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Combinatorics of labelling in higher-dimensional automata

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ABSTRACT

The main idea for interpreting concurrent processes as labelled precubical sets is that a given set of n actions running concurrently must be assembled to a labelled n-cube, in exactly one way. The main ingredient is the non-functorial construction called the labelled directed coskeleton. It is defined as a subobject of the labelled coskeleton, the latter coinciding in the unlabelled case with the right adjoint to the truncation functor. This non-functorial construction is necessary since the labelled coskeleton functor of the category of labelled precubical sets does not fulfil the above requirement. We prove in this paper that it is possible to force the labelled coskeleton functor to be well behaved by working with labelled transverse symmetric precubical sets. Moreover, we prove that this solution is the only one. A transverse symmetric precubical set is a precubical set equipped with symmetry maps and with a new kind of degeneracy map called transverse degeneracy. Finally, we also prove that the two settings are equivalent from a directed algebraic topological viewpoint. To illustrate, a new semantics of the calculus of communicating systems (CCS), equivalent to the old one, is given.

1. Introduction

1.1. Presentation of the results

Directed algebraic topology is a field of research aiming at modelling time flows of concurrent processes and their properties by various algebraic topological models [13,17,20,24,28,14,23] (see [36] for other references). In this work, we are interested in concurrent processes arising from process algebras [38,31,6], and more precisely in the labelling process of these objects, which is related to combinatorics in a non-trivial way. By borrowing several ideas from [39,22] (see also [34,8,36,11,12]), with several slight modifications, [18] presented a semantics of process algebras in terms of *labelled precubical sets*. We consider in this paper only the case of Milner's calculus of communicating systems (CCS). The adaptation to other synchronization algebras and therefore to other process algebras is straightforward and is left to the reader.

The principle of this semantics is that the concurrent execution of n actions is abstracted by a full labelled n-cube. Each coordinate corresponds to one of the n actions, and therefore two opposite faces are labelled by the same action (e.g., Fig. 1 represents the concurrent execution of two actions a and b). The core of the construction of [18] is the non-functorial notion of a *labelled directed coskeleton*. It is applied to the fibered product of the 1-dimensional parts of two full labelled cubes representing two higher-dimensional transitions. This construction is the key ingredient to defining the parallel composition with synchronization of CCS in [18]. It is defined as a subobject of the labelled (1-dimensional) coskeleton. The latter coincides with the usual coskeleton, i.e. the right adjoint to the truncation functor, when the set of labels is a singleton. The labelled directed coskeleton construction $\overrightarrow{cosk}^{\Sigma}$ takes a particular kind of 1-dimensional labelled precubical

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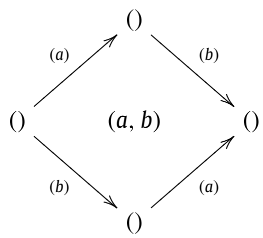


Fig. 1. Concurrent execution of a and b.

set K (the set of vertices K_0 must be a cube) to a higher-dimensional labelled precubical set $\overrightarrow{cosk}^{\Sigma}(K)$ such that each set of n actions running concurrently is assembled to an n-cube, in exactly one way. This role cannot be played by the full labelled 1-dimensional coskeleton functor $cosk_1^{\square,\Sigma}$ (see Proposition 2.3.4) of the category of labelled precubical sets since the latter may add several different n-cubes for the same set of n actions running concurrently.

The purpose of this paper is to introduce the notion of a *labelled transverse symmetric precubical set*. A transverse symmetric precubical set is a precubical set equipped with symmetry maps as in [21] and with a new kind of degeneracy map called transverse degeneracy. To the best of our knowledge, the latter maps seem to be new. In this new category of precubical sets, the labelled coskeleton functor is well behaved, as explained in Theorems 3.1.24 and 4.1.8. Indeed, the labelled transverse symmetric precubical set $\mathcal{L}(\overrightarrow{cosk}^{\Sigma}(K))$ freely generated by the labelled directed coskeleton $\overrightarrow{cosk}^{\Sigma}(K)$ of K is isomorphic to the labelled coskeleton functor $cosk_1^{\widehat{\square},\Sigma}(K)$ of the category of labelled transverse symmetric precubical sets applied to K if K is the 1-dimensional part of an n-cube or a fibered product over a synchronization algebra. Since the labelled transverse symmetric precubical set $\mathcal{L}(\overrightarrow{cosk}(K))$ and the labelled precubical set $\overrightarrow{cosk}(K)$ generate the same topological space of execution paths by Proposition 2.2.10 and Fig. 3, this result gives a functorial interpretation of the labelled directed coskeleton construction which is equivalent to the non-functorial construction from a directed algebraic topological point of view.

The labelled coskeleton functor in the category of labelled transverse symmetric precubical sets is therefore a categorical machinery allowing the understanding of the combinatorics of the labelling process in the parallel composition with synchronization of CCS. The advantage of this labelled coskeleton functor is twofold: (1) it is a functorial construction; (2) it is defined for *any* labelled 1-dimensional [transverse symmetric][1] precubical set, allowing future generalizations.

This enables us to give a semantics of CCS in terms of labelled transverse symmetric precubical sets which is equivalent to the one of [18] in terms of labelled precubical sets from a directed algebraic topological point of view: see Theorem 4.3.3 and Fig. 3.

1.2. Outline of the paper and reading guide

The paper is divided into three parts:

(1) Section 2.1, Section 2.2 and Section 2.3 generalize notions previously introduced in [18] to any category of cubes.
(2) Section 3.1 contains the mathematical treatment. A reader only interested in computer-scientific applications will only have to read the statement of Theorem 3.1.24.
(3) Section 4.1, Section 4.2 and Section 4.3 are the computer-scientific part of the paper.

The core of the paper is the categorical interpretation of the non-functorial labelled directed coskeleton construction using a generalization of the notion of a labelled precubical set. The notion of the category of cubes, and the generalized notion of labelled precubical sets are presented in Section 2.1. The main difficulty is the definition of the generalized precubical set of labels. Section 2.2 proves that all the notions of labelled precubical sets are equivalent from a directed algebraic topological point of view, in particular that they generate the same path space of execution paths. This section is the only topological one of the paper. Proposition 2.2.10 is only used in Theorem 4.3.3 to conclude that the two semantics of CCS generate the same spaces of execution paths. There is also a small application (Proposition 2.2.13) which is used inside the proof of Theorem 4.1.8. The topological material of Section 2.2 is not necessary for the proof of Proposition 2.2.13 but a pure combinatorial proof would be far more complicated. Section 2.3 generalizes the labelled coskeleton functor to all categories of precubical sets. It is defined as a right adjoint of a truncation functor, as in the setting of labelled precubical sets.

Section 3.1 is the mathematical core of the paper. It proves that all labelled coskeleton functors but one are defective. Indeed, the labelled coskeleton of the 1-dimensional part of the n-cube is never contractible in a directed algebraic topological sense, except for the unique *shell-complete* category of cubes, the maximal one containing all adjacency-preserving maps. This is the key property to obtaining a well-behaved labelled coskeleton functor (see Theorem 3.1.24). A presheaf over the unique shell-complete category of cubes is called a *transverse symmetric precubical set*.

[1] The words “transverse symmetric” can be omitted here by Propositions 2.1.19 and 2.1.28.

Section 4.1 is the first section of the computer-scientific part of the paper. It explains how one can use the preceding constructions to represent the parallel composition in CCS of an m-transition with an n-transition, modelled by a full labelled m-cube and a full labelled n-cube respectively. In other words, it studies parallel composition in the local case. It is shown that the definition of the fibered product in CCS must be slightly modified to allow the use of the labelled coskeleton functor of the category of labelled transverse symmetric precubical sets. Section 4.2 then studies parallel composition in CCS in the global case. It compares the two notions of synchronized tensor products in the category of labelled precubical sets and in that of labelled transverse symmetric ones. It is then proved in Section 4.3 that the two semantics of CCS in terms of labelled precubical sets and labelled transverse symmetric ones are equivalent from a directed algebraic topological point of view.

Finally, Section A.1 is an additional section treating the particular case of labelled symmetric precubical sets. This formalism will enable us to establish a link between concurrent processes viewed as precubical sets and Cattani–Sassone higher-dimensional transition systems in [19].

1.3. Prerequisites

The paper [18] contains an introduction to CCS for mathematicians which is sufficient to understand Sections 4.2 and 4.3 of this paper. Computer scientists might prefer [31,38]. For the rest of the paper, only general knowledge in category theory [32,33] is required, in particular in presheaf theory and in the theory of locally presentable categories [3]. A few model category techniques are used in Section 2.2. In fact, except for Section 2.2, the rest of the paper is purely combinatorial. Possible references for model categories are [10,26,25].

2. About labelled precubical sets over categories of cubes

2.1. Labelled precubical set over a category of cubes

We want to generalize the notion of labelled precubical set introduced in [18] by working on a category of cubes $\mathcal{A}$ (see Definition 2.1.7) instead on the reduced box category $\square$ (see Definition 2.1.2) as in [18]. The particular case $\mathcal{A} = \square$ will give back the notion of a labelled precubical set.

Category of cubes (definition and examples)

The category of partially ordered sets or posets together with the strictly increasing maps ($x < y$ implies $f(x) < f(y)$) is denoted by **PoSet**. It is worth noting that it is not the usual category of partially ordered sets since we restrict ourselves to strictly increasing maps. Let $[0] = \{()\}$ and $[n] = \{0, 1\}^n$ for $n \geqslant 1$. By convention, one has $\{0, 1\}^0 = [0] = \{()\}$. The set $[n]$ is equipped with the product ordering $\{0 < 1\}^n$: $(\epsilon_1, \dots, \epsilon_n) \leqslant (\epsilon'_1, \dots, \epsilon'_n)$ if and only if, for every $1 \leqslant i \leqslant n$, one has $\epsilon_i \leqslant \epsilon'_i$. The poset $[n]$ is also called the *n-cube*.

Definition 2.1.1. Let $\delta_i^\alpha : [n-1] \to [n]$ be the set map defined for $1 \leqslant i \leqslant n$ and $\alpha \in \{0, 1\}$ by $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$. These maps are called the *face maps*.

They satisfy the cocubical relations $\delta_j^\beta \delta_i^\alpha = \delta_i^\alpha \delta_{j-1}^\beta$ for $i < j$ and for all $(\alpha, \beta) \in \{0, 1\}^2$.

Definition 2.1.2. The *reduced box category*, denoted by $\square$, is the subcategory of **PoSet** with the set of objects $\{[n], n \geqslant 0\}$ and generated by the morphisms δ_i^α.

It is well known that the face maps together with the cocubical relations give a presentation by generators and relations of the small category $\square$ [21].

Proposition 2.1.3. *Let $n \geqslant 1$. Let $(\epsilon_1, \dots, \epsilon_n)$ and $(\epsilon'_1, \dots, \epsilon'_n)$ be two elements of the poset $[n]$ with $(\epsilon_1, \dots, \epsilon_n) \leqslant (\epsilon'_1, \dots, \epsilon'_n)$. Then there exist $i_1 > \dots > i_{n-r}$ and $\alpha_1, \dots, \alpha_{n-r} \in \{0, 1\}$ such that $(\epsilon_1, \dots, \epsilon_n) = \delta_{i_1}^{\alpha_1} \dots \delta_{i_{n-r}}^{\alpha_{n-r}}(0 \dots 0)$ and $(\epsilon'_1, \dots, \epsilon'_n) = \delta_{i_1}^{\alpha_1} \dots \delta_{i_{n-r}}^{\alpha_{n-r}}(1 \dots 1)$, where $r \geqslant 0$ is the number of 0s (resp. 1s) in the arguments $0 \dots 0$ (resp. $1 \dots 1$). In other words, $(\epsilon_1, \dots, \epsilon_n)$ is the bottom element and $(\epsilon'_1, \dots, \epsilon'_n)$ the top element of an r-dimensional subcube of $[n]$.*

Proof. The set $\{1, \dots, n\}$ is equal to the disjoint union

$$\{i \in \{1, \dots, n\}, \epsilon_i = \epsilon'_i\} \sqcup \{i \in \{1, \dots, n\}, \epsilon_i < \epsilon'_i\}.$$

In the latter case, one necessarily has $\epsilon_i = 0$ and $\epsilon'_i = 1$. $\square$

Definition 2.1.4. Let $n \geqslant 1$. Let $(\epsilon_1, \dots, \epsilon_n)$ and $(\epsilon'_1, \dots, \epsilon'_n)$ be two elements of the poset $[n]$. The integer r of Proposition 2.1.3 is called the *distance* between $(\epsilon_1, \dots, \epsilon_n)$ and $(\epsilon'_1, \dots, \epsilon'_n)$. Let us denote this situation by $r = d((\epsilon_1, \dots, \epsilon_n), (\epsilon'_1, \dots, \epsilon'_n))$. By definition, one has

$$r = \sum_{i=1}^{i=n} |\epsilon_i - \epsilon'_i|.$$

Definition 2.1.5. A set map $f : [m] \to [n]$ is *adjacency preserving* if it is strictly increasing and if $d((\epsilon_1, \dots, \epsilon_m), (\epsilon'_1, \dots, \epsilon'_m)) = 1$ implies $d(f(\epsilon_1, \dots, \epsilon_m), f(\epsilon'_1, \dots, \epsilon'_m)) = 1$.

An adjacency-preserving map does not necessarily preserve distance. For example, the map $\gamma_1 : [2] \to [2]$ defined by $\gamma_1(\epsilon_1, \epsilon_2) = (\max(\epsilon_1, \epsilon_2), \min(\epsilon_1, \epsilon_2))$ is adjacency preserving and not distance preserving because $\gamma_1(1, 0) = \gamma_1(0, 1) = (1, 0)$. We shall later see that γ_1 is an example of transverse degeneracy map (see Definition 3.1.11).

Proposition 2.1.6. *For any $n \geqslant 1$, the set map $\delta_i^\alpha : [n-1] \to [n]$ is adjacency preserving. Any strictly increasing map from $\{0 < 1\}^n$ to itself is adjacency preserving as well.*

Proof. That the set map $\delta_i^\alpha : [n-1] \to [n]$ is adjacency preserving is clear. Let f be a strictly increasing map from $\{0 < 1\}^n$ to itself. Let x and y be two elements of $\{0 < 1\}^n$ with $d(x, y) = 1$ and, for example, $x < y$. Then there exists a strictly increasing chain $(0, \ldots, 0) = x_0 < x_1 < \cdots < x_n = (1, \ldots, 1)$ of $\{0 < 1\}^n$ with $x = x_{i-1}$ and $y = x_i$ for some $i \geqslant 1$. Then $f(x_0) < f(x_1) < \cdots < f(x_n)$ is a strictly increasing chain of $\{0 < 1\}^n$. Therefore one has $f(x_0) = x_0$ and $f(x_n) = x_n$. It is easy to see that $n = d(f(x_0), f(x_n)) = \sum_{i=1}^{n} d(f(x_{i-1}), f(x_i))$. So for all $i \geqslant 1$, one has $d(f(x_{i-1}), f(x_i)) = 1$. Thus, f is adjacency preserving. □

Definition 2.1.7. A *category of cubes* $\mathcal{A}$ is a subcategory of **PoSet** such that:

- the set of objects is $\{[n], n \geqslant 0\}$,
- there is the inclusion $\square \subset \mathcal{A}$, and
- every morphism of $\mathcal{A}$ is adjacency preserving.

The minimal category of cubes for inclusion is the reduced box category $\square$.

Notation 2.1.8. *Let us denote by $\widehat{\square}$ the subcategory of* **PoSet** *containing all adjacency-preserving maps.*

The category $\widehat{\square}$ is the maximal category of cubes for inclusion. In other words, a small category $\mathcal{C}$ is a category of cubes if and only there are the inclusions $\square \subset \mathcal{C} \subset \widehat{\square}$.

Notation 2.1.9. *In what follows, $\mathcal{A}$ always denotes a category of cubes.*

Definition 2.1.10. [21] Let $\sigma_i : [n] \to [n]$ be the set map defined for $1 \leqslant i \leqslant n-1$ and $n \geqslant 2$ by $\sigma_i(\epsilon_1, \ldots, \epsilon_n) = (\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_i, \epsilon_{i+2}, \ldots, \epsilon_n)$. These maps are called the *symmetry maps*.

The symmetry maps are clearly adjacency preserving.

Notation 2.1.11. *Let us denote by $\square_S$ the smallest category of cubes containing the symmetry maps.*

We have the inclusions of categories of cubes $\square \subset \square_S \subset \widehat{\square}$.

Unlabelled $\mathcal{A}$-set

Definition 2.1.12. An *(unlabelled) $\mathcal{A}$-set* is a presheaf over $\mathcal{A}$. The corresponding category is denoted by $\mathcal{A}^{op}$**Set**.

Let K be an object of $\mathcal{A}^{op}$**Set**. The set $K([n])$ will be also denoted by K_n. A map $f : K \to L$ of $\mathcal{A}^{op}$**Set** will be also denoted by $(f_n)_{n \geqslant 0}$, where $f_n : K_n \to L_n$ is the corresponding set map. For any map $k : [m] \to [n]$ of $\mathcal{A}$ and any $\mathcal{A}$-set K, denote by $k^* : K_n \to K_m$ the set map induced by k.

Let $p \geqslant 0$. The *p-dimensional $\mathcal{A}$-cube or p-cube* $\mathcal{A}[p]$ is by definition the presheaf $\mathcal{A}(-, [p])$. In other words, $\mathcal{A}[p]_k$ is the set of maps from $[k]$ to $[p]$ in the category of cubes $\mathcal{A}$. The *boundary* $\partial\mathcal{A}[p]$ of the p-dimensional $\mathcal{A}$-cube is the presheaf defined by $\partial\mathcal{A}[p]_k = \mathcal{A}[p]_k$ if $k < p$ and $\partial\mathcal{A}[p]_p = \varnothing$ otherwise. In particular, the boundary of the 0-dimensional $\mathcal{A}$-cube is the empty presheaf.

Let $\mathcal{A}_n \subset \mathcal{A}$ be the full subcategory of $\mathcal{A}$ whose set of objects is $\{[k], k \leqslant n\}$. The category of presheaves over $\mathcal{A}_n$ is denoted by $\mathcal{A}_n^{op}$**Set**. Its objects are called the *n-dimensional $\mathcal{A}$-sets*. The category of n-dimensional $\mathcal{A}$-sets can be identified with the full subcategory of the category of $\mathcal{A}$-sets K such that $K_p = \varnothing$ for $p > n$.

Let K be an $\mathcal{A}$-set. Let $K_{\leqslant n}$ be the $\mathcal{A}$-set obtained from K by keeping the p-dimensional cubes of K only for $p \leqslant n$. In particular, $K_{\leqslant 0} = K_0$. Note that one has $\partial\mathcal{A}[n] = \mathcal{A}[n]_{\leqslant n-1}$ for every $n \geqslant 0$ since our precubical sets contain no degeneracy maps in the usual sense.

Definition 2.1.13. A $\square$-set is called a *precubical set* [5]. A $\square_S$-set is called a *symmetric precubical set* [21]. A $\widehat{\square}$-set is called a *transverse symmetric precubical set.*[2]

The inclusion functor $\square \subset \mathcal{A}$ induces a forgetful functor $\omega_{\mathcal{A}} : \mathcal{A}^{op}$**Set** $\to \square^{op}$**Set** which has both a left and a right adjoint obtained respectively as a left and a right Kan extension along the inclusion $\square^{op} \subset \mathcal{A}^{op}$. The right adjoint is denoted by $\mathcal{R}_{\mathcal{A}} : \square^{op}$**Set** $\to \mathcal{A}^{op}$**Set**. The left adjoint $\mathcal{L}_{\mathcal{A}} : \square^{op}$**Set** $\to \mathcal{A}^{op}$**Set** is of special interest since it formally adds all additional operators defining an $\mathcal{A}$-set. The two following propositions state some elementary remarks about $\mathcal{L}_{\mathcal{A}}$ which will be reused later.

