   We deduce that

   $$\gamma_A(T) \geqslant \frac{1}{\sqrt{m}\left(1+\|T\|_A^2\right)^{\frac{m-1}{2}}},$$

which is the desired result. $\square$

3. *A*-2-Isometric operators

Lemma 3.1. *Let $T \in \mathcal{L}(\mathbb{K})$ be an A-2-isometry then*

1. $\|Tu\|_A^2 \geqslant \frac{n-1}{n}\|u\|_A^2, \quad n \geqslant 1, \ u \in \mathbb{K}.$
2. $\|Tu\|_A \geqslant \|u\|_A, \quad u \in \mathbb{K}.$
3. $\|T^n u\|_A^2 + (n-1)\|u\|_A^2 = n\|Tu\|_A^2, \ u \in \mathbb{K}, \ n = 0, 1, 2, \ldots$
4. $\lim_{n\longrightarrow\infty} \|T^n u\|_A^{\frac{1}{n}} = 1, \ \text{for} \ \ u \in \mathbb{K}, u \neq 0.$

Proof. Using the fact that T is an A-2-isometry, we get

$$\|T^2u\|_A^2 - \|Tu\|_A^2 = \|Tu\|_A^2 - \|u\|_A^2.$$

Replacing u by T^ku leads to

$$\|T^{k+2}u\|_A^2 - \|T^{k+1}u\|_A^2 = \|T^{k+1}u\|_A^2 - \|T^ku\|_A^2,\ k \geqslant 0.$$

Hence,

$$\begin{aligned} 0 \leqslant \|T^nu\|_A^2 &= \sum_{k=1}^{n} \left(\|T^ku\|_A^2 - \|T^{k-1}u\|_A^2\right) + \|u\|_A^2 \\ &= n\left(\|Tu\|_A^2 - \|u\|_A^2\right) + \|u\|_A^2 \\ &= n\|Tu\|_A^2 + (1-n)\|u\|_A^2. \end{aligned}$$

Which implies 1. and 3. Letting $n \longrightarrow \infty$ in 1. yields to 2.

4. Take $u \in \mathbb{K}, u \neq 0$. It follows from 3. that

$$\limsup_{n\longrightarrow\infty} \|T^nu\|_A^{\frac{1}{n}} \leqslant 1.$$

However, according to 2., the sequence

$$(\|T^nu\|_A)_{n\in\mathbb{N}}$$

is monotonically increasing, so

$$\liminf_{n\longrightarrow\infty} \|T^nu\|_A^{\frac{1}{n}} \geqslant \lim_{n\longrightarrow\infty} \|u\|_A^{\frac{1}{n}} = 1,$$

which completes the proof. □

Remark 3.1. Assume that T is an A-2-isometry and using Lemma 3.1, we see that

$$\|T^{2n}u\|_A^2 = n\|T^{n+1}u\|_A^2 - n(n-1)\|Tu\|_A^2 + (n-1)^2\|u\|_A^2,\ n \geqslant 1,\ u \in \mathbb{K}.$$

Theorem 3.1. *Let $T, S \in \mathcal{L}(\mathbb{K})$ with $TS = ST$. If T is an A-2-isometry and S is an A-m-isometry, then the operator ST is an A-(m+1)-isometry,*

Proof. Let $u \in \mathbb{K}$, by using Lemma 3.1, one have

$$\begin{aligned} &\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \|(TS)^{m+1-j}u\|_A^2 \\ &= \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \|T^{m+1-j}S^{m+1-j}u\|_A^2 \\ &= \left\{ \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (m+1-j)\|TS^{m+1-j}u\|_A^2 - \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (m-j))\|(S^{m+1-j}u)\|_A^2 \right\} \\ &= (m+1) \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \|(S)^{m+1-j}Tu\|_A^2 - \sum_{j=1}^{m+1} (-1)^j \binom{m+1}{j} j\|S^{m+1-j}(Tu)\|_A^2 \\ &\quad -m \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \|S^{m+1-j}u\|_A^2 - (m+1) \sum_{j=0}^{m} (-1)^j \binom{m}{j} \|S^{m-j}u\|_A^2 = 0. \end{aligned}$$

Hence, the result. □

Theorem 3.2. *Let $T \in \mathcal{L}(\mathbb{K})$ be an A-2-isometry, then T^n is again a A-2-isometry for each $n \in \mathbb{N}$.*

Proof. Use Lemma 3.1, to write

$$\|T^{2n}u\|_A^2 = 2n\|Tu\|_A^2 - (2n-1)\|u\|_A^2$$

and

$$-2\|T^n u\|_A^2 = -2(n\|Tu\|_A^2 - (n-1)\|u\|_A^2).$$

Adding gives

$$\|T^{2n}u\|_A^2 - 2\|T^n u\|_A^2 + \|u\|_A^2 = 0$$

as required. □

Corollary 3.1. *Let $T, S \in \mathcal{L}(\mathbb{K})$ with $TS = ST$. If T is an A-2-isometry and S is an A-m-isometry, then the operator $T^k S$ is an A-(m+1)-isometry for all $k \geqslant 0$.*

Lemma 3.2. *Let $T \in \mathcal{L}(\mathbb{K})$. If T is an A-2-isometry, then*

$$T^{*p+1}AT^{p+1} - T^{*p}AT^p = \Delta_T, \ \forall \ p \geqslant 0.$$

In particular the sequence of operators

$$(T^{*p+1}AT^{p+1} - T^{*p}AT^p)_{p \geqslant 0}$$

is positive.