Proposition 2.1.14. *Let K be a precubical set. Then one has the isomorphism*

$$\mathcal{L}_{\mathcal{A}}(K) \cong \varinjlim_{\square[n] \to K} \mathcal{A}[n].$$

In particular, there is the isomorphism of $\mathcal{A}$-sets $\mathcal{L}_{\mathcal{A}}(\square[n]) \cong \mathcal{A}[n]$.

2 Note that the last notion is new.

Proof. For every $\mathcal{A}$-set K, one has $K_n = (\omega_{\mathcal{A}} K)_n$ for all $n \geqslant 0$ since the inclusion functor $\square \subset \mathcal{A}$ is the identity on objects. So one has the bijections of sets

$$\mathcal{A}^{op}\mathbf{Set}(\mathcal{L}_{\mathcal{A}}(\square[n]), K) \cong \square^{op}\mathbf{Set}(\square[n], \omega_{\mathcal{A}} K) = (\omega_{\mathcal{A}} K)_n = K_n = \mathcal{A}^{op}\mathbf{Set}(\mathcal{A}[n], K).$$

By the Yoneda lemma, one obtains the isomorphism $\mathcal{L}_{\mathcal{A}}(\square[n]) \cong \mathcal{A}[n]$ for all $n \geqslant 0$. Since $\mathcal{L}_{\mathcal{A}}$ is a left adjoint, it preserves colimits. So one obtains for every precubical set K

$$\mathcal{L}_{\mathcal{A}}(K) = \mathcal{L}_{\mathcal{A}}\left(\varinjlim_{\square[n]\to K} \square[n]\right) \cong \varinjlim_{\square[n]\to K} \mathcal{A}[n]. \quad \square$$

Proposition 2.1.15. *Let K be a precubical set. The identity map $Id_{\mathcal{L}_{\mathcal{A}}(K)}$ induces by adjunction an inclusion of presheaves $i_K : K \subset \omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)$.*

Proof. Since the functor $K \mapsto K_p$ from precubical sets to sets is colimit preserving for every $p \geqslant 0$, one has the bijections

$$K_p \cong \varinjlim_{\square[n]\to K} \square[n]_p$$

and

$$\omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)_p \cong \varinjlim_{\square[n]\to K} \omega_{\mathcal{A}}\mathcal{A}[n]_p = \varinjlim_{\square[n]\to K} \mathcal{A}([p],[n]).$$

Each set map $\square[n]_p \to \omega_{\mathcal{A}}\mathcal{A}[n]_p$ is one-to-one because of the inclusions of sets $\square([p],[n]) \subset \mathcal{A}([p],[n])$ for every $p \geqslant 0$. For any map $g : [n] \to [n']$ of $\square$, one has the commutative diagram of sets

$$\begin{array}{ccc} \square[n]_p & \xrightarrow{\subset} & \square[n']_p \\ \downarrow{\subset} & & \downarrow{\subset} \\ \omega_{\mathcal{A}}\mathcal{A}[n]_p = \mathcal{A}([p],[n]) & \longrightarrow & \omega_{\mathcal{A}}\mathcal{A}[n']_p = \mathcal{A}([p],[n']). \end{array}$$

The bottom map is one-to-one since it consists of composing by g which is one-to-one as any map of $\square$. So each set map $\omega_{\mathcal{A}}\mathcal{A}[n]_p \to \omega_{\mathcal{A}}\mathcal{A}[n']_p$ of the diagram calculating $\omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)_p$ is one-to-one as well. One deduces that the map $K_p \to \omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)_p$ is one-to-one. $\square$

Notation 2.1.16. *Since $K \subset \omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)$ is an inclusion, $i_K(y)$ will be simply denoted by y for any $y \in K$.*

The 1*-dimensional case*

This paragraph proves that the 1-dimensional case does not depend on the choice of the category of cubes. The crucial facts are that a category of cubes contains all face maps and that all morphisms are adjacency preserving.

Proposition 2.1.17. *For every $m > n$, one has $\mathcal{A}([m],[n]) = \varnothing$. For every $n \geqslant 0$, the inclusion $\square \subset \mathcal{A}$ implies the bijections $[n] \cong \square([0],[n]) \cong \mathcal{A}([0],[n])$ and $\square([1],[n]) \cong \mathcal{A}([1],[n])$.*

Note that this implies that $\mathcal{A}$ cannot have any degeneracies.

Proof. It is clear that $\mathcal{A}([m],[n]) \neq \varnothing$ implies $m \leqslant n$. One has the inclusions

$$\square([0],[n]) \subset \mathcal{A}([0],[n]) \subset \widehat{\square}([0],[n]) = \square([0],[n]) = \{\delta_n^{\epsilon_n} \ldots \delta_1^{\epsilon_1}, (\epsilon_1, \ldots, \epsilon_n) \in [n]\};$$

hence the second assertion. For every $n \geqslant 0$, the inclusion $\square([1],[n]) \subset \mathcal{A}([1],[n])$ is a bijection since every map of $\mathcal{A}$ is adjacency preserving by definition of a category of cubes; hence the third assertion. $\square$

Proposition 2.1.18. *Let K be a precubical set. Then the inclusion of precubical sets $K \subset \omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)$ induces the isomorphism of* 1*-dimensional precubical sets $K_{\leqslant 1} \cong \omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)_{\leqslant 1}$.*

Proof. We already know by Proposition 2.1.17 that, for every $n \geqslant 0$, the inclusions $\square([0],[n]) \subset \mathcal{A}([0],[n])$ and $\square([1],[n]) \subset \mathcal{A}([1],[n])$ are bijective. So the inclusion of presheaves $\square[n]_{\leqslant 1} \subset \omega_{\mathcal{A}}\mathcal{A}[n]_{\leqslant 1}$ is an isomorphism for every $n \geqslant 0$. Since the forgetful functor $\omega_{\mathcal{A}} : \mathcal{A}^{op}\mathbf{Set} \to \square^{op}\mathbf{Set}$ is a left adjoint, it is colimit preserving. Hence the proof is complete. $\square$

Proposition 2.1.19. *The category of* 1*-dimensional precubical sets is equivalent to the category of* 1*-dimensional $\mathcal{A}$-sets.*

Proof. The adjunction $\mathcal{L}_{\mathcal{A}} : \square^{op}\mathbf{Set} \leftrightarrows \mathcal{A}^{op}\mathbf{Set} : \omega_{\mathcal{A}}$ induces an adjunction $(\mathcal{L}_{\mathcal{A}})_{\leqslant 1} : \square_1^{op}\mathbf{Set} \leftrightarrows \mathcal{A}_1^{op}\mathbf{Set} : (\omega_{\mathcal{A}})_{\leqslant 1}$ by Proposition 2.1.14. We already know by Proposition 2.1.18 that there is the isomorphism $(\omega_{\mathcal{A}})_{\leqslant 1}(\mathcal{L}_{\mathcal{A}})_{\leqslant 1} \cong Id_{\square_1^{op}\mathbf{Set}}$. One has $(\mathcal{L}_{\mathcal{A}})_{\leqslant 1}(\omega_{\mathcal{A}})_{\leqslant 1}(\mathcal{A}[0]) \cong \mathcal{A}[0]$ by Proposition 2.1.14 and $(\mathcal{L}_{\mathcal{A}})_{\leqslant 1}(\omega_{\mathcal{A}})_{\leqslant 1}(\mathcal{A}[1]) \cong \mathcal{A}[1]$ by Proposition 2.1.17 and Proposition 2.1.14. Hence the isomorphism of functors $(\mathcal{L}_{\mathcal{A}})_{\leqslant 1}(\omega_{\mathcal{A}})_{\leqslant 1} \cong Id_{\mathcal{A}_1^{op}\mathbf{Set}}$. $\square$

Labelled $\mathcal{A}$-set

We fix a non-empty set Σ of *labels* or of *actions*. It always contains a distinguished label denoted by τ. We want to label the cubes of a $\mathcal{A}$-set with the elements of Σ. A labelled $\mathcal{A}$-set will be a map of $\mathcal{A}$-sets $K \to L$, where L is the $\mathcal{A}$-set of labels. Let us start by recalling the construction of the precubical set of labels.

Proposition 2.1.20 (*Variant of Goubault's Construction [22]*). *Let*

- $(!\Sigma)_0 = \{()\}$ *(the empty word),*
- *for* $n \geqslant 1$, $(!\Sigma)_n = \Sigma^n$, *and*
- $\partial_i^0(a_1, \ldots, a_n) = \partial_i^1(a_1, \ldots, a_n) = (a_1, \ldots, \widehat{a_i}, \ldots, a_n)$, *where the notation* $\widehat{a_i}$ *means that* a_i *is removed.*

Then these data generate a precubical set denoted by $!\Sigma$.

Definition 2.1.21. Let K be an $\mathcal{A}$-set. Let $x \in K_p$ with $p \geqslant 1$. The *boundary of* x is the composite map $\partial x : \partial\mathcal{A}[p] \subset \mathcal{A}[p] \xrightarrow{x} K$.

The main feature of the precubical set $!\Sigma$ is that, for every $p \geqslant 2$, a p-cube of $!\Sigma$, which labels the concurrent execution of p actions like in Fig. 1, is determined by its boundary. In other words, a commutative square of precubical sets of the form

$$\begin{array}{ccc} \partial\square[p] & \longrightarrow & !\Sigma \\ {\scriptstyle\subset}\downarrow & \overset{k}{\nearrow} & \downarrow \\ \square[p] & \longrightarrow & \mathbf{1} \end{array}$$

with $p \geqslant 2$, where $\mathbf{1}$ is the terminal precubical set, admits at most one lift k. An equivalent mathematical formulation of the preceding condition is that for every commutative square of precubical sets of the form

$$\begin{array}{ccc} \partial\square[p] & \xrightarrow{\subset} & \square[p] \\ {\scriptstyle\subset}\downarrow & & \downarrow{\scriptstyle f} \\ \square[p] & \xrightarrow{g} & !\Sigma, \end{array}$$

one has $f = g$. So every commutative square of precubical sets of the form

$$\begin{array}{ccc} \square[p] \sqcup_{\partial\square[p]} \square[p] & \xrightarrow{(f,g)} & !\Sigma \\ \downarrow & \overset{k}{\nearrow} & \downarrow \\ \square[p] & \longrightarrow & \mathbf{1} \end{array}$$

with $p \geqslant 2$ admits exactly one lift $k = f = g$. In other words, the precubical set $!\Sigma$ turns out to be orthogonal to the set of maps $\{\square[p] \sqcup_{\partial\square[p]} \square[p] \to \square[p], p \geqslant 2\}$ in the sense of [3, Definition 1.32].

Because of the inclusion $K \subset \omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)$ for every precubical set K, we need more cubes for the $\mathcal{A}$-set of labels as soon as the inclusion $\square \subset \mathcal{A}$ is strict. Indeed, we must be able to label all cubes of $\mathcal{L}_{\mathcal{A}}(K)$ for every labelled precubical set $K \to !\Sigma$. The first candidate for the $\mathcal{A}$-set of labels is then the $\mathcal{A}$-set $\mathcal{L}_{\mathcal{A}}(!\Sigma)$ freely generated by $!\Sigma$. However, it is not well behaved. Consider the two set involutions $\sigma_1 : [2] \to [2]$ and $Id_{[2]} : [2] \to [2]$. Let us suppose that $\sigma_1 \in \mathcal{A}$. Then the two 2-cubes $(\sigma_1)^*(\tau, \tau)$ and (τ, τ) of $\mathcal{L}_{\mathcal{A}}(!\Sigma)$ have the same boundary. This means that the commutative square of $\mathcal{A}$-sets

$$\begin{array}{ccc} \partial\mathcal{A}[p] & \xrightarrow{\partial(\sigma_1)^*(\tau,\tau) = \partial(\tau,\tau)} & \mathcal{L}_{\mathcal{A}}(!\Sigma) \\ {\scriptstyle\subset}\downarrow & \overset{k}{\nearrow} & \downarrow \\ \mathcal{A}[p] & \longrightarrow & \mathbf{1} \end{array}$$

has two distinct lifts $k = (\sigma_1)^*(\tau,\tau)$ and $k = (\tau,\tau)$. In other words, the $\mathcal{A}$-set $\mathcal{L}_\mathcal{A}(!\Sigma)$ is never orthogonal to the set of morphisms $\{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}$ as soon as $\sigma_1 : [2] \to [2]$ belongs to $\mathcal{A}$. In fact, the $\mathcal{A}$-set $\mathcal{L}_\mathcal{A}(!\{\tau\})$ is even not the terminal $\mathcal{A}$-set in this case. Yet, the notion of $\mathcal{A}$-set must coincide with the unlabelled notion if the set of labels is equal to $\{\tau\}$. The full subcategory $\{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}^\perp$ of $\mathcal{A}$-sets orthogonal to the set of maps $\{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}$ is a full reflective subcategory of the locally presentable category of $\mathcal{A}$-sets by [3, Theorem 1.39]. Let

$$\boxed{Sh_\mathcal{A} : \mathcal{A}^{op}\mathbf{Set} \to \{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}^\perp}$$

be the left adjoint to the inclusion functor $\{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}^\perp \subset \mathcal{A}^{op}\mathbf{Set}$.

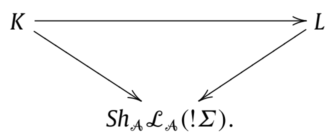
Definition 2.1.22. The *$\mathcal{A}$-set of labels* is the $\mathcal{A}$-set $Sh_\mathcal{A}\mathcal{L}_\mathcal{A}(!\Sigma)$.

In $Sh_\mathcal{A}\mathcal{L}_\mathcal{A}(!\Sigma)$, the two 2-cubes $(\sigma_1)^*(\tau,\tau)$ and (τ,τ) are forced to be equal. Note that there is the isomorphism of precubical sets $Sh_\square\mathcal{L}_\square(!\Sigma) \cong !\Sigma$.

Definition 2.1.23. A *labelled $\mathcal{A}$-set (over Σ)* is an object of the comma category

$$\mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_\mathcal{A}\mathcal{L}_\mathcal{A}(!\Sigma).$$

That is, an object is a map of $\mathcal{A}$-sets $\ell : K \to Sh_\mathcal{A}\mathcal{L}_\mathcal{A}(!\Sigma)$ and a morphism is a commutative diagram



The ℓ map is called the *labelling map*. The $\mathcal{A}$-set K is sometimes called the *underlying $\mathcal{A}$-set* of the labelled $\mathcal{A}$-set.

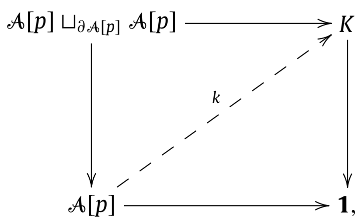
The functor $\mathcal{L}_\mathcal{A} : \square^{op}\mathbf{Set} \to \mathcal{A}^{op}\mathbf{Set}$ induces a functor (denoted in the same way)

$$\mathcal{L}_\mathcal{A} : \square^{op}\mathbf{Set}{\downarrow}!\Sigma \to \mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_\mathcal{A}\mathcal{L}_\mathcal{A}(!\Sigma)$$

which takes $\ell : K \to !\Sigma$ to the composite $\mathcal{L}_\mathcal{A}(K) \xrightarrow{\mathcal{L}_\mathcal{A}(\ell)} \mathcal{L}_\mathcal{A}(!\Sigma) \to Sh_\mathcal{A}\mathcal{L}_\mathcal{A}(!\Sigma)$.

Proposition 2.1.24. *Let K be an $\mathcal{A}$-set. Then the map of $\mathcal{A}$-sets $K \to Sh_\mathcal{A}(K)$ induces the isomorphism of 1-dimensional $\mathcal{A}$-sets $K_{\leqslant 1} \cong Sh_\mathcal{A}(K)_{\leqslant 1}$.*

Proof. For every $p \geqslant 2$ and for every commutative diagram of solid arrows



there exists at most one lift k. So an $\mathcal{A}$-set K is orthogonal to the set of morphisms $\{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}$ if and only if the canonical map $K \to \mathbf{1}$ satisfies the right lifting property with respect to the same set of morphisms. So the $\mathcal{A}$-set $Sh_\mathcal{A}(K)$ can be obtained by a small object argument by factoring the map $K \to \mathbf{1}$ as a composite $K \to Sh_\mathcal{A}(K) \to \mathbf{1}$, where $K \to Sh_\mathcal{A}(K)$ is a relative $\{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}$-cell complex and where the map $Sh_\mathcal{A}(K) \to \mathbf{1}$ satisfies the right lifting property with respect to the same set of morphisms. The small object argument is possible by [4, Proposition 1.3] since the category of $\mathcal{A}$-sets is locally presentable, as every presheaf category. Since, for every $p \geqslant 2$, the map of $\mathcal{A}$-sets $\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p]$ induces an isomorphism

$$\left(\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p]\right)_{\leqslant 1} \cong \mathcal{A}[p]_{\leqslant 1},$$

one deduces that the canonical map $K \to Sh_\mathcal{A}(K)$ induces an isomorphism $K_{\leqslant 1} \cong Sh_\mathcal{A}(K)_{\leqslant 1}$. □

Proposition 2.1.25. *There is the isomorphism $Sh_\mathcal{A}\mathcal{L}_\mathcal{A}(!\{\tau\}) \cong \mathbf{1}$. Therefore, when $\Sigma = \{\tau\}$, the category of labelled $\mathcal{A}$-sets is equivalent to the category of unlabelled $\mathcal{A}$-sets.*

Proof. Indeed, both the functors $Sh_\mathcal{A}$ and $\mathcal{L}_\mathcal{A}$ do not modify the set of 0-cubes and the set of 1-cubes by Proposition 2.1.18 and Proposition 2.1.24. Moreover, for any $\mathcal{A}$-set K such that K_1 is a singleton, it is clear by induction on $p \geqslant 1$ that the set $(Sh_\mathcal{A}K)_p$ is a singleton. So $Sh_\mathcal{A}\mathcal{L}_\mathcal{A}(!\{\tau\}) \cong \mathbf{1}$ (the terminal object of $\mathcal{A}^{op}\mathbf{Set}$). □

Notation 2.1.26. *Let $(a_1, \ldots, a_n) \in \Sigma^n$ with $n \geqslant 1$. The labelled precubical set $\square[a_1, \ldots, a_n]$ denotes the map $\ell : \square[n] \to !\Sigma$ such that $\ell(Id_{[n]}) = (a_1, \ldots, a_n)$.*

Fig. 1 gives the example of the labelled 2-cube $\square[a, b]$. It represents the concurrent execution of a and b. It is important to notice that two opposite faces of Fig. 1 have the same label.

Notation 2.1.27. *Let $(a_1, \ldots, a_n) \in \Sigma^n$ with $n \geqslant 1$. The labelled $\mathcal{A}$-set $\mathcal{A}[a_1, \ldots, a_n]$ denotes the labelled $\mathcal{A}$-set $\mathcal{L}_{\mathcal{A}}(\square[a_1, \ldots, a_n])$.*

Proposition 2.1.28. *The category of labelled 1-dimensional precubical sets is equivalent to the category of labelled 1-dimensional $\mathcal{A}$-sets.*

Proof. This is a consequence of Proposition 2.1.19 and Proposition 2.1.24. □

2.2. Geometric realization of a labelled $\mathcal{A}$-set

The purpose of this section is to prove that the geometric realization functor $\square^{op}\mathbf{Set}{\downarrow}!\Sigma \to \mathbf{Flow}{\downarrow}?\Sigma$ of [18] which takes a labelled precubical set to the corresponding labelled flow factors as a composite $\square^{op}\mathbf{Set}{\downarrow}!\Sigma \to \mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma) \to \mathbf{Flow}{\downarrow}?\Sigma$, where the left-hand functor is induced by $\mathcal{L}_{\mathcal{A}}$. This result ensures that all the notions of labelled $\mathcal{A}$-sets are equivalent from a directed algebraic topological point of view. The results of this section are used only in Theorem 4.3.3 and in Theorem 4.1.8.

Unlabelled flow

The category **Top** of *compactly generated topological spaces* (i.e. of weak Hausdorff k-spaces) is complete, cocomplete and cartesian closed (more details for these kinds of topological spaces are in [7,30], the Appendix of [29] and also in the preliminaries of [17]). In what follows, all topological spaces will be supposed to be compactly generated. A *compact space* is always Hausdorff.

Definition 2.2.1 (*[17]*). A *(time) flow* X is a small topological category without identity maps. The set of objects is denoted by X^0. The topological space of morphisms from α to β is denoted by $\mathbb{P}_{\alpha,\beta}X$. The elements of X^0 are also called the *states* of X. The elements of $\mathbb{P}_{\alpha,\beta}X$ are called the *(non-constant) execution paths from α to β*. A flow X is *loopless* if, for every $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.

Notation 2.2.2. *Let $\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}_{\alpha,\beta}X$. The topological space $\mathbb{P}X$ is called the path space of X. The source map (resp. the target map) $\mathbb{P}X \to X^0$ is denoted by s (resp. t).*

Definition 2.2.3. Let X be a flow, and let $\alpha \in X^0$ be a state of X. The state α is *initial* if $\alpha \notin t(\mathbb{P}X)$, and the state α is *final* if $\alpha \notin s(\mathbb{P}X)$.

Definition 2.2.4. A morphism of flows $f : X \to Y$ consists in a set map $f^0 : X^0 \to Y^0$ and a continuous map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ such that $s(\mathbb{P}f(x)) = f^0(s(x))$, $t(\mathbb{P}f(x)) = f^0(t(x))$ and $\mathbb{P}f(x * y) = \mathbb{P}f(x) * \mathbb{P}f(y)$ for every $x, y \in \mathbb{P}X$. The corresponding category is denoted by **Flow**.

The strictly associative composition law

$$\begin{cases} \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X \\ (x, y) \mapsto x * y \end{cases}$$

models the composition of non-constant execution paths. The composition law $*$ is extended in a usual way to states, that is to constant execution paths, by $x * t(x) = x$ and $s(x) * x = x$ for every non-constant execution path x.

Here are two fundamental examples of flows:

(1) Let S be a set. The flow associated with S, still denoted by S, has S as a set of states and the empty space as path space. This construction induces a functor **Set** $\to$ **Flow** from the category of sets to that of flows. The flow associated with a set is loopless.
(2) Let $(P, \leqslant)$ be a poset. The flow associated with $(P, \leqslant)$, and still denoted by P is defined as follows: the set of states of P is the underlying set of P; the space of morphisms from α to β is empty if $\alpha \geqslant \beta$ and equal to $\{(\alpha, \beta)\}$ if $\alpha < \beta$ and the composition law is defined by $(\alpha, \beta) * (\beta, \gamma) = (\alpha, \gamma)$. This construction induces a functor **PoSet** $\to$ **Flow** from the category of posets together with the strictly increasing maps to the category of flows. The flow associated with a poset is loopless as well.[3]

There is an important model structure on **Flow** which is characterized as follows [17]:

3 And must be loopless! This is one of the reasons for working with small categories without identity maps.

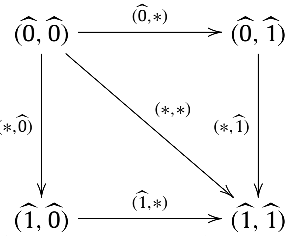


Fig. 2. The flow $\{\widehat{0} < \widehat{1}\}^2$ (Note that $(*, *) = (\widehat{0}, *) * (*, \widehat{1}) = (*, \widehat{0}) * (\widehat{1}, *)$).

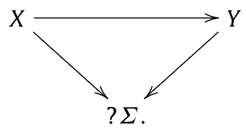
- The weak equivalences are the *weak S-homotopy equivalences*, i.e. the morphisms of flows $f : X \longrightarrow Y$ such that $f^0 : X^0 \longrightarrow Y^0$ is a bijection of sets and such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence.
- The fibrations are the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration.[4]

This model structure is cofibrantly generated. The cofibrant replacement functor is denoted by $(-)^{cof}$.

Labelled flow

Definition 2.2.5 (*[18]*). The *flow of labels* $?\Sigma$ is defined as follows: $(?\Sigma)^0 = \{0\}$ and $\mathbb{P}?\Sigma$ is the discrete free commutative semigroup generated by the elements of Σ.

Definition 2.2.6 (*[18]*). A *labelled flow* is an object of the comma category **Flow**$\downarrow ?\Sigma$. That is an object is a map of flows $\ell : X \to ?\Sigma$ and a morphism is a commutative diagram



The ℓ map is called the *labelling map*. The flow X is sometimes called the *underlying flow* of the labelled flow.

Geometric realization of a labelled precubical set

A state of the flow associated with the poset $\{\widehat{0} < \widehat{1}\}^n$ (i.e. the product of n copies of $\{\widehat{0} < \widehat{1}\}$) is denoted by an n-tuple of elements of $\{\widehat{0}, \widehat{1}\}$. By convention, $\{\widehat{0} < \widehat{1}\}^0 = \{()\}$. The unique morphism/execution path from $(x_1, \dots, x_n)$ to $(y_1, \dots, y_n)$ is denoted by an n-tuple $(z_1, \dots, z_n)$ of $\{\widehat{0}, \widehat{1}, *\}$ with $z_i = x_i$ if $x_i = y_i$ and $z_i = *$ if $x_i < y_i$. For example, in the flow $\{\widehat{0} < \widehat{1}\}^2$ (see Fig. 2), one has the algebraic relation $(*, *) = (\widehat{0}, *) * (*, \widehat{1}) = (*, \widehat{0}) * (\widehat{1}, *)$.

Let $\square \to$ **PoSet** $\subset$ **Flow** be the functor defined on objects by the mapping $[n] \mapsto \{\widehat{0} < \widehat{1}\}^n$ and on morphisms by the mapping

$$\delta_i^\alpha \mapsto ((\epsilon_1, \dots, \epsilon_{n-1}) \mapsto (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})),$$

where the ϵ_i are elements of $\{\widehat{0}, \widehat{1}, *\}$. The functor $[n] \mapsto \{\widehat{0} < \widehat{1}\}^n$ from $\square$ to **Flow** induces a bad realization functor from $\square^{op}$**Set** to **Flow** defined by

$$\boxed{|K|_{bad} := \varinjlim_{\square[n]\to K} \{\widehat{0} < \widehat{1}\}^n}.$$

Theorem 2.2.7 (*[18, Theorem 7.1] and [18, Proposition 8.1]*). *For all $n \geqslant 3$, the inclusion $\partial\square[n] \subset \square[n]$ induces an isomorphism of flows $|\partial\square[n]|_{bad} \cong |\square[n]|_{bad}$. One has the isomorphism of flows $|!\Sigma|_{bad} \cong ?\Sigma$.*

Definition 2.2.8 (*[18]*). Let K be a precubical set. By definition, the *geometric realization* of K is the flow

$$\boxed{|K| := \varinjlim_{\square[n]\to K} (\{\widehat{0} < \widehat{1}\}^n)^{cof}}.$$

The natural trivial fibrations $(\{\widehat{0} < \widehat{1}\}^n)^{cof} \longrightarrow \{\widehat{0} < \widehat{1}\}^n$ for $n \geqslant 0$ induce a natural map $|K| \longrightarrow |K|_{bad}$ for any precubical set K. Let $K \to !\Sigma$ be a labelled precubical set. Then the composition $|K| \to |!\Sigma| \to |!\Sigma|_{bad} \cong ?\Sigma$ gives rise to a labelled flow.

[4] That is, a continuous map having the right lifting property with respect to the inclusion $\mathbf{D}^n \times 0 \subset \mathbf{D}^n \times [0, 1]$ for any $n \geqslant 0$, where $\mathbf{D}^n$ is the n-dimensional disk.

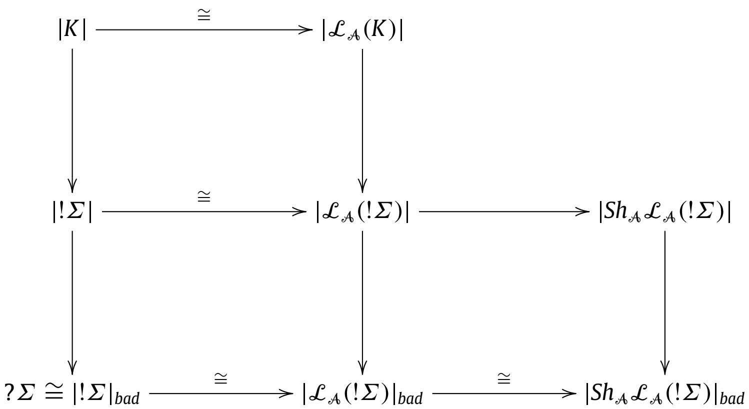


Fig. 3. Labelled precubical sets and labelled $\mathcal{A}$-sets equivalent from the directed algebraic topological point of view.

Geometric realization of a labelled $\mathcal{A}$-set

Let $\mathcal{A} \to \mathbf{PoSet} \subset \mathbf{Flow}$ be the functor defined on objects by the mapping $[n] \mapsto \{\widehat{0} < \widehat{1}\}^n$ and on morphisms as follows. Let $f : [m] \to [n]$ be a map of $\mathcal{A}$ with $m, n \geqslant 0$. Let $(\epsilon_1, \ldots, \epsilon_m) \in \{\widehat{0}, \widehat{1}, *\}^m$ be an r-cube. Since f is adjacency preserving, the two elements $f(s(\epsilon_1, \ldots, \epsilon_m))$ and $f(t(\epsilon_1, \ldots, \epsilon_m))$ are respectively the initial and final states of a unique r-dimensional subcube denoted by $f(\epsilon_1, \ldots, \epsilon_m)$ of $[n]$ with $f(\epsilon_1, \ldots, \epsilon_m) \in \{\widehat{0}, \widehat{1}, *\}^n$. Note that the composite functor $\square \subset \mathcal{A} \to \mathbf{PoSet} \subset \mathbf{Flow}$ is the functor defined above. The functor $[n] \mapsto \{\widehat{0} < \widehat{1}\}^n$ from $\mathcal{A}$ to $\mathbf{Flow}$ induces a bad realization functor from $\mathcal{A}^{op}\mathbf{Set}$ to $\mathbf{Flow}$ defined by

$$\boxed{|K|_{bad} := \varinjlim_{\mathcal{A}[n]\to K} \{\widehat{0} < \widehat{1}\}^n}.$$

Definition 2.2.9. Let K be an $\mathcal{A}$-set. By definition, the *geometric realization* of K is the flow

$$\boxed{|K| := \varinjlim_{\mathcal{A}[n]\to K} (\{\widehat{0} < \widehat{1}\}^n)^{cof}}.$$

Note that the two geometric realizations of $\mathcal{A}$-sets are colimit preserving. In fact, it is easy to prove that both are left adjoints.

Proposition 2.2.10. *Let K be a precubical set. Then there are the natural isomorphisms of flows $|\mathcal{L}_{\mathcal{A}}(K)|_{bad} \cong |K|_{bad}$ and $|\mathcal{L}_{\mathcal{A}}(K)| \cong |K|$.*

Proof. Since all functors involved in the statement of the proposition are left adjoint and therefore colimit preserving, it suffices to check the isomorphism for $K = \square[n]$. The proof is complete after Proposition 2.1.14. $\square$

Corollary 2.2.11. *For all $n \geqslant 3$, the inclusion $\partial\mathcal{A}[n] \subset \mathcal{A}[n]$ induces an isomorphism of flows $|\partial\mathcal{A}[n]|_{bad} \cong |\mathcal{A}[n]|_{bad}$.*

Proof. Since $\mathcal{L}_{\mathcal{A}}$ is colimit preserving, one has $\mathcal{L}_{\mathcal{A}}(\partial\square[n]) \cong \partial\mathcal{A}[n]$. So by Proposition 2.2.10 and Theorem 2.2.7, one obtains $|\partial\mathcal{A}[n]|_{bad} \cong |\partial\square[n]|_{bad} \cong |\square[n]|_{bad} \cong |\mathcal{A}[n]|_{bad}$. $\square$

Proposition 2.2.12. *Let K be an $\mathcal{A}$-set. The canonical map $K \to Sh_{\mathcal{A}}(K)$ induces an isomorphism of flows $|K|_{bad} \cong |Sh_{\mathcal{A}}(K)|_{bad}$.*

Proof. We already know that the map $K \to Sh_{\mathcal{A}}(K)$ is obtained by factoring the canonical map $K \to \mathbf{1}$ as the composite $K \to Sh_{\mathcal{A}}(K) \to \mathbf{1}$, where $K \to Sh_{\mathcal{A}}(K)$ is a relative $\{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}$-cell complex and the map $Sh_{\mathcal{A}}(K) \to \mathbf{1}$ satisfies the right lifting property with respect to the same set of morphisms. So the map $|K|_{bad} \to |Sh_{\mathcal{A}}(K)|_{bad}$ is a relative $\{|\mathcal{A}[2]|_{bad} \sqcup_{|\partial\mathcal{A}[2]|_{bad}} |\mathcal{A}[2]|_{bad} \to |\mathcal{A}[2]|_{bad}\}$-cell complex by Corollary 2.2.11. Fig. 2 explains why the map of flows $|\mathcal{A}[2]|_{bad} \sqcup_{|\partial\mathcal{A}[2]|_{bad}} |\mathcal{A}[2]|_{bad} \to |\mathcal{A}[2]|_{bad}$ is in fact an isomorphism.[5] Hence the proof is complete. $\square$

The commutative diagram of flows of Fig. 3 concludes the section. It proves that labelled precubical sets and labelled $\mathcal{A}$-sets are equivalent from a directed algebraic topological point of view, K being any labelled precubical set.

5 Intuitively, adding an algebraic relation is an idempotent operation.

An application

We give now a small application of the notion of geometric realization of labelled $\mathcal{A}$-set which will be reused later. The following proposition could of course be proved without using the topological material of this section. However, the proof would be more complicated (see the proof of [18, Theorem 7.1]).

Proposition 2.2.13. *Let $\ell : \mathcal{A}[p] \to Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$ be a full labelled p-dimensional $\mathcal{A}$-cube with $p \geqslant 2$. Then there exists $(a_1, \ldots, a_p) \in \Sigma^p$ such that, for every maximal path $(c_1, \ldots, c_p)$ of $\mathcal{A}[p]$, i.e. for any p-tuple of 1-cubes of $\mathcal{A}[p]$ with $\partial_1^0(c_1) = (0, \ldots, 0)$, $\partial_1^1(c_i) = \partial_1^0(c_{i+1})$ for $1 \leqslant i \leqslant p-1$ and $\partial_1^1(c_p) = (1, \ldots, 1)$, one has $\ell(c_1) * \cdots * \ell(c_p) = a_1 * \cdots * a_p$.*

Proof. Let $(c_1, \ldots, c_p)$ and $(c'_1, \ldots, c'_p)$ be two maximal paths. Since there is a unique morphism from $(\widehat{0}, \ldots, \widehat{0})$ to $(\widehat{1}, \ldots, \widehat{1})$ in $|\mathcal{A}[p]|_{bad}$ (this is the key point!), one has $\ell(c_1) * \cdots * \ell(c_p) = \ell(c'_1) * \cdots * \ell(c'_p)$ in the flow $|Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)|_{bad}$. But the semigroup $\mathbb{P}(|Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)|_{bad}) \cong \mathbb{P}(?\Sigma)$ is the free commutative semigroup generated by the elements of Σ. Hence the result. □

2.3. Labelled coskeleton over a category of cubes

In this section we give the generalization of the notion of a labelled coskeleton to any category of labelled precubical sets. The particular case $\mathcal{A} = \square$ will give back the situation of [18]. The unlabelled version, i.e. when $\Sigma = \{\tau\}$, is the classical coskeleton functor, right adjoint to the truncation functor [5].

The unlabelled case

Proposition 2.3.1. *Let $n \geqslant 0$.*

(1) *The functor $K \mapsto K_{\leqslant n}$ from $\mathcal{A}_{n+1}^{op}$**Set** to $\mathcal{A}_n^{op}$**Set** has a right adjoint denoted by $cosk_{n,n+1}^{\mathcal{A}} : \mathcal{A}_n^{op}\mathbf{Set} \to \mathcal{A}_{n+1}^{op}\mathbf{Set}$. There is an inclusion of presheaves*

$$K \subset cosk_{n,n+1}^{\mathcal{A}}(K)$$

natural with respect to the n-dimensional $\mathcal{A}$-set K. This inclusion induces the isomorphism $K \cong cosk_{n,n+1}^{\mathcal{A}}(K)_{\leqslant n}$.

(2) *The functor $K \mapsto K_{\leqslant n}$ from $\mathcal{A}^{op}$**Set** to $\mathcal{A}_n^{op}$**Set** has a right adjoint denoted by $cosk_n^{\mathcal{A}} : \mathcal{A}_n^{op}\mathbf{Set} \to \mathcal{A}^{op}\mathbf{Set}$. There is an inclusion of presheaves $K \subset cosk_n^{\mathcal{A}}(K)$ natural with respect to the n-dimensional $\mathcal{A}$-set K. This inclusion induces the isomorphism $K \cong cosk_n^{\mathcal{A}}(K)_{\leqslant n}$.*

(3) *Let $cosk_{n,n+p}^{\mathcal{A}} = cosk_{n+p-1,n+p}^{\mathcal{A}} \circ \cdots \circ cosk_{n,n+1}^{\mathcal{A}} \circ cosk_{n,n}^{\mathcal{A}}$, where the functor*

$$cosk_{n,n}^{\mathcal{A}} : \mathcal{A}_n^{op}\mathbf{Set} \to \mathcal{A}_n^{op}\mathbf{Set}$$

denotes the identity functor. Then there is an isomorphism of functors

$$cosk_n^{\mathcal{A}} \cong \varinjlim cosk_{n,n+p}^{\mathcal{A}}.$$

Proof. Let us prove the first assertion. The functor $K \mapsto K_{\leqslant n}$ from $\mathcal{A}_{n+1}^{op}$**Set** to $\mathcal{A}_n^{op}$**Set** is induced by the inclusion of categories $\mathcal{A}_n^{op} \subset \mathcal{A}_{n+1}^{op}$. Thus, the right adjoint is obtained by taking the right Kan extension along $\mathcal{A}_n^{op} \subset \mathcal{A}_{n+1}^{op}$. The isomorphism of presheaves $K_{\leqslant n} \cong K$ for an n-dimensional $\mathcal{A}$-set K induces by adjunction a natural map $K \to cosk_{n,n+1}^{\mathcal{A}}(K)$. Let $p \leqslant n$. There is a bijection $\mathcal{A}^{op}\mathbf{Set}(\mathcal{A}[p], K) \cong \mathcal{A}^{op}\mathbf{Set}(\mathcal{A}[p], cosk_{n,n+1}^{\mathcal{A}}(K))$ because of the isomorphism $\mathcal{A}[p]_{\leqslant n} \cong \mathcal{A}[p]$. Hence we obtain the desired inclusion. The proof of the second assertion is similar to the above proof. The third assertion is obvious. □

Definition 2.3.2. Let K be an $\mathcal{A}$-set. An $(n+1)$-cube of $cosk_{n,n+1}^{\mathcal{A}}(K_{\leqslant n})$, i.e. a map $\partial\mathcal{A}[n+1] \to K$, is called an *n-dimensional shell or n-shell* of K.

The labelled case

Before giving the labelled version of Proposition 2.3.1, let us prove the following general categorical fact.