Proof. We prove the assertion by induction. Since T is an A-2-isometry the result is true for $p = 0$ and $p = 1$. Now assume that the result is true for p, i.e.,

$$(T^*)^p AT^p - (T^*)^{p-1}AT^{p-1} = \Delta_T.$$

Then

$$T^*\left((T^*)^p AT^p - (T^*)^{p-1}AT^{p-1}\right)T = T^*\Delta_T T = \Delta_T.$$

Hence, the result.

On the other hand, from Theorem 2.1, we have

$$\forall \ u \in \mathbb{K}, \quad \langle (T^*)^{k+1}AT^{k+1} - (T^*)^k AT^k)u \mid u\rangle = \langle \Delta_T u \mid u\rangle \geqslant 0. \quad \square$$

Lemma 3.3. *Let $T \in \mathcal{L}(\mathbb{K})$ be an A-2-isometry, then*

$$T^{*p}AT^p = pT^*AT - (p-1)A, \quad \forall \ p \geqslant 0.$$

Moreover if $T \in \mathcal{L}_A(\mathbb{K})$, we have

$$T^{\sharp p}T^p = pT^\sharp T - (p-1)I, \quad \forall \ p \geqslant 0.$$

Proof. We prove the assertion by induction. For $p = 0, 1$it is trivial. Since T is an A-2-isometry the result is true for $p = 2$. Now assume that the result is true for p, i.e.,

$$T^{*p}AT^p = pT^*AT - (p-1)A.$$

Then

$$\begin{aligned} T^{*p+1}AT^{p+1} &= T^*T^{*p}AT^P T \\ &= T^*(pT^*T - (p-1)A)T = pT^{*2}AT^2 - (p-1)T^*AT \\ &= p(2T^*AT - A) - (p-1)T^*AT \\ &= 2pT^*AT - pA - (p-1)T^*AT = (p+1)T^*AT - pA. \end{aligned}$$

This finishes the proof. □

Definition 3.1. Let $T \in \mathcal{L}(\mathbb{K})$. T is said to be:

1. A-bounded, if $\|T\|_A < \infty$.
2. A-power bounded, if $\sup_n \|T^n\|_A < \infty$.

The following result generalizes Theorem 2.4 proved in [11], for 2-isometries.

Theorem 3.3. *Let $T \in \mathcal{L}(\mathbb{K})$. If T is an A-power bounded A-2-isometry, then T is an A-isometry.*

Proof. Let T be an A-power bounded A-2-isometry. Then there exists a positive real number C such that

$$\|T^n\|_A \leqslant C \text{ for } \; n = 1, 2, 3, \ldots$$

The definition of an A-2-isometry yields

$$\|T^2\|_A^2 + 1 = 2\|T\|_A^2$$

Since T^n is also a A-2-isometry by Theorem 3.2, an induction argument shows that

$$\|T^{2^n}\|_A^2 = 2^n\|T\|_A^2 - (2^n - 1) \quad \text{for every positive integer} n.$$

Thus we obtain

$$\frac{C^2}{2^n} \geqslant \|T\|_A^2 - 1 + \frac{1}{2^n} \geqslant 0 :$$

Letting $n \longrightarrow \infty$, we find $\|T\|_A = 1$. In particular, $A \geqslant T^*AT$. Since from Theorem 2.1 $T^*AT \geqslant A$, by we conclude $T^*AT = A$. □

4. Spectral properties of *A*-m-isometry

We describe now the some spectral properties of an A-m-isometry.

Proposition 4.1. *Assume that $0 \notin \sigma_{ap}(A)$. If T is an A-m-isometry, then the approximate point spectrum of T lies in the unit circle. Thus, either $\sigma(T) \subset \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$. In particular, T is injective and $R(T)$ is closed.*

Proof. If $\lambda \in \mathbb{C}$ is in the approximate point spectrum of T, then there exists a sequence $(x_j) \subset \mathbb{K}$ such that for all j, $||x_j|| = 1$, and $(T - \lambda)x_j \to 0$ as $j \to \infty$ (see [3]). Thus for each integer k, $\lim_{j\to\infty}(T^k - \lambda^k)x_j \to 0$.