Proposition 2.3.3. *Let $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ be a categorical adjunction, where L is the left adjoint and R the right one. Let us suppose that $\mathcal{C}$ has all pullbacks. Let A be an object of $\mathcal{C}$. Then the functor $L_A : \mathcal{C}{\downarrow}A \to \mathcal{D}{\downarrow}L(A)$ defined by $L_A(X \to A) := L(X) \to L(A)$ has a right adjoint R_A defined by the following pullback diagram of $\mathcal{C}$:*

$$\begin{array}{ccc} R_A(Y) & \longrightarrow & R(Y) \\ \downarrow & & \downarrow \\ A & \longrightarrow & R(L(A)) \end{array},$$

where the map $A \to R(L(A))$ is the unit of the adjunction.

Note that we are going to use Proposition 2.3.3 with $\mathcal{C}$ and $\mathcal{D}$ locally presentable. In this situation, the categories $\mathcal{C}{\downarrow}A$ and $\mathcal{D}{\downarrow}L(A)$ are both locally presentable as well by [3, Proposition 1.57]. In particular, the category $\mathcal{C}{\downarrow}A$ has a generator and is co-wellpowered. The functor $L_A : \mathcal{C}{\downarrow}A \to \mathcal{D}{\downarrow}L(A)$ is colimit preserving since L is colimit preserving. So by the opposite of the Special Adjoint Functor Theorem, the functor L_A has a right adjoint.

Proof. Let $X \to A$ be an object of $\mathcal{C}{\downarrow}A$. Let $Y \to L(A)$ be an object of $\mathcal{D}{\downarrow}L(A)$. There is a bijection between the commutative diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & R_A(Y) \\ \downarrow & & \downarrow \\ A & = & A \end{array}$$

and the commutative diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & R(Y) \\ \downarrow & & \downarrow \\ A & \longrightarrow & R(L(A)) \end{array}$$

because of the universal property of pullback. And there is a bijection between the latter diagrams and the commutative diagrams of the form

$$\begin{array}{ccc} L(X) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ L(A) & = & L(A) \end{array}$$

by universality of adjunction. Hence the result. □

Here is now the labelled analogue of Proposition 2.3.1.

Proposition 2.3.4. *Let $n \geqslant 0$.*

(1) *The functor $K \mapsto K_{\leqslant n}$ from $\mathcal{A}_{n+1}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$ to $\mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$ has a right adjoint denoted by $cosk_{n,n+1}^{\mathcal{A},\Sigma} : \mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma) \to \mathcal{A}_{n+1}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$. There is an inclusion of presheaves*

$$K \subset cosk_{n,n+1}^{\mathcal{A},\Sigma}(K)$$

natural with respect to the n-dimensional labelled $\mathcal{A}$-set K. This inclusion induces the isomorphism $K \cong cosk_{n,n+1}^{\mathcal{A},\Sigma}(K)_{\leqslant n}$.

(2) *The functor $K \mapsto K_{\leqslant n}$ from $\mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$ to $\mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$ has a right adjoint denoted by $cosk_{n}^{\mathcal{A},\Sigma} : \mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma) \to \mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$. There is an inclusion of presheaves $K \subset cosk_{n}^{\mathcal{A},\Sigma}(K)$ natural with respect to the n-dimensional labelled $\mathcal{A}$-set K. This inclusion induces the isomorphism $K \cong cosk_{n}^{\mathcal{A},\Sigma}(K)_{\leqslant n}$.*

(3) *Let $cosk_{n,n+p}^{\mathcal{A},\Sigma} = cosk_{n+p-1,n+p}^{\mathcal{A},\Sigma} \circ \cdots \circ cosk_{n,n+1}^{\mathcal{A},\Sigma} \circ cosk_{n,n}^{\mathcal{A},\Sigma}$, where the functor $cosk_{n,n}^{\mathcal{A},\Sigma} : \mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma) \to \mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$ denotes the identity functor. Then there is an isomorphism of functors $cosk_{n}^{\mathcal{A},\Sigma} \cong \varinjlim cosk_{n,n+p}^{\mathcal{A},\Sigma}$.*

Proof. We note that the categories $\mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$ and $\mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}(Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma))_{\leqslant n}$ are isomorphic. So the theorem is a consequence of Propositions 2.3.1 and 2.3.3. □

Note that, for every $n \geqslant 1$ and for every n-dimensional labelled $\mathcal{A}$-set K, one has the pullback diagram of $\mathcal{A}$-sets

$$\begin{array}{ccc} cosk_n^{\mathcal{A},\Sigma}(K) & \longrightarrow & cosk_n^{\mathcal{A}}(K) \\ \downarrow & \lrcorner & \downarrow \\ Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma) & \longrightarrow & cosk_n^{\mathcal{A}}((Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma))_{\leqslant n}). \end{array}$$

Intuitively, this means that the labelled coskeleton functor keeps from the unlabelled one only the shells which are compatibly labelled. For example, the boundary of a square is compatibly labelled if and only if opposite sides are labelled in the same way.

Definition 2.3.5. Let K be a labelled $\mathcal{A}$-set. An $(n+1)$-cube of $cosk_{n,n+1}^{\mathcal{A},\Sigma}(K_{\leqslant n})$ is called a *labelled n-dimensional shell or n-shell* of K.

The following proposition generalizes [18, Definition 3.12 and Proposition 3.13].

Proposition 2.3.6. *Let K be a labelled $\mathcal{A}$-set. The set of labelled n-dimensional shells of K is in bijection with the set of commutative diagrams of the form*

$$\begin{array}{ccc} \partial\mathcal{A}[n+1] & \longrightarrow & K \\ \downarrow & & \downarrow \\ \mathcal{A}[n+1] & \longrightarrow & Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma). \end{array}$$

Since $Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\{\tau\})$ is the terminal $\mathcal{A}$-set by Proposition 2.1.25, the case $\Sigma = \{\tau\}$ coincides with the unlabelled notion of Definition 2.3.2.

Proof. Let $\mathcal{A}[n+1] \to cosk_{n,n+1}^{\mathcal{A},\Sigma}(K_{\leqslant n})$ be a labelled n-shell of K. By adjunction, one obtains the commutative diagram of labelled $\mathcal{A}$-sets

$$\begin{array}{ccc} \partial\mathcal{A}[n+1] & \longrightarrow & K_{\leqslant n} \\ \downarrow & & \downarrow \\ \mathcal{A}[n+1] & \longrightarrow & cosk_{n,n+1}^{\mathcal{A},\Sigma}(K_{\leqslant n}). \end{array}$$

By composing with the labelling map $cosk_{n,n+1}^{\mathcal{A},\Sigma}(K_{\leqslant n}) \to Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)$, one obtains the commutative diagram of $\mathcal{A}$-sets

$$\begin{array}{ccc} \partial\mathcal{A}[n+1] & \longrightarrow & K \\ \downarrow & & \downarrow \\ \mathcal{A}[n+1] & \longrightarrow & Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma). \end{array}$$

Conversely, from such a diagram, one obtains the commutative diagram of $\mathcal{A}$-sets

$$\begin{array}{ccc} \mathcal{A}[n+1]_{\leqslant n} & \longrightarrow & K_{\leqslant n} \\ \downarrow & & \downarrow \\ Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma) & = & Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma); \end{array}$$

hence the result by adjunction. □

3. Mathematical treatment

3.1. Shell-complete category of cubes

The purpose of this combinatorial section is to address the following question. Is it possible to find a category of cubes $\mathcal{A}$ such that $cosk_1^{\mathcal{A},\Sigma}(\mathcal{A}[a_1,\ldots,a_n]_{\leqslant 1})$ is exactly the labelled n-cube $\mathcal{A}[a_1,\ldots,a_n]$ for every $n \geqslant 0$ and every $a_1,\ldots,a_n \in \Sigma$? Let us repeat one more time that there is always a strict inclusion $\square[a_1,\ldots,a_n] \subset cosk_1^{\square,\Sigma}(\square[a_1,\ldots,a_n]_{\leqslant 1})$ for every $n \geqslant 2$ by [18, Proposition 3.15] and that this is the reason for introducing in [18] the non-functorial subobject of $cosk_1^{\square,\Sigma}(\square[a_1,\ldots,a_n]_{\leqslant 1})$ called the labelled directed coskeleton of $\square[a_1,\ldots,a_n]_{\leqslant 1}$ (see Definition 3.1.22). For $\Sigma = \{\tau\}$, i.e. for the unlabelled case, the previous equality reduces to finding a category of cubes $\mathcal{A}$ such that $cosk_1^{\mathcal{A}}(\mathcal{A}[n]_{\leqslant 1}) \cong \mathcal{A}[n]$ for every $n \geqslant 0$. Such a category $\mathcal{A}$ will be called a shell-complete category of cubes. We will see in Theorem 3.1.24 that such a category of cubes answers the question above. We will see in Theorem 3.1.15 that there exists one and only one such category of cubes.

Definition and elementary properties

Proposition 3.1.1. *Let $p, q \geqslant 0$. The natural bijection*

$$\mathcal{A}^{op}\mathbf{Set}(\mathcal{A}[p], \mathcal{A}[q]) \cong \mathcal{A}([p],[q])$$

induced by the mapping $f \mapsto f_p(Id_{[p]})$ given by the Yoneda lemma takes $f : \mathcal{A}[p] \to \mathcal{A}[q]$ to $f_0 : [p] \cong \mathcal{A}([0],[p]) \to \mathcal{A}([0],[q]) \cong [q]$.

Proof. Let $f : \mathcal{A}[p] \to \mathcal{A}[q]$ be a map of $\mathcal{A}^{op}\mathbf{Set}$. The map $\delta_p^{\epsilon_p} \ldots \delta_1^{\epsilon_1} : [0] \to [p]$ induces a commutative square of sets

$$\begin{array}{ccc}
\mathcal{A}([p],[p]) & \xrightarrow{f_p} & \mathcal{A}([p],[q]) \\
\big\downarrow {\scriptstyle (\delta_p^{\epsilon_p}\ldots\delta_1^{\epsilon_1})^*} & & \big\downarrow {\scriptstyle (\delta_p^{\epsilon_p}\ldots\delta_1^{\epsilon_1})^*} \\
\mathcal{A}([0],[p]) & \xrightarrow{f_0} & \mathcal{A}([0],[q])
\end{array}$$

for any $\epsilon_1,\ldots,\epsilon_p \in \{0,1\}$ since $\mathcal{A}$ is a category of cubes. So

$$f_0(\delta_p^{\epsilon_p} \ldots \delta_1^{\epsilon_1})^*(Id_{[p]}) = (\delta_p^{\epsilon_p} \ldots \delta_1^{\epsilon_1})^*(f_p(Id_{[p]})).$$

Therefore $f_0 = f_p(Id_{[p]})$. □

The following proposition motivates the notion of shell-complete category of cubes.

Proposition 3.1.2. *For any $q \geqslant 0$, the canonical map $\mathcal{A}[q] \to cosk_1^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1})$ induced by the isomorphism $\mathcal{A}[q]_{\leqslant 1} \cong \mathcal{A}[q]_{\leqslant 1}$ is an inclusion of presheaves. For $q = 0$ or $q = 1$, this inclusion is always an equality for any category of cubes $\mathcal{A}$.*

Proof. Let x and y be two k-cubes of $\mathcal{A}[q]$ having the same image by the map

$$\mathcal{A}[q] \longrightarrow cosk_1^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1}).$$

So one has the commutative diagram of $\mathcal{A}$-sets

$$\mathcal{A}[k] \underset{y}{\overset{x}{\rightrightarrows}} \mathcal{A}[q] \longrightarrow cosk_1^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1}).$$

By adjunction, one obtains the commutative diagram of $\mathcal{A}$-sets

$$\mathcal{A}[k]_{\leqslant 1} \underset{y_{\leqslant 1}}{\overset{x_{\leqslant 1}}{\rightrightarrows}} \mathcal{A}[q]_{\leqslant 1} \xrightarrow{\cong} \mathcal{A}[q]_{\leqslant 1}.$$

In particular, the two set maps $x_0, y_0 : [k] \rightrightarrows [q]$ are equal. Thus, by Proposition 3.1.1, one obtains $x = y$. The last assertion is a consequence of Proposition 2.3.1. □

Hence the definition:

Definition 3.1.3. A category of cubes $\mathcal{A}$ is *shell complete* if, for every $p \geqslant 2$, the canonical inclusion of presheaves $\mathcal{A}[p] \subset cosk_1^{\mathcal{A}}(\mathcal{A}[p]_{\leqslant 1})$ is an isomorphism.

The category of cubes $\square$ is of course not shell complete by [18, Proposition 3.15]. For example, the precubical set $cosk_1^{\square}(\square[2]_{\leqslant 1})$ contains the 2-cube $x : \square[2] \to cosk_1^{\square}(\square[2]_{\leqslant 1})$ corresponding by adjunction to the map $\partial x : \partial\square[2] = \square[2]_{\leqslant 1} \to \square[2]_{\leqslant 1}$ characterized by $x_0(\epsilon_1,\epsilon_2) = (\epsilon_2,\epsilon_1)$. It is not a 2-cube of $\square[2]$ since the only 2-cube of the precubical set $\square[2]$ is the identity of $[2]$.

In general, for any $p, q \geqslant 2$, there exists at most one lift $\overline{x}$ in the commutative diagram of solid arrows

$$\begin{array}{ccc} \partial\mathcal{A}[p] & \xrightarrow{x} & \mathcal{A}[q] \\ \downarrow & \overset{\overline{x}}{\nearrow} & \downarrow \\ \mathcal{A}[p] & \longrightarrow & \mathbf{1}, \end{array}$$

where **1** is the terminal object. Indeed, by Proposition 3.1.1, the bijection of sets

$$\mathcal{A}^{op}\mathbf{Set}(\mathcal{A}[p], \mathcal{A}[q]) \cong \mathcal{A}([p], [q])$$

takes $\overline{x}$ to x_0. Shell-completeness means that this lift always exists.

Theorem 3.1.4. *Let $\mathcal{A}$ be a category of cubes. The following conditions are equivalent:*

(1) *The category $\mathcal{A}$ is shell complete.*
(2) *For any $p, q \geqslant 2$, for any map $x : \partial\mathcal{A}[p] \to \mathcal{A}[q]$, the set map $x_0 : [p] \to [q]$ belongs to $\mathcal{A}$.*
(3) *For any $p, q \geqslant 2$, any map $x : \partial\mathcal{A}[p] \to \mathcal{A}[q]$ factors uniquely as a composite $x : \partial\mathcal{A}[p] \to \mathcal{A}[p] \to \mathcal{A}[q]$.*

Proof. Let us prove the implication (1) $\Longrightarrow$ (2). Let $x : \partial\mathcal{A}[p] \to \mathcal{A}[q]$ be a map of $\mathcal{A}$-sets with $p, q \geqslant 2$. One can suppose that $p \leqslant q$ by Proposition 2.1.17. Then x factors (uniquely) as a composite

$$x : \partial\mathcal{A}[p] = \mathcal{A}[p]_{\leqslant p-1} \longrightarrow \mathcal{A}[q]_{\leqslant p-1} \longrightarrow \mathcal{A}[q].$$

One has the isomorphisms

$$\mathcal{A}[q]_{\leqslant p-1} \cong cosk_1^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1})_{\leqslant p-1} \cong cosk_{1,p-1}^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1})$$

since $\mathcal{A}$ is shell complete and by Proposition 2.3.1. So x factors as a composite

$$x : \partial\mathcal{A}[p] \longrightarrow \mathcal{A}[p] \longrightarrow cosk_{p-1,p}^{\mathcal{A}}(cosk_{1,p-1}^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1})) = \mathcal{A}[q]_{\leqslant p} \longrightarrow \mathcal{A}[q].$$

So $x_0 : [p] \to [q]$ is a morphism of $\mathcal{A}$ by Proposition 3.1.1.

Let us prove now the implication (2) $\Longrightarrow$ (1). Propositions 2.3.1 and 3.1.2 imply that there is an inclusion of presheaves

$$\mathcal{A}[q]_{\leqslant p} \subset (cosk_1^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1}))_{\leqslant p} = cosk_{1,p}^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1})$$

for any $p \geqslant 1$. This inclusion is trivially an equality for $p = 1$. Let us prove by induction on p that this inclusion is an equality. This will establish the shell-completeness of $\mathcal{A}$. Let us suppose the equality proved for $p \geqslant 1$. Let $x : \mathcal{A}[p+1] \to cosk_{1,p+1}^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1})$ be a $(p+1)$-dimensional $\mathcal{A}$-cube of $cosk_{1,p+1}^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1})$. By adjunction and by induction hypothesis, one obtains a map

$$\partial x : \mathcal{A}[p+1]_{\leqslant p} = \partial\mathcal{A}[p+1] \longrightarrow cosk_{1,p}^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1}) \cong \mathcal{A}[q]_{\leqslant p} \subset \mathcal{A}[q].$$

By hypothesis, the map $x_0 : [p+1] \to [q]$ belongs to $\mathcal{A}$. Thus, by Proposition 3.1.1, there is a commutative diagram of $\mathcal{A}$-sets

$$\begin{array}{ccc} \partial\mathcal{A}[p+1] & \longrightarrow & cosk_{1,p}^{\mathcal{A}}(\mathcal{A}[q]_{\leqslant 1}) \cong \mathcal{A}[q]_{\leqslant p} \\ \downarrow & & \downarrow \subset \\ \mathcal{A}[p+1] & \longrightarrow & \mathcal{A}[q]_{\leqslant p+1}. \end{array}$$

Hence the equality for $p+1$.

The equivalence (2) $\Longleftrightarrow$ (3) is a consequence of Proposition 3.1.1. $\square$

Examples of shell-complete categories of cubes

Theorem 3.1.5. *The category of cubes $\widehat{\square}$ (i.e. the maximal category of cubes containing all adjacency-preserving maps) is shell complete.*

Proof. Let $x : \partial\widehat{\square}[p] \to \widehat{\square}[q]$ be a morphism of $\widehat{\square}^{op}\mathbf{Set}$ with $p, q \geqslant 2$. For all k such that $1 \leqslant k \leqslant p-1$, one has the commutative diagram of sets

$$\begin{array}{ccc}
\widehat{\square}([k],[p]) & \xrightarrow{x_k} & \widehat{\square}([k],[q]) \\
\downarrow{\scriptstyle \partial_1^{\epsilon_1}\ldots\partial_k^{\epsilon_k}} & & \downarrow{\scriptstyle \partial_1^{\epsilon_1}\ldots\partial_k^{\epsilon_k}} \\
\widehat{\square}([0],[p]) & \xrightarrow{x_0} & \widehat{\square}([0],[q])
\end{array}$$

for all $\epsilon_1, \ldots, \epsilon_k \in \{0,1\}$ since $x : \partial\widehat{\square}[p] \to \widehat{\square}[q]$ is a map of $\widehat{\square}^{op}\mathbf{Set}$ and where the set map $\partial_1^{\epsilon_1}\ldots\partial_k^{\epsilon_k}$ is induced by the morphism $\delta_k^{\epsilon_k}\ldots\delta_1^{\epsilon_1} : [0] \to [k]$ of $\widehat{\square}$. With $\phi \in \widehat{\square}([k],[p])$, that means that $x_k(\phi)(\delta_k^{\epsilon_k}\ldots\delta_1^{\epsilon_1}) = x_0(\phi(\delta_k^{\epsilon_k}\ldots\delta_1^{\epsilon_1}))$. Thus, one obtains $x_k(\phi)(\epsilon_1,\ldots,\epsilon_k) = x_0(\phi(\epsilon_1,\ldots,\epsilon_k))$. So $x_k(\phi) = x_0\phi$ with the identification $\widehat{\square}([0],[p]) \cong [p]$. Let $(\epsilon_1,\ldots,\epsilon_p)$ and $(\epsilon'_1,\ldots,\epsilon'_p)$ be two elements of $[p]$ with $\epsilon_i = \epsilon'_i$ for all i but one denoted by i_0. Suppose moreover that $\epsilon_{i_0} = 0$ and $\epsilon'_{i_0} = 1$. Since $p \geqslant 2$, there exists $i_1 \in \{1,\ldots,p\}\backslash\{i_0\}$. Consider $\delta_{i_1}^{\epsilon_{i_1}} : [p-1] \to [p]$. Then $x_{p-1}(\delta_{i_1}^{\epsilon_{i_1}}) = x_0\delta_{i_1}^{\epsilon_{i_1}}$. The preceding equality applied to $(\epsilon_1,\ldots,\widehat{\epsilon_{i_1}},\ldots,\epsilon_p)$ gives

$$\begin{aligned}
x_0(\epsilon_1,\ldots,\epsilon_p) &= x_{p-1}(\delta_{i_1}^{\epsilon_{i_1}})(\epsilon_1,\ldots,\widehat{\epsilon_{i_1}},\ldots,\epsilon_p) \\
&< x_{p-1}(\delta_{i_1}^{\epsilon_{i_1}})(\epsilon'_1,\ldots,\widehat{\epsilon_{i_1}},\ldots,\epsilon'_p) \\
&= x_0(\epsilon'_1,\ldots,\epsilon'_p)
\end{aligned}$$

since the map $x_{p-1}(\delta_{i_1}^{\epsilon_{i_1}}) : [p-1] \to [q]$ is a morphism of the small category $\widehat{\square}$. So the set map $x_0 : [p] \to [q]$ is adjacency preserving, i.e. it belongs to the small category $\widehat{\square}$. Thus, the small category $\widehat{\square}$ is shell complete by Theorem 3.1.4. $\square$

Proposition 3.1.6. *Let $\mathcal{A}$ be a category of cubes. Let $p, q \geqslant 2$. The set $\mathcal{A}^{op}\mathbf{Set}(\partial\mathcal{A}[p], \mathcal{A}[q])$ is equal to the set of families $(f_i^\alpha : \mathcal{A}[p-1] \to \mathcal{A}[q])$ of morphisms of $\mathcal{A}^{op}\mathbf{Set}$ with $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$ with $(f_j^\beta)_0\delta_i^\alpha = (f_i^\alpha)_0\delta_{j-1}^\beta$ for any $i < j$ and any $\alpha, \beta \in \{0,1\}$.*

Proof. Let $f : \partial\mathcal{A}[p] \to \mathcal{A}[q]$ be a morphism of $\mathcal{A}^{op}\mathbf{Set}$. The $2p$ inclusions $\mathcal{A}[p-1] \subset \partial\mathcal{A}[p]$ with $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$ induce $2p$ maps $f_i^\alpha : \mathcal{A}[p-1] \subset \partial\mathcal{A}[p] \to \mathcal{A}[q]$ such that $(f_i^\alpha)_0 = f_0\delta_i^\alpha$ with $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$. The equalities $(f_j^\beta)_0\delta_i^\alpha = (f_i^\alpha)_0\delta_{j-1}^\beta$ for any $i < j$ and any $\alpha, \beta \in \{0,1\}$ are then a consequence of the cocubical relations.

Conversely, let $(f_i^\alpha : \mathcal{A}[p-1] \to \mathcal{A}[q])$ be a family of morphisms of $\mathcal{A}^{op}\mathbf{Set}$ with $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$ such that $(f_j^\beta)_0\delta_i^\alpha = (f_i^\alpha)_0\delta_{j-1}^\beta$ for any $i < j$ and any $\alpha, \beta \in \{0,1\}$. Consider the set map $g : [p] \to [q]$ defined by $g(\epsilon_1,\ldots,\epsilon_p) := (f_p^{\epsilon_p})_0(\epsilon_1,\ldots,\epsilon_{p-1})$. Then $g\delta_p^\alpha = (f_p^\alpha)_0$ by definition of g, and for any $0 \leqslant i < p$, one has

$$g\delta_i^\alpha(\epsilon_1,\ldots,\epsilon_{p-1}) = (f_p^{\epsilon_{p-1}})_0\delta_i^\alpha(\epsilon_1,\ldots,\epsilon_{p-2}) = (f_i^\alpha)_0\delta_{p-1}^{\epsilon_{p-1}}(\epsilon_1,\ldots,\epsilon_{p-2}) = (f_i^\alpha)_0(\epsilon_1,\ldots,\epsilon_{p-1})$$

for any $\alpha, \epsilon_{p-1} \in \{0,1\}$ thanks to the cocubical relations. So one obtains $g\delta_i^\alpha = (f_i^\alpha)_0$ for $0 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$. The mapping $\phi \mapsto g\phi$ gives rise for each $0 \leqslant k \leqslant p-1$ to a set map $g_k : \partial\mathcal{A}[p]_k := \mathcal{A}([k],[p]) \to \mathcal{A}[q]_k := \mathcal{A}([k],[q])$. For any morphism $\psi : [k'] \to [k]$ of $\mathcal{A}$ with $0 \leqslant k' \leqslant k \leqslant p-1$, one obtains a diagram of sets

$$\begin{array}{ccc}
\partial\mathcal{A}[p]_k & \longrightarrow & \mathcal{A}[q]_k \\
\downarrow & & \downarrow \\
\partial\mathcal{A}[p]_{k'} & \longrightarrow & \mathcal{A}[q]_{k'}
\end{array}$$

which is commutative since the two boundaries of the square take $\phi \in \mathcal{A}[p]_k$ to $g\phi\psi$. $\square$

Proposition 3.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two categories of cubes such that $\mathcal{A} \subset \mathcal{B}$. Let $p, q \geqslant 2$. Then one has the inclusion $\mathcal{A}^{op}\mathbf{Set}(\partial\mathcal{A}[p], \mathcal{A}[q]) \subset \mathcal{B}^{op}\mathbf{Set}(\partial\mathcal{B}[p], \mathcal{B}[q])$ by identifying the maps f with the corresponding set maps f_0 from $[p]$ to $[q]$.*

Proof. This is a corollary of Proposition 3.1.6 and of the fact that $\mathcal{A}([p-1],[q]) \subset \mathcal{B}([p-1],[q])$. $\square$

Theorem 3.1.8. *There exists a smallest shell-complete category, denoted by $\widetilde{\square}$.*

Proof. Let $(\square^{(i)})_{i\in I}$ be the class of all shell-complete small categories of cubes. This class is non-empty by Theorem 3.1.5, and small since, for any i, there is the inclusion $\square^{(i)} \subset \mathbf{PoSet}$. Consider the small category $\widetilde{\square} = \bigcap_{i\in I}\square^{(i)}$. Let $f : \partial\widetilde{\square}[p] \to \widetilde{\square}[q]$ be a map of $\widetilde{\square}^{op}\mathbf{Set}$ with $p, q \geqslant 2$. By Proposition 3.1.7, the morphism of presheaves f gives rise for each $i \in I$ to a morphism of presheaves $f^{(i)} : \partial\square^{(i)}[p] \to \square^{(i)}[q]$. By Theorem 3.1.4, $f_0 = (f^{(i)})_0$ is a morphism of $\square^{(i)}$ for each $i \in I$. So, by Theorem 3.1.4 again, the category $\widetilde{\square}$ is shell complete. $\square$

Some combinatorial lemmas

Let us recall that $\sigma_i : [n] \to [n]$ is the set map defined for $1 \leqslant i \leqslant n-1$ and $n \geqslant 2$ by $\sigma_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_i, \epsilon_{i+2}, \dots, \epsilon_n)$ (see Definition 2.1.10).

Proposition 3.1.9 ([21] p195). *Let $\sigma_i : [n] \to [n]$ be the set map defined for $1 \leqslant i \leqslant n-1$ and $n \geqslant 2$ by $\sigma_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_i, \epsilon_{i+2}, \dots, \epsilon_n)$. One has the relations $\sigma_i\delta_j^\alpha = \delta_j^\alpha\sigma_{i-1}$ for $j < i$, $\sigma_i\delta_j^\alpha = \delta_{i+1}^\alpha$ for $j = i$, $\sigma_i\delta_j^\alpha = \delta_i^\alpha$ for $j = i+1$ and $\sigma_i\delta_j^\alpha = \delta_j^\alpha\sigma_i$ for $j > i+1$.*

Proposition 3.1.10. $\sigma_i \in \widetilde{\square}$.

Proof. Let us prove by induction on $n \geqslant 2$ that the set maps $\sigma_i : [n] \to [n]$ for $1 \leqslant i \leqslant n-1$ belong to $\widetilde{\square}$. The composite map $\partial\sigma_1 : \partial\square[2] \subset \square[2] \xrightarrow{\sigma_1} \square[2]$ induces a map $\partial\sigma_1 : \partial\widetilde{\square}[2] \to \widetilde{\square}[2]$ by Proposition 3.1.7 since $\square \subset \widetilde{\square}$. So $\sigma_1 : [2] \to [2]$ is a map of $\widetilde{\square}$ by Theorem 3.1.4 since $\widetilde{\square}$ is shell complete. Hence the proof is complete for $n = 2$. Now assume that $n > 2$. By Proposition 3.1.9 and by induction hypothesis, the $2n$ set maps $\sigma_i\delta_j^\alpha : [n-1] \subset [n] \to [n]$ belong to $\widetilde{\square}$. These $2n$ morphisms of $\widetilde{\square}$ induce a morphism $\partial\widetilde{\square}[n] \to \widetilde{\square}[n]$ by Proposition 3.1.6. So $\sigma_i : [n] \to [n]$ belongs to $\widetilde{\square}$ by shell-completeness. □

To our knowledge, the structure maps introduced below are new. They are related to the notion of *connection* in the setting of *cubical* sets, see [1,2]; indeed, with their notation of ε_i for degeneracies and Γ_i^α for connections, one has $\Gamma_i^+ = \varepsilon_i\gamma_i$ and $\Gamma_i^- = \varepsilon_{i+1}\gamma_i$. An example of use of these connections in directed algebraic topology can be found in [15,16].

Definition 3.1.11. Let $\gamma_i : [n] \to [n]$ be the set map defined for $1 \leqslant i \leqslant n-1$ and $n \geqslant 2$ by

$$\gamma_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, \max(\epsilon_i, \epsilon_{i+1}), \min(\epsilon_i, \epsilon_{i+1}), \epsilon_{i+2}, \dots, \epsilon_n).$$

These maps are called the *transverse degeneracy maps*.

Proposition 3.1.12. *One has the relations $\gamma_j\delta_i^\alpha = \delta_i^\alpha\gamma_j$ for $j < i-1$, $\gamma_j\delta_i^\alpha = \delta_i^\alpha\gamma_{j-1}$ for $j \geqslant i+1$, $\gamma_j\delta_i^\alpha = \delta_{i-\alpha}^\alpha$ for $j = i-1$ and $\gamma_j\delta_i^\alpha = \delta_{i+1-\alpha}^\alpha$ for $j = i$.*

Proof. The relation $\gamma_j\delta_i^\alpha = \delta_i^\alpha\gamma_j$ for $j < i-1$ is obvious. One has

$$\gamma_j\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = \gamma_j(\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1}) = \delta_i^\alpha\gamma_{j-1}(\epsilon_1, \dots, \epsilon_{n-1})$$

for $j \geqslant i+1$. For $j = i-1$, one has

$$\gamma_j\delta_i^1(\epsilon_1, \dots, \epsilon_{n-1}) = \gamma_j(\epsilon_1, \dots, \epsilon_{i-1}, 1, \epsilon_i, \dots, \epsilon_{n-1}) = \delta_{i-1}^1(\epsilon_1, \dots, \epsilon_{n-1})$$

and

$$\gamma_j\delta_i^0(\epsilon_1, \dots, \epsilon_{n-1}) = \gamma_j(\epsilon_1, \dots, \epsilon_{i-1}, 0, \epsilon_i, \dots, \epsilon_{n-1}) = \delta_i^0(\epsilon_1, \dots, \epsilon_{n-1}).$$

Finally, for $j = i$, one has

$$\gamma_j\delta_i^1(\epsilon_1, \dots, \epsilon_{n-1}) = \gamma_j(\epsilon_1, \dots, \epsilon_{i-1}, 1, \epsilon_i, \dots, \epsilon_{n-1}) = \delta_i^1(\epsilon_1, \dots, \epsilon_{n-1})$$

and

$$\gamma_j\delta_i^0(\epsilon_1, \dots, \epsilon_{n-1}) = \gamma_j(\epsilon_1, \dots, \epsilon_{i-1}, 0, \epsilon_i, \dots, \epsilon_{n-1}) = \delta_{i+1}^0(\epsilon_1, \dots, \epsilon_{n-1}). \quad \square$$

Proposition 3.1.13. $\gamma_i \in \widetilde{\square}$.

Proof. The proof is mutatis mutandis the one of Proposition 3.1.10. □

Proposition 3.1.14. *Let $0 \leqslant m \leqslant n$. Every adjacency-preserving (resp. adjacency-preserving one-to-one) map $f : [m] \to [n]$ factors uniquely as a composite $[m] \xrightarrow{\psi} [m] \xrightarrow{\phi} [n]$ with $\phi \in \square$ and ψ adjacency preserving (resp. adjacency preserving one-to-one).*

Note that, by a cardinality argument, if $\psi : [m] \to [m]$ is one-to-one, then it is bijective.

Proof. One has $d(f(0, \dots, 0), f(1, \dots, 1)) = m$. So, by Proposition 2.1.3, $f([m])$ is an m-subcube of $[n]$. So the assertion is a consequence of Proposition 2.1.6. □

The uniqueness and a negative result

Theorem 3.1.15. *The category of cubes $\widehat{\square}$ is the unique shell-complete category of cubes.*

Proof. It suffices to prove that the inclusion $\widetilde{\square} \subset \widehat{\square}$ is an equality since the category of cubes $\widehat{\square}$ is the maximal category of cubes.

For any $p, q \geqslant 0$, there is the inclusion $\square([p], [q]) \subset \widetilde{\square}([p], [q]) \subset \widehat{\square}([p], [q])$, and one wants to prove the equality $\widetilde{\square}([p], [q]) = \widehat{\square}([p], [q])$.

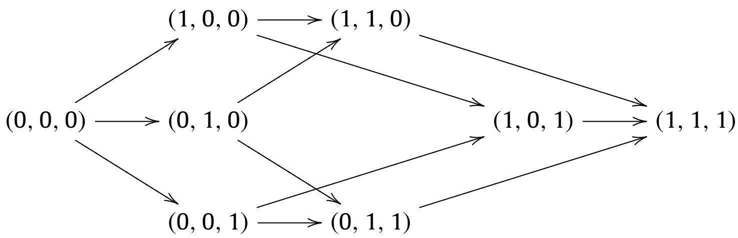


Fig. 4. Poset of vertices of the 3-cube.

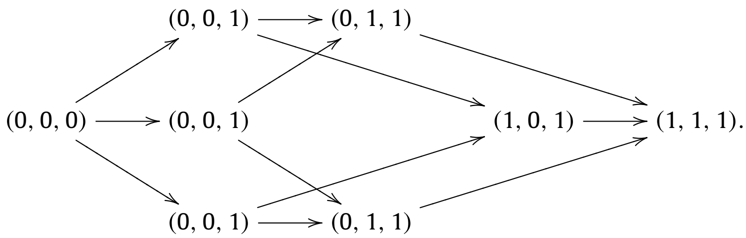


Fig. 5. Image of the vertices of the 3-cube by f.

For $p > q$, one has $\square([p],[q]) = \widetilde{\square}([p],[q]) = \widehat{\square}([p],[q]) = \varnothing$ by Proposition 2.1.17. One has $\widetilde{\square}([0],[q]) = \widehat{\square}([0],[q]) = [q]$ and $\widetilde{\square}([1],[q]) = \widehat{\square}([1],[q])$ by Proposition 2.1.17 again. It remains to prove the equality $\widetilde{\square}([p],[q]) = \widehat{\square}([p],[q])$ for $2 \leqslant p \leqslant q$ for a fixed q by induction on p.

First of all, let us treat the case $p = 2$. Let $f \in \widehat{\square}([2],[q])$. By Proposition 3.1.14, the set map f factors uniquely as a composite of set maps $f : [2] \xrightarrow{\phi} [2] \xrightarrow{\psi} [q]$ with $\phi \in \widehat{\square}$ and $\psi \in \square$. It is easy to see that the set $\widehat{\square}([2],[2])$ consists of the four set maps $Id_{[2]} : (\epsilon_1, \epsilon_2) \mapsto (\epsilon_1, \epsilon_2)$, $\sigma_1 : (\epsilon_1, \epsilon_2) \mapsto (\epsilon_2, \epsilon_1)$, $\gamma_1 : (\epsilon_1, \epsilon_2) \mapsto (\max(\epsilon_1, \epsilon_2), \min(\epsilon_1, \epsilon_2))$ and $\sigma_1\gamma_1 : (\epsilon_1, \epsilon_2) \mapsto (\min(\epsilon_1, \epsilon_2), \max(\epsilon_1, \epsilon_2))$. So $\widetilde{\square}([2],[2]) = \widehat{\square}([2],[2])$ by Propositions 3.1.10 and 3.1.13. Therefore one obtains $\widetilde{\square}([2],[q]) = \widehat{\square}([2],[q])$ for any $q \geqslant 0$.

Let us now treat the case $p \geqslant 3$. Every set map $f \in \widehat{\square}([p],[q])$ for $p \geqslant 3$ gives rise to a map $x : \widehat{\square}[p] \to \widehat{\square}[q]$ such that $x_0 = f$ by Proposition 3.1.1. By composition, one obtains a map $\partial x : \partial\widehat{\square}[p] \subset \widehat{\square}[p] \to \widehat{\square}[q]$. By Proposition 3.1.6, one obtains $2p$ maps $x_i^\alpha : \widehat{\square}[p-1] \to \widehat{\square}[q]$ with $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$ such that $(x_j^\beta)_0 \delta_i^\alpha = (x_i^\alpha)_0 \delta_{j-1}^\beta$ for any $i < j$ and any $\alpha, \beta \in \{0,1\}$. By Proposition 3.1.1, the $2p$ set maps $(x_i^\alpha)_0 : [p-1] \to [q]$ for $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$ belong to $\widehat{\square}$. So, by induction hypothesis, the latter set maps belong to $\widetilde{\square}$ as well. By Proposition 3.1.1 again, one obtains $2p$ maps $y_i^\alpha : \widetilde{\square}[p-1] \to \widetilde{\square}[q]$ with $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$ such that $(y_j^\beta)_0 \delta_i^\alpha = (y_i^\alpha)_0 \delta_{j-1}^\beta$ for any $i < j$ and any $\alpha, \beta \in \{0,1\}$ and such that $(y_i^\alpha)_0 = (x_i^\alpha)_0$ for all $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$. So, by Proposition 3.1.6, one obtains a map $\partial y : \partial\widetilde{\square}[p] \to \widetilde{\square}[q]$ such that $(\partial y)_0 \delta_i^\alpha = y_i^\alpha$ for all $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$. By Theorem 3.1.4 and since $\widetilde{\square}$ is shell complete, the set map $(\partial y)_0 = x_0 = f : [p] \to [q]$ then belongs to $\widetilde{\square}$. The induction on p is complete. □

Theorem 3.1.16. *The category of cubes $\overline{\square}$ generated by the δ_i^α, σ_i and γ_i operators is not shell complete. In other words, the inclusion of small categories $\overline{\square} \subset \widehat{\square}$ is strict.*

Proof. It suffices to find a morphism of $\widehat{\square}$ which does not belong to $\overline{\square}$. Consider the set map $f : [3] \to [3]$ sending the poset of vertices of the 3-cube (Fig. 4) to the poset depicted in Fig. 5.