So

$$\begin{aligned}
0 &= \sum_{k=0}^{m}(-1)^k \binom{m}{k} \langle T^{*(m-k)}AT^{m-k}x_j, x_j\rangle \\
&= \sum_{k=0}^{m}(-1)^k \binom{m}{k} \langle AT^{m-k}x_j, T^{(m-k)}x_j\rangle \\
&= \sum_{k=0}^{m}(-1)^k \binom{m}{k} |\lambda|^{2(m-k)} \lim_{j\to+\infty}\langle Ax_j, x_j\rangle \\
&= (|\lambda|^2 - 1)^m \lim_{j\to+\infty}\langle Ax_j, x_j\rangle
\end{aligned}$$

and so $|\lambda| = 1$. Moreover $\partial\sigma(T) \subset \sigma_{ap}(T) \subset \partial\mathbb{D}$ and thus $\sigma(T) \subset \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$. In particular $0 \notin \sigma_{ap}(T)$, or T is bounded below that is equivalent to T is injective and it has a closed range. □

Corollary 4.1. *Assume that* $0 \notin \sigma_{ap}(A)$. *If both* T *and* T^* *are A-m-isometries, then* $\sigma(T) \subset \partial\mathbb{D}$.

Proof. We reason with contradiction. From the previous proposition, If $\sigma(T)\nsubseteq\partial\mathbb{D}$, then $\sigma(T) = \overline{\mathbb{D}}$. Since $0 \notin \sigma_{ap}(T)$, so $\overline{R(T)} = R(T) \neq \mathbb{K}$ and also $N(T^*) \neq \{0\}$. Hence $0 \in \sigma_p(T) \subset \sigma_{ap}(T)$, which contradicts the fact that T^* is an A-m-isometry. □

Remark 4.1. If $0 \notin \sigma_{ap}(A)$ and T is an A-m-isometry. Thus there two cases either T is invertible and $\sigma(T) \subset \partial\mathbb{D}$ or T is not invertible and $\sigma(T) = \overline{\mathbb{D}}$. Hence $r(T) = 1$ from [5].

In the rest of this article, we shall obtain additional spectral properties of 2-A-isometries.

Theorem 4.1. *Assume that* $0 \notin \sigma_{ap}(A)$. *If* T *is an A-2-isometry. Then*

1. $\lambda \in \sigma_{ap}(T)$ *implies* $\overline{\lambda} \in \sigma_{ap}(T^*)$.
2. $\lambda \in \sigma_p(T)$ *implies* $\overline{\lambda} \in \sigma_p(T^*)$.
3. *Eigenvectors of* T *corresponding to distinct eigenvalues are A-orthogonal.*

Proof

1. Let $\lambda \in \sigma_{ap}(T)$, Since $0 \notin \sigma_{ap}(A)$ we can choose a sequence $(x_n)_n$ of unit vectors satisfying $(T - \lambda I)x_n \to 0$ and such that $||Ax_n|| \geqslant \delta$ for some $\delta > 0$. Thus $T^*A(T - \lambda I)x_n \to 0$ and $T^{*2}A(T^2 - \lambda^2 I)x_n \to 0$. If T is an A-2-isometry then
$$\begin{aligned} 0 &= (T^{*2}AT^2 - 2T^*AT + A)x_n \\ &= (T^{*2}AT^2x_n - \lambda^2T^{*2}Ax_n) - (2T^*ATx_n - 2\lambda T^*Ax_n) + (\lambda^2T^{*2}x_n - 2\lambda T^*Ax_n + Ax_n). \end{aligned}$$
This implies that $(\lambda^2T^{*2}Ax_n - 2\lambda T^*Ax_n + Ax_n) = (\lambda T^* - I)^2Ax_n \to 0$. Since $\sigma_{ap}(T)$ is a subset of the unit circle $\partial\mathbb{D}$, this is equivalent to $(T^* - \overline{\lambda}I)^2Ax_n \to 0$ and we deduce that $(T^* - \overline{\lambda}I)^2\frac{Ax_n}{||Ax_n||} \to 0$. From which it follows that $\overline{\lambda} \in \sigma_{ap}(T^*)$.
2. We use a similar argument as in 1.
3. Let λ and μ be two distinct eigenvalues of T and suppose that $Tx = \lambda x$ and $Ty = \mu y$. Since T is an A-2-isomerty then $0 = \langle (T^{*2}AT^2 - 2T^*AT + A)x, y\rangle = \langle AT^2x, T^2y\rangle - 2\langle ATx, Ty\rangle + \langle Ax, y\rangle = (\lambda^2\overline{\mu}^2 - 2\lambda\overline{\mu} + 1)\langle Ax, y\rangle$. As $\lambda \neq \mu$ and $|\lambda| = |\mu| = 1$, we obtain $(\lambda^2\overline{\mu}^2 - 2\lambda\overline{\mu} + 1) = \left(\frac{\lambda}{\mu} - 1\right)^2 \neq 0$. This leads to $\langle Ax, y\rangle = 0$. which finishes the proof. The proof of Theorem 4.1 shows also that if $0 \notin \sigma_{ap}(A)$ and $T \in \mathcal{L}_A(\mathbb{K})$ is an A-2-isometry, then $\overline{\sigma_{ap}(T)} \subset \sigma_{ap}(T^\sharp)$ and $\overline{\sigma_p(T)} \subset \sigma_P(T^\sharp)$. Moreover our results improve most of those established in [11]. □

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