It is clear that f is adjacency preserving, i.e. $f \in \widehat{\square}$. One has

- $f(0,0,0) = (0,0,0), f(0,1,0) = (0,0,1), f(0,0,1) = (0,0,1), f(0,1,1) = (0,1,1)$, so $f\delta_1^0 = \delta_1^0\sigma_1\gamma_1$.
- $f(1,0,0) = (0,0,1), f(1,0,1) = (1,0,1), f(1,1,0) = (0,1,1), f(1,1,1) = (1,1,1)$, so $f\delta_1^1 = \delta_3^1\sigma_1$.
- $f(0,0,0) = (0,0,0), f(1,0,0) = (0,0,1), f(0,0,1) = (0,0,1), f(1,0,1) = (1,0,1)$, so $f\delta_2^0 = \delta_2^0\sigma_1\gamma_1$.
- $f(0,1,0) = (0,0,1), f(1,1,0) = (0,1,1), f(0,1,1) = (0,1,1), f(1,1,1) = (1,1,1)$, so $f\delta_2^1 = \delta_3^1\sigma_1\gamma_1$.
- $f(0,0,0) = (0,0,0), f(1,0,0) = (0,0,1), f(0,1,0) = (0,0,1), f(1,1,0) = (0,1,1)$, so $f\delta_3^0 = \delta_1^0\sigma_1\gamma_1$.
- $f(0,0,1) = (0,0,1), f(1,0,1) = (1,0,1), f(0,1,1) = (0,1,1), f(1,1,1) = (1,1,1)$, so $f\delta_3^1 = \delta_3^1$.

The six set maps $f\delta_1^0, f\delta_1^1, f\delta_2^0, f\delta_2^1, f\delta_3^0$ and $f\delta_3^1$ belong to $\overline{\square}$, giving rise to a morphism of presheaves $\partial\overline{\square}[3] \to \overline{\square}[3]$ by Proposition 3.1.6. Any set map $g : [2] \to [3]$ of $\widehat{\square}$ factors uniquely as a composite $[2] \xrightarrow{g_1} [2] \xrightarrow{g_2} [3]$ with $g_1 \in \widehat{\square}$ and $g_2 \in \square$ by Proposition 3.1.14. So the set map $f_0 = \sigma_1\gamma_1 : [2] \to [2]$ is the unique set map of $\widehat{\square}$ such that $f\delta_1^0 = \delta_1^0 f_0$. And the set map $f_1 = \sigma_1 : [2] \to [2]$ is the unique set map of $\widehat{\square}$ such that $f\delta_1^1 = \delta_3^1 f_1$. Since $f_0 \neq f_1$, the set map $f : [3] \to [3]$ cannot be a composite of $\sigma_i : [3] \to [3]$ and $\gamma_i : [3] \to [3]$ with $i = 1, 2$ by Propositions 3.1.9 and 3.1.12. Therefore $f \notin \overline{\square}$. □

In fact, we do not know any "small" presentation by generators and relations of the small category $\widehat{\square}$. This is an interesting and open question. It seems to be related to similar questions arising in combinatorics.

Definition 3.1.17. An *extremal path* of $[n]$ is an n-tuple $(A_1, \ldots, A_n)$ of $[n]$ such that $A_1 = 0_n < A_2 < \cdots < A_{n-1} < A_n = 1_n$. The set of extremal paths of $[n]$ is denoted by P_n.

Notation 3.1.18. *Let e_I be the element $(\epsilon_1, \ldots, \epsilon_n)$ of $[n]$ such that $\epsilon_i = 1$ if and only if $i \in I$.*

There is a bijection $p : \Sigma_n \to P_n$ from the n-th symmetric group Σ_n to P_n defined by $p(\sigma) = (e_\varnothing, e_{\{\sigma(1)\}}, e_{\{\sigma(1),\sigma(2)\}}, \ldots, e_{\{1,\ldots,n\}})$.

Proposition 3.1.19. *Let $\sigma \in \Sigma_n$. Let s_i be the transposition $(i\, i+1)$. Then one has the equalities*

$$p(\sigma s_i) = (\sigma_i(e_\varnothing), \sigma_i(e_{\{\sigma(1)\}}), \sigma_i(e_{\{\sigma(1),\sigma(2)\}}), \ldots, \sigma_i(e_{\{1,\ldots,n\}}))$$

and

$$p(\sigma.\overline{\pi}_i) = (\gamma_i(e_\varnothing), \gamma_i(e_{\{\sigma(1)\}}), \gamma_i(e_{\{\sigma(1),\sigma(2)\}}), \ldots, \gamma_i(e_{\{1,\ldots,n\}})),$$

where $\overline{\pi}_i$ are the elementary increasing bubble sort operators (see [27]) defined by $\sigma.\overline{\pi}_i = \sigma$ if $\sigma(i) < \sigma(i+1)$ and $\sigma.\overline{\pi}_i = \sigma s_i$ otherwise.

Proof. Trivial. □

As a corollary, the monoid $\overline{\square}([n], [n])$ is isomorphic to the monoid $\langle s_i, \overline{\pi}_i\rangle_{i=1,\ldots,n}$ of set maps from Σ_n to itself generated by the operators s_i and $\overline{\pi}_i$. In particular, it satisfies the following relations.

Proposition 3.1.20. *Let $n \geqslant 1$. The monoid of set maps from $[n]$ to itself generated by the σ_i and γ_i operators satisfies the relations:*

- *$\sigma_i\sigma_i = Id$, $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $i = j-1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i < j-1$ (the Moore relations for symmetry operators).*
- *$\gamma_i\gamma_i = \gamma_i$, $\gamma_i\gamma_j\gamma_i = \gamma_j\gamma_i\gamma_j$ for $i = j-1$ and $\gamma_i\gamma_j = \gamma_j\gamma_i$ for $i < j-1$ (the Moore relations for transverse degeneracy).*
- *$\gamma_j\sigma_i = \sigma_i\gamma_j$ for $j > i+1$ and $j < i-1$, $\gamma_i\sigma_i = \gamma_i$, $\sigma_{i+1}\gamma_i\sigma_{i+1} = \sigma_i\gamma_{i+1}\sigma_i$.*

Proof. The Moore relations for symmetry operators are explained for example in [21, Theorem 8.1]. Let us prove the Moore relations for transverse degeneracy maps. The relations $\gamma_i\gamma_i = \gamma_i$ and $\gamma_i\gamma_j = \gamma_j\gamma_i$ for $i < j-1$ are obvious. One wants to prove that $\gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}$. It suffices to prove the identity $\gamma_1\gamma_2\gamma_1(\epsilon_1, \epsilon_2, \epsilon_3) = \gamma_2\gamma_1\gamma_2(\epsilon_1, \epsilon_2, \epsilon_3)$. One has

$$\begin{aligned}\gamma_1\gamma_2\gamma_1(\epsilon_1, \epsilon_2, \epsilon_3) &= \gamma_1\gamma_2(\max(\epsilon_1, \epsilon_2), \min(\epsilon_1, \epsilon_2), \epsilon_3)\\ &= \gamma_1(\max(\epsilon_1, \epsilon_2), \max(\min(\epsilon_1, \epsilon_2), \epsilon_3), \min(\epsilon_1, \epsilon_2, \epsilon_3))\\ &= (\max(\epsilon_1, \epsilon_2, \min(\epsilon_1, \epsilon_2), \epsilon_3), \min(\max(\epsilon_1, \epsilon_2), \max(\min(\epsilon_1, \epsilon_2), \epsilon_3)), \min(\epsilon_1, \epsilon_2, \epsilon_3))\\ &= (\max(\epsilon_1, \epsilon_2, \epsilon_3), \min(\max(\epsilon_1, \epsilon_2), \max(\min(\epsilon_1, \epsilon_2), \epsilon_3)), \min(\epsilon_1, \epsilon_2, \epsilon_3))\end{aligned}$$

and

$$\begin{aligned}\gamma_2\gamma_1\gamma_2(\epsilon_1, \epsilon_2, \epsilon_3) &= \gamma_2\gamma_1(\epsilon_1, \max(\epsilon_2, \epsilon_3), \min(\epsilon_2, \epsilon_3))\\ &= \gamma_2(\max(\epsilon_1, \epsilon_2, \epsilon_3), \min(\epsilon_1, \max(\epsilon_2, \epsilon_3)), \min(\epsilon_2, \epsilon_3))\\ &= (\max(\epsilon_1, \epsilon_2, \epsilon_3), \max(\min(\epsilon_1, \max(\epsilon_2, \epsilon_3)), \min(\epsilon_2, \epsilon_3)), \min(\epsilon_1, \max(\epsilon_2, \epsilon_3), \epsilon_2, \epsilon_3))\\ &= (\max(\epsilon_1, \epsilon_2, \epsilon_3), \max(\min(\epsilon_1, \max(\epsilon_2, \epsilon_3)), \min(\epsilon_2, \epsilon_3)), \min(\epsilon_1, \epsilon_2, \epsilon_3)).\end{aligned}$$

It remains to check the equality

$$\min(\max(\epsilon_1, \epsilon_2), \max(\min(\epsilon_1, \epsilon_2), \epsilon_3)) = \max(\min(\epsilon_1, \max(\epsilon_2, \epsilon_3)), \min(\epsilon_2, \epsilon_3))$$

for any $(\epsilon_1, \epsilon_2, \epsilon_3) \in \{0, 1\}^3$. By distributivity of min and max over each other, one has

$$\begin{aligned}\min(\max(\epsilon_1, \epsilon_2), \max(\min(\epsilon_1, \epsilon_2), \epsilon_3)) &= \max(\min(\epsilon_1, \epsilon_2), \min(\epsilon_1, \epsilon_3), \min(\epsilon_2, \epsilon_3))\\ &= \max(\min(\epsilon_1, \max(\epsilon_2, \epsilon_3)), \min(\epsilon_2, \epsilon_3)).\end{aligned}$$

The proof will be complete by establishing the relations between transverse degeneracy maps and symmetry operators. The equalities $\gamma_j\sigma_i = \sigma_i\gamma_j$ for $j > i+1$ and $j < i-1$ and $\gamma_i\sigma_i = \gamma_i$ are obvious. One wants to prove that $\sigma_{i+1}\gamma_i\sigma_{i+1} = \sigma_i\gamma_{i+1}\sigma_i$. It suffices to prove the identity $\sigma_2\gamma_1\sigma_2(\epsilon_1, \epsilon_2, \epsilon_3) = \sigma_1\gamma_2\sigma_1(\epsilon_1, \epsilon_2, \epsilon_3)$. One has

$$\begin{aligned}\sigma_2\gamma_1\sigma_2(\epsilon_1, \epsilon_2, \epsilon_3) &= \sigma_2\gamma_1(\epsilon_1, \epsilon_3, \epsilon_2)\\ &= \sigma_2(\max(\epsilon_1, \epsilon_3), \min(\epsilon_1, \epsilon_3), \epsilon_2)\\ &= (\max(\epsilon_1, \epsilon_3), \epsilon_2, \min(\epsilon_1, \epsilon_3))\end{aligned}$$

and

$$\begin{aligned}\sigma_1\gamma_2\sigma_1(\epsilon_1, \epsilon_2, \epsilon_3) &= \sigma_1\gamma_2(\epsilon_2, \epsilon_1, \epsilon_3)\\ &= \sigma_1(\epsilon_2, \max(\epsilon_1, \epsilon_3), \min(\epsilon_1, \epsilon_3))\\ &= (\max(\epsilon_1, \epsilon_3), \epsilon_2, \min(\epsilon_1, \epsilon_3)).\quad\square\end{aligned}$$

[27, Conjecture 3.5 and Paragraph 3.1.1] suggest the following conjecture:

Conjecture 3.1.21. *Proposition 3.1.20 gives a presentation by generators and relations of the monoid $\overline{\square}([n], [n])$ for every $n \geqslant 2$.*

Functorial interpretation of the labelled directed coskeleton

For $n \geqslant 2$, and for every $a_1, \ldots, a_n \in \Sigma$, the inclusion $\square[a_1, \ldots, a_n] \subset cosk_1^{\square,\Sigma}(\square[a_1, \ldots, a_n]_{\leqslant 1})$ is strict by [18, Proposition 3.15]. The strictness of the latter inclusion means that the concurrent execution of n actions always assemble in $cosk_1^{\square,\Sigma}(\square[a_1, \ldots, a_n]_{\leqslant 1})$ to *several* labelled n-cubes. To remedy this problem, the labelled directed coskeleton construction is introduced in [18]. Its main feature is to select one n-cube (the *non-twisted* one) for each multiset of n actions running concurrently:

Definition 3.1.22. Let K be a 1-dimensional labelled precubical set with $K_0 = [p]$ for some $p \geqslant 0$. The *labelled directed coskeleton* of K is the labelled precubical set $\overrightarrow{cosk}^{\Sigma}(K)$ defined as the subobject of $cosk_1^{\square,\Sigma}(K)$ such that:

- $\overrightarrow{cosk}^{\Sigma}(K)_{\leqslant 1} = cosk_1^{\square,\Sigma}(K)_{\leqslant 1}$,
- for every $n \geqslant 2$, $x \in cosk_1^{\square,\Sigma}(K)_n$ is an n-cube of $\overrightarrow{cosk}^{\Sigma}(K)$ if and only if the set map $x_0 : [n] \to [p]$ is *non-twisted*, i.e. $x_0 : [n] \to [p]$ is a composite[6]

  $$x_0 : [n] \xrightarrow{\phi} [q] \xrightarrow{\psi} [p],$$

  where ψ is a morphism of the small category $\square$ and where ϕ is of the form

  $$(\epsilon_1, \ldots, \epsilon_n) \mapsto (\epsilon_{i_1}, \ldots, \epsilon_{i_q})$$

  such that $\{1, \ldots, n\} \subset \{i_1, \ldots, i_q\}$ and such that the first appearance of ϵ_i is before the first appearance of ϵ_{i+1} in $(\epsilon_{i_1}, \ldots, \epsilon_{i_q})$ for any $1 \leqslant i \leqslant n$ by reading from the left to the right.

The fundamental property of the labelled directed coskeleton is then:

Theorem 3.1.23 (*[18, Proposition 3.21]*). *Let $n \geqslant 1$. Let $(a_1, \ldots, a_n) \in \Sigma^n$. Then one has the isomorphism of labelled precubical sets*

$$\overrightarrow{cosk}^{\Sigma}(\square[a_1, \ldots, a_n]_{\leqslant 1}) \cong \square[a_1, \ldots, a_n].$$

The following theorem gives the functorial interpretation of the labelled directed coskeleton construction.

Theorem 3.1.24. *The category of cubes $\mathcal{A} = \widehat{\square}$ (i.e. the maximal category of cubes containing all adjacency-preserving maps) is the only category of cubes such that, for every $n \geqslant 1$ and every $(a_1, \ldots, a_n) \in \Sigma^n$, there is the isomorphism of labelled $\mathcal{A}$-sets*

$$\mathcal{L}_{\mathcal{A}}(\overrightarrow{cosk}^{\Sigma}(\square[a_1, \ldots, a_n]_{\leqslant 1})) \cong cosk_1^{\mathcal{A},\Sigma}(\mathcal{A}[a_1, \ldots, a_n]_{\leqslant 1})(\cong \mathcal{A}[a_1, \ldots, a_n]).$$

Proof. This is a consequence of Theorem 3.1.23, Proposition 2.1.14 and Theorem 3.1.15. $\square$

The commutative diagram of Fig. 3 proves that the labelled directed coskeleton construction and the labelled transverse symmetric coskeleton functor are equivalent from a directed algebraic topological point of view.

4. Computer-scientific application

A short introduction to process algebra can be found in [38]. An introduction to CCS (Milner's calculus of communicating systems [31]) for mathematicians is available in [18].

4.1. Parallel composition (local case)

We want to explain in this section how it is possible to use the labelled transverse symmetric coskeleton functor to model the parallel composition in CCS of two labelled cubes representing two higher-dimensional transitions.

The fibered product in CCS

Notation 4.1.1. $\mathcal{L} := \mathcal{L}_{\widehat{\square}}$, $Sh := Sh_{\widehat{\square}}$ *and* $\omega = \omega_{\widehat{\square}}$.

The set $\Sigma\backslash\{\tau\}$, which may be empty, is now supposed to be equipped with an involution $a \mapsto \overline{a}$. In Milner's calculus of communicating systems (CCS) [31], which is the only case treated of this paper, one has $a \neq \overline{a}$. However, this mathematical hypothesis is not used in this paper. The involution on $\Sigma\backslash\{\tau\}$ is used only in Definition 4.1.2 of the fibered product (and in the new definition given with the proof of Theorem 4.1.8) of two 1-dimensional labelled (transverse symmetric) precubical sets over Σ. For other examples of fibered products over other synchronization algebras than the one of CCS, see [31,38].

Definition 4.1.2. Let K and L be two 1-dimensional labelled (transverse symmetric) precubical sets. The *fibered product* of K and L *over* Σ is the 1-dimensional labelled precubical set $K \times_{\Sigma} L$ defined as follows:

- $(K \times_{\Sigma} L)_0 = K_0 \times L_0$,
- $(K \times_{\Sigma} L)_1 = (K_1 \times L_0) \sqcup (K_0 \times L_1) \sqcup \{(x, y) \in K_1 \times L_1, \overline{\ell(x)} = \ell(y)\}$,

[6] The factorization is necessarily unique.

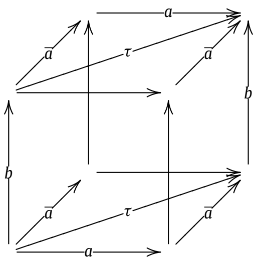


Fig. 6. Representation of $\square[a, b]_{\leqslant 1} \times_\Sigma \square[\overline{a}]$, labelled over Σ.

- $\partial_1^\alpha(x, y) = (\partial_1^\alpha(x), y)$ for any $(x, y) \in K_1 \times L_0$,
- $\partial_1^\alpha(x, y) = (x, \partial_1^\alpha(y))$ for any $(x, y) \in K_0 \times L_1$,
- $\partial_1^\alpha(x, y) = (\partial_1^\alpha(x), \partial_1^\alpha(y))$ for any $(x, y) \in K_1 \times L_1$,
- $\ell(x, y) = \ell(x)$ for any $(x, y) \in K_1 \times L_0$,
- $\ell(x, y) = \ell(y)$ for any $(x, y) \in K_0 \times L_1$,
- $\ell(x, y) = \tau$ for any $(x, y) \in K_1 \times L_1$ with $\overline{\ell(x)} = \ell(y)$.

The 1-cubes (x, y) of $(K \times_\Sigma L)_1 \cap (K_1 \times L_1)$ are called *synchronizations* of x and y.

The 1-dimensional labelled precubical set $K \times_\Sigma L$ is the key ingredient in the definition of the synchronized tensor product of labelled precubical sets given in [18], and recalled in Section 4.2. Fig. 6 describes $\square[a, b]_{\leqslant 1} \times_\Sigma \square[\overline{a}]$.

We want to prove in this section that, for every $m \geqslant 0$ and $n \geqslant 0$, for every $a_1, \ldots, a_{m+n} \in \Sigma$, the labelled precubical set $\overrightarrow{cosk}^\Sigma(\square[a_1, \ldots, a_m]_{\leqslant 1} \times_\Sigma \square[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1})$ can be interpreted as a full labelled coskeleton in the category of labelled transverse symmetric precubical sets.

Proposition 4.1.3. *Let K be a precubical set. For any p-cube x of $\omega\mathcal{L}(K)$ with $p \geqslant 0$, there exists a p-cube y of $K \subset \omega\mathcal{L}(K)$ and a map $\mu \in \widehat{\square}([p], [p])$ such that $x = \mu^*(y)$, where $\mu^* : \mathcal{L}(K)_p \to \mathcal{L}(K)_p$ is the image of μ by the presheaf $\mathcal{L}(K) \in \widehat{\square}^{op}$**Set**.*

Proof. Let

$$x \in \omega\mathcal{L}(K)_p \cong \varinjlim_{\square[n]\to K} \widehat{\square}[n]_p \cong \varinjlim_{\square[n]\to K} \widehat{\square}([p], [n]).$$

Then there exists an n-cube $z : \square[n] \to K$ and $\overline{x} \in \widehat{\square}([p], [n])$ (the copy corresponding to z) such that $z \circ \overline{x} = x$. By Proposition 3.1.14, $\overline{x} : [p] \to [n]$ factors as a composite $[p] \xrightarrow{\mu} [p] \xrightarrow{\phi} [n]$ with $\mu \in \widehat{\square}$ and $\phi \in \square$. Then $\phi^*(z)$ is a p-cube of K and $\mu^*(\phi^*(z)) = x$. So $y = \phi^*(z)$ is a solution. $\square$

Note that the decomposition $x = \mu^*(y)$ is unique. But this fact will not be used in what follows. Indeed, let us consider another decomposition $x = \mu'^*(y')$, $z' : \square[n'] \to K$, $z' \circ \overline{x}' = x$, where $\overline{x}' : [p] \xrightarrow{\mu'} [p] \xrightarrow{\phi'} [n']$ belongs to the copy of $\widehat{\square}([p], [n'])$ corresponding to z' and $y' = \phi'^*(z')$. Since $\overline{x} \in \widehat{\square}([p], [n])$ and $\overline{x}' \in \widehat{\square}([p], [n'])$ are equal in the colimit calculating $\omega\mathcal{L}(K)_p$, the two sets $\widehat{\square}([p], [n])$ and $\widehat{\square}([p], [n'])$ are related in the colimit by a zig-zag sequence of maps of $\square$ (this is the crucial point) relating $\overline{x}$ and $\overline{x}'$. We can suppose that there exists a map $h : [n] \to [n']$ such that $h \circ \overline{x} = \overline{x}'$ and such that $z = z' \circ h$. Then the composite $[p] \xrightarrow{\mu} [p] \xrightarrow{\phi} [n] \xrightarrow{h} [n']$ gives the unique decomposition of $\overline{x}'$ as the composite of a map of $\widehat{\square}([p], [p])$ followed by a map of $\square$ by Proposition 3.1.14. Thus, $\mu = \mu'$ and $h \circ \phi = \phi'$. Therefore $y' = \phi'^*(z') = z' \circ h \circ \phi = z \circ \phi = y$.

We will need the following combinatorial lemma twice in what follows.

Proposition 4.1.4. *Let $x : [p] \to [r]$ be a strictly increasing set map. Then there exists a unique decomposition of x as*

$$[p] \xrightarrow{\mu} [p'] \xrightarrow{\phi} [q] \xrightarrow{\psi} [r]$$

such that ϕ is non-twisted, $\psi \in \square$ and $\mu = (g_1, \ldots, g_{p'})$, where the $g_i : [p] \to [1]$ are non-constant and mutually distinct (i.e. $g_i = g_j$ implies $i = j$). Moreover, $p \leqslant p'$, μ is strictly increasing, and it is also adjacency preserving if and only if $p = p'$.

Proof. Let $x = (x^{(1)}, \ldots, x^{(r)})$, where the maps $x^{(i)} : [p] \to [1]$ are the r projection maps. The map ψ is necessarily the composite $\delta_{i_1}^{\alpha_1} \ldots \delta_{i_s}^{\alpha_s}$, where $\{i_1 > \cdots > i_s\} = \{i \in \{1, \ldots, r\} \mid x^{(i)} = 0 \text{ or } x^{(i)} = 1\}$ and where $x^{(i_k)}(\epsilon_1, \ldots, \epsilon_p) = \alpha_k$ for all $(\epsilon_1, \ldots, \epsilon_p) \in [p]$. Let $A \subset \{1, \ldots, r\}$ be the subset of i such that $x^{(i)}$ is a non-constant map. Consider the equivalence relation on the set A defined by $i \sim j$ if and only if $x^{(i)} = x^{(j)}$. Let $p' = \mathrm{card}(A/\sim)$, where $\mathrm{card}(S)$ denotes the cardinality of the set S. The map $\mu = (x^{(j_1)}, \ldots, x^{(j_{p'})})$ is obtained by taking in each equivalence class of $\sim$ the representative $x^{(j)}$ with the smallest j and by imposing $j_1 < \cdots < j_{p'}$. The non-twisted map ϕ is then defined so that the repetitions encode the equivalence relation $\sim$. Since x is strictly increasing, the set map μ is also strictly increasing. Therefore $p \leqslant p'$. Since none of the set maps $x^{(j_k)}$ are constant, one has $\mu(0, \ldots, 0) = (0, \ldots, 0)$ and $\mu(1, \ldots, 1) = (1, \ldots, 1)$. Thus, if $p < p'$, then μ cannot be adjacency preserving. And if $p = p'$, then μ is adjacency preserving by Proposition 2.1.6. This decomposition is clearly unique. $\square$

Theorem 4.1.5. *Let $\square[a_1,\ldots,a_m]$ and $\square[a_{m+1},\ldots,a_{m+n}]$ be two labelled cubes with $m \geqslant 0$ and $n \geqslant 0$. Then there is an inclusion of presheaves*

$$\mathcal{L}\left(\overrightarrow{cosk}^{\Sigma}\left(\square[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\square[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\right)\subset cosk_1^{\widehat{\square},\Sigma}\left(\widehat{\square}[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right).$$

Moreover, when $\Sigma\backslash\{\tau\}$ is non-empty, there exist two labelled cubes such that the above inclusion is strict.

Proof. Let K be a labelled precubical set. Consider the composite set map, natural with respect to K:

$$\begin{aligned}
&\left(\square^{op}\mathbf{Set}{\downarrow}!\Sigma\right)\left(K,\overrightarrow{cosk}^{\Sigma}\left(\square[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\square[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\right)\\
&\rightarrow \left(\square^{op}\mathbf{Set}{\downarrow}!\Sigma\right)\left(K,cosk_1^{\square,\Sigma}\left(\square[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\square[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\right)\\
&\cong \left(\square_1^{op}\mathbf{Set}{\downarrow}!\Sigma\right)\left(K_{\leqslant 1},\square[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\square[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\\
&\cong \left(\widehat{\square}_1^{op}\mathbf{Set}{\downarrow}Sh\mathcal{L}(!\Sigma)\right)\left(\mathcal{L}(K)_{\leqslant 1},\widehat{\square}[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\\
&\cong \left(\widehat{\square}^{op}\mathbf{Set}{\downarrow}Sh\mathcal{L}(!\Sigma)\right)\left(\mathcal{L}(K),cosk_1^{\widehat{\square},\Sigma}\left(\widehat{\square}[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\right),
\end{aligned}$$

the first and last isomorphisms by adjunction and the second one by Proposition 2.1.28. Take $K=\overrightarrow{cosk}^{\Sigma}(\square[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\square[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1})$. The identity of K yields a map of labelled transverse symmetric precubical sets

$$\begin{aligned}
f:\mathcal{L}&\left(\overrightarrow{cosk}^{\Sigma}\left(\square[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\square[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\right)\\
&\longrightarrow cosk_1^{\widehat{\square},\Sigma}\left(\widehat{\square}[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right).
\end{aligned}$$

The case $K=\square[p]$ for $p\geqslant 0$ gives the injection of sets

$$\left(\overrightarrow{cosk}^{\Sigma}\left(\square[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\square[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\right)_p\subset\left(cosk_1^{\widehat{\square},\Sigma}\left(\widehat{\square}[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\right)_p.$$

The set map f_p is therefore one-to-one for every $p\geqslant 0$ by Propositions 4.1.3 and 4.1.4. Suppose now that $\Sigma\backslash\{\tau\}$ is non-empty. Let $a\in\Sigma\backslash\{\tau\}$. The transverse symmetric precubical set

$$cosk_1^{\widehat{\square},\Sigma}(\widehat{\square}[a,a]_{\leqslant 1}\times_{\Sigma}\widehat{\square}[\overline{a},\overline{a}]_{\leqslant 1})$$

contains a 2-cube x such that $x_0(0,0)=(0,0,0,0)$, $x_0(0,1)=(1,0,0,1)$, $x_0(1,0)=(1,0,1,0)$ and $x_0(1,1)=(1,1,1,1)$ since all 1-cubes of x are labelled by τ. The set map $x_0:[2]\rightarrow[4]$ cannot be written as a composite $[2]\xrightarrow{\mu}[2]\xrightarrow{\phi}[4]$ with $\mu\in\widehat{\square}([2],[2])$ and $\phi:[2]\rightarrow[4]$ non-twisted (see Definition 3.1.22) since $x_0=(x^{(1)},x^{(2)},x^{(3)},x^{(4)})$, where the set maps $x^{(i)}:[2]\rightarrow[1]$ are four different set maps. So, by Proposition 4.1.3, one obtains

$$x\notin\mathcal{L}\left(\overrightarrow{cosk}^{\Sigma}(\square[a,a]_{\leqslant 1}\times_{\Sigma}\square[\overline{a},\overline{a}]_{\leqslant 1})\right).$$

Therefore the inclusion of presheaves

$$\mathcal{L}\left(\overrightarrow{cosk}^{\Sigma}\left(\square[a,a]_{\leqslant 1}\times_{\Sigma}\square[\overline{a},\overline{a}]_{\leqslant 1}\right)\right)\subset cosk_1^{\widehat{\square},\Sigma}\left(\widehat{\square}[a,a]_{\leqslant 1}\times_{\Sigma}\widehat{\square}[\overline{a},\overline{a}]_{\leqslant 1}\right)$$

is strict. □

Functorial construction of the parallel composition

Theorem 4.1.5 does not mean that the labelled coskeleton functor of the category of labelled transverse symmetric precubical sets is badly behaved. The coskeleton functor of $\widehat{\square}^{op}\mathbf{Set}$ does the job it is designed for: filling all compatibly labelled shells. To avoid this problem, we have to keep the memory of what is synchronized by τ, as depicted in Fig. 7. By labelling the 1-cube $x(0,*)$ by $(2,3)$ instead of τ, the 1-cube $x(1,*)$ by $(2,4)$ instead of τ, the 1-cube $x(*,0)$ by $(1,3)$ instead of τ and the 1-cube $x(*,1)$ by $(1,4)$ instead of τ, it becomes impossible to fill the new shell since the opposite faces are not labelled anymore in the same way. Hence the definition of the new labelling:

Notation 4.1.6. *Let $a_1,\ldots,a_{m+n}\in\Sigma$ with $m\geqslant 0$ and $n\geqslant 0$. Let $\overline{\Sigma}:=\Sigma\sqcup(\mathbb{N}^*\times\mathbb{N}^*)$, where $\mathbb{N}^*$ is the set of strictly positive integers. Let us define the 1-dimensional labelled (transverse symmetric) precubical set $\widehat{\square}[a_1,\ldots,a_m]_{\leqslant 1}\overline{\times}_{\Sigma}\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}$ as follows (the boxed part is the only new part):*

- *The underlying 1-dimensional precubical set is the one of*

  $$\widehat{\square}[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}$$

- *The labelling map is defined by:*
  - $\ell(x,y)=\ell(x)$ *for any* $(x,y)\in\widehat{\square}[a_1,\ldots,a_m]_1\times\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_0$,

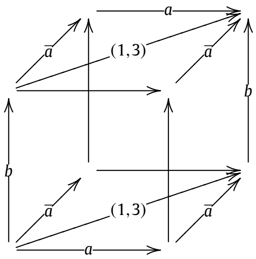


Fig. 7. Representation of $\square[a,b]_{\leqslant 1}\overline{\times}_\Sigma\square[\overline{a}]$, labelled over $\overline{\Sigma}=\Sigma\sqcup(\mathbb{N}^*\times\mathbb{N}^*)$.

- $\ell(x,y)=\ell(y)$ *for any* $(x,y)\in\widehat{\square}[a_1,\dots,a_m]_0\times\widehat{\square}[a_{m+1},\dots,a_{m+n}]_1$,
- $\boxed{\ell(x,y)=(r,s)\in\mathbb{N}^*\times\mathbb{N}^*}$ [7] *for any* $(x,y)\in\widehat{\square}[a_1,\dots,a_m]_1\times\widehat{\square}[a_{m+1},\dots,a_{m+n}]_1$ *with* $\overline{\ell(x)}=\ell(y)$, *where* $1\leqslant r\leqslant m$ *and* $m+1\leqslant s\leqslant m+n$ *are the unique integers such that* $(x_0(\alpha),y_0(\alpha))=\delta_s^\alpha\delta_r^\alpha(X)$ *for some* $X\in[m+n-2]$ *and for* $\alpha=0,1$.

Lemma 4.1.7. *Let* $c:\widehat{\square}[1]\to\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\overline{\times}_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}$ *be a 1-cube of* $\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\overline{\times}_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}$. *Then the set map* $c_0:[1]\to[m+n]$ *satisfies* $c_0(0)<c_0(1)$ *and there are two mutually exclusive possibilities:*

- $d(c_0(0),c_0(1))=1$ *and* $c_0(\alpha)=\delta_r^\alpha(X)$ *for some* $X\in[m+n-1]$ *with* $1\leqslant r\leqslant m+n$ *and for* $\alpha=0,1$. *In this case,* $\ell(c)=a_r\in\Sigma$.
- $d(c_0(0),c_0(1))=2$ *and* $c_0(\alpha)=\delta_s^\alpha\delta_r^\alpha(X)$ *for some* $X\in[m+n-2]$ *with* $1\leqslant r\leqslant m$ *and* $m+1\leqslant s\leqslant m+n$ *and for* $\alpha=0,1$. *In this case,* $\ell(c)=(r,s)\in\mathbb{N}^*\times\mathbb{N}^*$.

Proof. Obvious. □

Note that Lemma 4.1.7 holds for $\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\times_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}$ as well by replacing $\ell(c)=(r,s)\in\mathbb{N}^*\times\mathbb{N}^*$ in the last sentence by $\ell(c)=\tau$.

We are now ready to give the categorical interpretation of the labelled directed coskeleton construction when applied to the fibered product of two 1-dimensional labelled precubical sets.

Theorem 4.1.8. *Let* $\square[a_1,\dots,a_m]$ *and* $\square[a_{m+1},\dots,a_{m+n}]$ *be two labelled cubes with* $m\geqslant 0$ *and* $n\geqslant 0$. *Then one has the isomorphism of labelled transverse symmetric precubical sets*

$$\mathcal{L}\left(\overrightarrow{cosk}^\Sigma\left(\square[a_1,\dots,a_m]_{\leqslant 1}\times_\Sigma\square[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}\right)\right)\cong cosk_1^{\widehat{\square},\overline{\Sigma}}\left(\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\overline{\times}_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}\right),$$

where the right-hand labelled transverse symmetric precubical set over $\overline{\Sigma}$ *is viewed as labelled over* Σ *by composing its labelling map with the morphism of transverse symmetric precubical sets* $Sh\mathcal{L}(!\overline{\Sigma})\to Sh\mathcal{L}(!\Sigma)$, *where the set map* $\overline{\Sigma}\to\Sigma$ *is defined as the identity on* Σ *and by the mapping* $(p,q)\mapsto\tau$ *on the complement.*

Note that, with $m=0$ or $n=0$, we have the isomorphism of Theorem 3.1.24.

Proof. $\boxed{\text{Injectivity}}$. There is an inclusion of presheaves

$$cosk_1^{\widehat{\square},\overline{\Sigma}}\left(\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\overline{\times}_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}\right)\subset cosk_1^{\widehat{\square},\Sigma}\left(\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\times_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}\right)$$

since fewer shells are filled in the left-hand term than in the right-hand term because of the labelling over $\overline{\Sigma}$. Moreover, one has the equalities

$$\begin{aligned}&\left(\mathcal{L}\left(\overrightarrow{cosk}^\Sigma\left(\square[a_1,\dots,a_m]_{\leqslant 1}\times_\Sigma\square[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}\right)\right)\right)_{\leqslant 1}\\&=\left(cosk_1^{\widehat{\square},\overline{\Sigma}}\left(\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\overline{\times}_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}\right)\right)_{\leqslant 1}\\&=\left(cosk_1^{\widehat{\square},\Sigma}\left(\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\times_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}\right)\right)_{\leqslant 1}\end{aligned}$$

by Propositions 2.3.4 and 2.1.18. Let

$$x\in\left(\mathcal{L}\left(\overrightarrow{cosk}^\Sigma(\square[a_1,\dots,a_m]_{\leqslant 1}\times_\Sigma\square[a_{m+1},\dots,a_{m+n}]_{\leqslant 1})\right)\right)_p$$

[7] Instead of $\ell(x,y)=\tau$.

with $p \geqslant 2$. Then $x = \mu^*(y)$, where $\mu : [p] \to [p]$ is an adjacency-preserving map and where y is a p-cube of the labelled precubical set

$$\overrightarrow{cosk}^{\Sigma}(\square[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \square[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1})$$

by Proposition 4.1.3. The map of 1-dimensional precubical sets

$$y_{\leqslant 1} : \square[p]_{\leqslant 1} \longrightarrow (\overrightarrow{cosk}^{\Sigma}(\square[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \square[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}))_{\leqslant 1}$$

induces by Proposition 2.1.28 a map of transverse symmetric 1-dimensional precubical sets

$$y_{\leqslant 1} : \widehat{\square}[p]_{\leqslant 1} \to \widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}.$$

The latter induces a unique map $\overline{y} : \widehat{\square}[p]_{\leqslant 1} \to \widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \overline{\times}_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}$ of 1-dimensional precubical sets which is this time labelled over $\overline{\Sigma}$ since the underlying precubical sets of $\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}$ and $\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \overline{\times}_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}$ are equal. The map $\overline{y}$ induces by adjunction a unique p-dimensional transverse symmetric cube of

$$cosk_1^{\widehat{\square}, \overline{\Sigma}}(\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \overline{\times}_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}).$$

Thus the inclusion

$$\mathcal{L}\left(\overrightarrow{cosk}^{\Sigma}\left(\square[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \square[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}\right)\right) \subset cosk_1^{\widehat{\square}, \Sigma}\left(\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}\right)$$

factors uniquely as a composite of inclusions

$$\begin{aligned}\mathcal{L}\left(\overrightarrow{cosk}^{\Sigma}\left(\square[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \square[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}\right)\right) &\subset cosk_1^{\widehat{\square}, \overline{\Sigma}}\left(\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \overline{\times}_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}\right) \\ &\subset cosk_1^{\widehat{\square}, \Sigma}\left(\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}\right).\end{aligned}$$

Let us call f again the inclusion

$$\mathcal{L}\left(\overrightarrow{cosk}^{\Sigma}\left(\square[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \square[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}\right)\right) \subset cosk_1^{\widehat{\square}, \overline{\Sigma}}\left(\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \overline{\times}_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}\right).$$

It then remains to prove that, for every $p \geqslant 2$, the set map f_p is onto.

Surjectivity. Let $x : \widehat{\square}[p] \to cosk_1^{\widehat{\square}, \overline{\Sigma}}(\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \overline{\times}_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1})$ be a p-dimensional transverse symmetric cube of $cosk_1^{\widehat{\square}, \overline{\Sigma}}(\widehat{\square}[a_1, \ldots, a_m]_{\leqslant 1} \overline{\times}_{\Sigma} \widehat{\square}[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1})$ with $p \geqslant 2$. Let $x_0 = (x^{(1)}, \ldots, x^{(m+n)})$, where the $x^{(i)} : [p] \to [1]$ are the $m + n$ projections. Let us apply the decomposition of Proposition 4.1.4. Let $\mu = (x^{(j_1)}, \ldots, x^{(j_{p'})})$. If one had $p < p'$, then there would exist a 1-cube $c : [1] \to [p]$ such that $d(\mu(c(0)), \mu(c(1))) > 1$. By Lemma 4.1.7, one would have

$$1 < d(\mu(c(0)), \mu(c(1))) \leqslant d(\psi\phi\mu(c(0)), \psi\phi\mu(c(1))) \leqslant 2,$$

and therefore $d(\mu(c(0)), \mu(c(1))) = 2$. Thus, one would have $\mu c(\alpha) = \delta_v^\alpha \delta_u^\alpha(X)$ for some $u < v$, $X \in [p' - 2]$ and for $\alpha = 0, 1$. By Lemma 4.1.7, one obtains $\psi\phi\mu c(\alpha) = \delta_{j_v}^\alpha \delta_{j_u}^\alpha(Z)$ for some cube $Z \in [m + n - 2]$ and for $\alpha = 0, 1$,[8] and finally $\ell(\psi\phi\mu c) = (j_u, j_v)$.

Use of the particular labelling of $\overline{\times}_\Sigma$. The crucial point is that the labelling of $\overline{\times}_\Sigma$ implies $x^{(j_u)} = x^{(j_v)}$, which contradicts the definition of μ. By Proposition 2.2.13, the commutative word $W = \ell(xc_1) \ldots \ell(xc_p)$ of the free commutative monoid without unit generated by $\overline{\Sigma}$ does not depend on the maximal path $(c_1, \ldots, c_p)$ of $\widehat{\square}[p]$. And one of the labels is necessarily (j_u, j_v). If $(\epsilon_1, \ldots, \epsilon_p) = (0, \ldots, 0)$, then $x^{(j_u)}(\epsilon_1, \ldots, \epsilon_p) = x^{(j_v)}(\epsilon_1, \ldots, \epsilon_p) = 0$. Let us suppose now that $(\epsilon_1, \ldots, \epsilon_p) \neq (0, \ldots, 0)$. By Proposition 2.2.13, for every maximal path $(c_1, \ldots, c_r)$ of the r-subcube from $(0, \ldots, 0)$ to $(\epsilon_1, \ldots, \epsilon_p)$, the commutative word $W' = \ell(xc_1) \ldots \ell(xc_r)$ is a subword of W which does not depend on $(c_1, \ldots, c_r)$. If (j_u, j_v) belongs to W', then $x^{(j_u)}(\epsilon_1, \ldots, \epsilon_p) = x^{(j_v)}(\epsilon_1, \ldots, \epsilon_p) = 1$. If (j_u, j_v) does not belong to W', then it belongs to the complement of W' in W. So $x^{(j_u)}(\epsilon_1, \ldots, \epsilon_p) = x^{(j_v)}(\epsilon_1, \ldots, \epsilon_p) = 0$. Hence, $x^{(j_u)} = x^{(j_v)}$, which is the desired contradiction.

End of the proof. Hence, one obtains the equality $p = p'$ thanks to the particular labelling of $\overline{\times}_\Sigma$. The map μ is therefore adjacency preserving by Proposition 2.1.6. Note that x_0 has no reason to be adjacency preserving. By definition of the labelled directed coskeleton, there exists a p-cube

$$y : \square[p] \longrightarrow \overrightarrow{cosk}^{\Sigma}\left(\square[a_1, \ldots, a_m]_{\leqslant 1} \times_{\Sigma} \square[a_{m+1}, \ldots, a_{m+n}]_{\leqslant 1}\right)$$

8 So far, the particular labelling of $\overline{\times}_\Sigma$ has not been used in the surjectivity part of the proof. In the counterexample of Theorem 4.1.5, one has $p = 2$ and $p' = 4$. So we cannot yet conclude that $p = p'$.

such that $y_0 = \psi\phi$. Then $\mathcal{L}(y)$ is a p-cube of the labelled transverse symmetric precubical set $\mathcal{L}(\overrightarrow{cosk}^{\Sigma}(\square[a_1,\dots,a_m]_{\leqslant 1} \times_\Sigma \square[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}))$ such that $\mathcal{L}(y)_0 = \psi\phi$. Then $\mu^*(\mathcal{L}(y))$ is a p-cube of the labelled transverse symmetric precubical set

$$\mathcal{L}\left(\overrightarrow{cosk}^{\Sigma}(\square[a_1,\dots,a_m]_{\leqslant 1} \times_\Sigma \square[a_{m+1},\dots,a_{m+n}]_{\leqslant 1})\right)$$

such that $(\mu^*(\mathcal{L}(y)))_0 = \psi\phi\mu$. By construction of f, the p-cube $f(\mu^*(\mathcal{L}(y)))$ of the labelled transverse symmetric precubical set $cosk_1^{\widehat{\square},\overline{\Sigma}}(\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\overline{\times}_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1})$ satisfies $(f(\mu^*(\mathcal{L}(y))))_0 = \psi\phi\mu = x_0$. Since there is at most one 1-cube between two vertices of $\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\overline{\times}_\Sigma\widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}$, this implies $(f(\mu^*(\mathcal{L}(y))))_{\leqslant 1} = x_{\leqslant 1}$, and therefore $f(\mu^*(\mathcal{L}(y))) = x$ by adjunction. So f is an isomorphism of labelled transverse symmetric precubical sets. □

Theorem 4.1.8 is of course false for any other category of cubes than $\widehat{\square}$. Indeed, the particular case $n = 0$ and $a_1 = \cdots = a_m = \tau$ gives back the inclusion of presheaves

$$\mathcal{A}[m] \cong \mathcal{L}_{\mathcal{A}}\left(\overrightarrow{cosk}^{\Sigma}\left(\square[m]_{\leqslant 1}\right)\right) \subset cosk_1^{\mathcal{A}}\left(\mathcal{A}[m]_{\leqslant 1}\right),$$

which is an equality if and only if the category of cubes $\mathcal{A}$ is shell complete, so if and only if $\mathcal{A} = \widehat{\square}$ by Theorem 3.1.15. The crucial point in the proof of Theorem 4.1.8 is that the map $\mu : [p] \to [p]$ must belong to $\mathcal{A}$. Therefore, it is really needed to work with the whole category $\widehat{\square}$ of all adjacency-preserving maps.

4.2. Parallel composition (global case)

We can now relate the synchronized tensor product of labelled precubical sets with the synchronized tensor product of labelled transverse symmetric precubical sets. First of all, let us give the definition of these two synchronized tensor products.

Definition

Definition 4.2.1 (*[18]*). Let K and L be two labelled precubical sets. The *tensor product with synchronization* (or *synchronized tensor product*) of K and L is

$$K \otimes_\Sigma L := \varinjlim_{\square[m]\to K}\ \varinjlim_{\square[n]\to L} \overrightarrow{cosk}^{\Sigma}(\square[m]_{\leqslant 1} \times_\Sigma \square[n]_{\leqslant 1}).$$

Definition 4.2.2. Let K and L be two labelled transverse symmetric precubical sets. The *tensor product with synchronization* (or *synchronized tensor product*) of K and L is

$$K \otimes_\Sigma L := \varinjlim_{\widehat{\square}[m]\to K}\ \varinjlim_{\widehat{\square}[n]\to L} cosk_1^{\widehat{\square},\overline{\Sigma}}(\widehat{\square}[m]_{\leqslant 1}\overline{\times}_\Sigma\widehat{\square}[n]_{\leqslant 1}).$$

The two constructions coincide

In which follows, the category of small categories is denoted by **Cat**. Let $H : I \longrightarrow$ **Cat** be a functor from a small category I to **Cat**. The *Grothendieck construction* $I \int H$ is the category defined as follows [35]: the objects are the pairs (i, a), where i is an object of I and a is an object of $H(i)$; a morphism $(i, a) \to (j, b)$ consists in a map $\phi : i \to j$ and in a map $h : H(\phi)(a) \to b$.

Lemma 4.2.3. *Let $\mathcal{A}$ be a category of cubes. Let I be a small category, and $i \mapsto K^i$ be a functor from I to the category of labelled $\mathcal{A}$-sets. Let $K = \varinjlim_i K^i$. Let $H : I \to$ **Cat** be the functor defined by $H(i) = \mathcal{A}{\downarrow}K^i$. Then the functor $\iota : I \int H \to \mathcal{A}{\downarrow}K$ defined by $\iota(i, \mathcal{A}[m] \to K^i) = (\mathcal{A}[m] \to K)$ is final in the sense of [32]; that is to say the comma category $k{\downarrow}\iota$ is non-empty and connected for all objects k of $\mathcal{A}{\downarrow}K$.*

Proof. The proof is similar to the proof of [18, Lemma A.1]. □

Proposition 4.2.4. *Let $\mathcal{A}$ be a category of cubes. Let $F : \mathcal{A} \times \mathcal{A} \to \mathcal{C}$ be a functor, where $\mathcal{C}$ is a cocomplete category. Let $\widehat{F} : (\mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)) \times (\mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma)) \to \mathcal{C}$ be the functor defined by*

$$\widehat{F}(K, L) := \varinjlim_{\mathcal{A}[m]\to K}\ \varinjlim_{\mathcal{A}[n]\to L} F([m], [n]).$$

Then, for any labelled $\mathcal{A}$-set L, the two functors

$$\widehat{F}(L, -) : \mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma) \longrightarrow \mathcal{C}$$

and

$$\widehat{F}(-, L) : \mathcal{A}^{op}\mathbf{Set}{\downarrow}Sh_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\Sigma) \longrightarrow \mathcal{C}$$

are colimit preserving.

Proof. The proof is similar to the proof of [18, Proposition A.2]. Let $K = \varinjlim_i K^i$ be a colimit of labelled $\mathcal{A}$-sets. By definition, one has the isomorphism

$$\varinjlim_i \widehat{F}(K^i, L) \cong \varinjlim_i \varinjlim_{\mathcal{A}[m]\to K^i} \varinjlim_{\mathcal{A}[n]\to L} F([m],[n]).$$

Consider the functor $H : I \longrightarrow \mathbf{Cat}$ defined by $H(i) = \mathcal{A}{\downarrow}K^i$. Consider the functors $F_i : H(i) \longrightarrow \mathcal{C}$ defined by

$$F_i(\mathcal{A}[m] \to K^i) = \varinjlim_{\mathcal{A}[n]\to L} F([m],[n]).$$

Consider the functor $\overline{F} : I \int H \longrightarrow \mathcal{C}$ defined by

$$\overline{F}(i, \mathcal{A}[m] \to K^i) = \varinjlim_{\mathcal{A}[n]\to L} F([m],[n]).$$

Then the composite $H(i) \subset I \int H \to \mathcal{C}$ is exactly F_i. Therefore one has the isomorphism

$$\varinjlim_i \varinjlim_{\mathcal{A}[m]\to K^i} \varinjlim_{\mathcal{A}[n]\to L} F([m],[n]) \cong \varinjlim_{(i,\mathcal{A}[m]\to K^i)} \varinjlim_{\mathcal{A}[n]\to L} F([m],[n])$$

by [9, Proposition 40.2]. The functor $\iota : I \int H \to \mathcal{A}{\downarrow}K$ defined by $\iota(i, \mathcal{A}[m] \to K^i) = (\mathcal{A}[m] \to K)$ is final in the sense of [32] by Lemma 4.2.3. Therefore, by [32, p. 213, Theorem 1] or [25, Theorem 14.2.5], one has the isomorphism

$$\varinjlim_{(i,\mathcal{A}[m]\to K^i)} \varinjlim_{\mathcal{A}[n]\to L} F([m],[n]) \cong \varinjlim_{\mathcal{A}[m]\to K} \varinjlim_{\mathcal{A}[n]\to L} F([m],[n]) =: \widehat{F}(K, L). \quad \square$$

Corollary 4.2.5. *Let I be a small category. Let $i \mapsto K^i$ be a functor from I to the category of labelled transverse symmetric precubical sets, and let L be a labelled transverse symmetric precubical set. Then one has the natural isomorphism*

$$(\varinjlim_i K^i) \otimes_\Sigma L \cong \varinjlim_i (K^i \otimes_\Sigma L).$$

Theorem 4.2.6. *Let K and L be two labelled precubical sets. Then there is the natural isomorphism of labelled transverse symmetric precubical sets*

$$\mathcal{L}(K \otimes_\Sigma L) \cong \mathcal{L}(K) \otimes_\Sigma \mathcal{L}(L).$$

Proof. One has

$$\begin{aligned}
\mathcal{L}(K \otimes_\Sigma L) &\cong \mathcal{L}\left(\varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} \overrightarrow{cosk}^\Sigma(\square[m]_{\leqslant 1} \times_\Sigma \square[n]_{\leqslant 1})\right) \\
&\cong \varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} \mathcal{L}\left(\overrightarrow{cosk}^\Sigma(\square[m]_{\leqslant 1} \times_\Sigma \square[n]_{\leqslant 1})\right) \quad \text{since } \mathcal{L} \text{ is a left adjoint} \\
&\cong \varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} cosk_1^{\widehat{\square},\overline{\Sigma}}(\widehat{\square}[m]_{\leqslant 1} \overline{\times}_\Sigma \widehat{\square}[n]_{\leqslant 1}) \quad \text{by Theorem 4.1.8} \\
&\cong \varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} \widehat{\square}[m] \otimes_\Sigma \widehat{\square}[n] \quad \text{by definition of } \otimes_\Sigma \\
&\cong \varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} \mathcal{L}(\square[m]) \otimes_\Sigma \mathcal{L}(\square[n]) \quad \text{by Proposition 2.1.14} \\
&\cong \left(\varinjlim_{\square[m]\to K} \mathcal{L}(\square[m])\right) \otimes_\Sigma \left(\varinjlim_{\square[n]\to L} \mathcal{L}(\square[n])\right) \quad \text{by Corollary 4.2.5} \\
&\cong \mathcal{L}(K) \otimes_\Sigma \mathcal{L}(L) \quad \text{since } \mathcal{L} \text{ is a left adjoint.} \quad \square
\end{aligned}$$

Associativity

As in [18], it is also possible to prove that the synchronized tensor product of labelled transverse symmetric precubical sets is associative.

Theorem 4.2.7. *Let K, L and M be three labelled transverse symmetric precubical sets. Then there is a canonical isomorphism of labelled transverse symmetric precubical sets*

$$(K \otimes_\Sigma L) \otimes_\Sigma M \cong K \otimes_\Sigma (L \otimes_\Sigma M).$$

Proof. One has

$$\begin{aligned}
K \otimes_\Sigma (L \otimes_\Sigma M) &\cong \left(\varinjlim_{\widehat{\square}[p]\to K} \widehat{\square}[p] \right) \otimes_\Sigma \left(\varinjlim_{\widehat{\square}[q]\to L} \varinjlim_{\widehat{\square}[r]\to M} \widehat{\square}[q] \otimes_\Sigma \widehat{\square}[r] \right) \\
&\cong \varinjlim_{\widehat{\square}[p]\to K} \varinjlim_{\widehat{\square}[q]\to L} \varinjlim_{\widehat{\square}[r]\to M} \widehat{\square}[p] \otimes_\Sigma (\widehat{\square}[q] \otimes_\Sigma \widehat{\square}[r]) \quad \text{by Corollary 4.2.5} \\
&\cong \varinjlim_{\widehat{\square}[p]\to K} \varinjlim_{\widehat{\square}[q]\to L} \varinjlim_{\widehat{\square}[r]\to M} \mathcal{L}\,(\square[p] \otimes_\Sigma (\square[q] \otimes_\Sigma \square[r])) \quad \text{by Theorem 4.2.6} \\
&\cong \varinjlim_{\widehat{\square}[p]\to K} \varinjlim_{\widehat{\square}[q]\to L} \varinjlim_{\widehat{\square}[r]\to M} \mathcal{L}\,((\square[p] \otimes_\Sigma \square[q]) \otimes_\Sigma \square[r]) \quad \text{by [18, Proposition A.3]} \\
&\cong \varinjlim_{\widehat{\square}[p]\to K} \varinjlim_{\widehat{\square}[q]\to L} \varinjlim_{\widehat{\square}[r]\to M} (\widehat{\square}[p] \otimes_\Sigma \widehat{\square}[q]) \otimes_\Sigma \widehat{\square}[r] \quad \text{by Theorem 4.2.6} \\
&\cong (K \otimes_\Sigma L) \otimes_\Sigma M \quad \text{by Corollary 4.2.5.} \quad \square
\end{aligned}$$

As already pointed out in [18], it is false in general that the two labelled precubical sets $K \otimes_\Sigma L$ and $L \otimes_\Sigma K$ are isomorphic as labelled precubical sets. Indeed, let us suppose that $\Sigma\backslash\{\tau\}$ contains an element a. Then $\square[a] \otimes_\Sigma \square[\tau] \cong \square[a, \tau]$ and $\square[\tau] \otimes_\Sigma \square[a] \cong \square[\tau, a]$. Because of the lack of symmetry operators, the two labelled 2-cubes $\square[a, \tau]$ and $\square[\tau, a]$ cannot be isomorphic as labelled precubical sets. However, the two underlying precubical sets are of course isomorphic, as already pointed out in [18]. In the category of transverse symmetric precubical sets, the situation is much better. Indeed, one has the isomorphisms of labelled transverse symmetric precubical sets

$$\widehat{\square}[a, \tau] \cong \mathcal{L}(\square[a, \tau]) \cong \widehat{\square}[\tau, a] \cong \mathcal{L}(\square[\tau, a]).$$

Proposition 4.2.8. *Let K and L be two labelled transverse symmetric precubical sets. Then there is a natural isomorphism of labelled transverse symmetric precubical sets $K \otimes_\Sigma L \cong L \otimes_\Sigma K$.*

Proof (*Sketch of proof*). It suffices to use Corollary 4.2.5 together with the isomorphism

$$\widehat{\square}[a_1, \ldots, a_m] \otimes_\Sigma \widehat{\square}[b_1, \ldots, b_n] \cong \widehat{\square}[b_1, \ldots, b_n] \otimes_\Sigma \widehat{\square}[a_1, \ldots, a_m]$$

(built using the symmetry operators) for all labelled full cubes $\widehat{\square}[a_1, \ldots, a_m]$ and $\widehat{\square}[b_1, \ldots, b_n]$. $\square$

4.3. Comparison of the two semantics of CCS

Interpreting CCS as labelled precubical sets

The *CCS process names* are generated by the following syntax:

$$P ::= nil \mid a.P \mid (\nu a)P \mid P + P \mid P||P \mid rec(x)P(x),$$

where $P(x)$ means a process name with one free variable x. The variable x must be *guarded*; that is, it must lie in a prefix term $a.P'(x)$ for some $a \in \Sigma$.

Definition 4.3.1. A labelled precubical set $\ell : K \to !\Sigma$ *decorated by process names* is a labelled precubical set together with a set map $d : K_0 \to \mathbf{Proc}_\Sigma$ called the *decoration*.

Let us define by induction on the syntax of the CCS process name P the decorated labelled precubical set $\square[\![P]\!]$ (see [18] for further explanations). The labelled precubical set $\square[\![P]\!]$ has a unique initial state canonically decorated by the process name P, and its other states will be decorated in an inductive way. Therefore, for every process name P, $\square[\![P]\!]$ is an object of the double comma category $\{i\}{\downarrow}\square^{op}\mathbf{Set}{\downarrow}!\Sigma$. One has $\square[\![nil]\!] := \square[0]$, $\square[\![\mu.nil]\!] := \mu.nil \xrightarrow{(\mu)} nil$, $\square[\![P + Q]\!] := \square[\![P]\!] \oplus \square[\![Q]\!]$ with the binary coproduct taken in $\{i\}{\downarrow}\square^{op}\mathbf{Set}{\downarrow}!\Sigma$, the pushout diagram of precubical sets

$$\begin{array}{ccc}
\square[0] = \{0\} & \xrightarrow{0 \mapsto nil} & \square[\![\mu.nil]\!] \\
\Big\downarrow{\scriptstyle 0 \mapsto P} & & \Big\downarrow \\
\square[\![P]\!] & \longrightarrow & \square[\![\mu.P]\!],
\end{array}$$

the pullback diagram of precubical sets

$$\begin{array}{ccc}
\square[\![(\nu a)P]\!] & \longrightarrow & \square[\![P]\!] \\
\Big\downarrow & & \Big\downarrow \\
!(\Sigma\backslash\{a, \overline{a}\}) & \longrightarrow & !\Sigma,
\end{array}$$

the formula giving the interpretation of the parallel composition with synchronization

$$\square[\![P||Q]\!] := \square[\![P]\!] \otimes_\Sigma \square[\![Q]\!]$$

and finally $\square[\![rec(x)P(x)]\!]$ defined as the least fixed point of $P(-)$.

The prefix operator, the direct sum and the restriction operator are ω-continuous; that is to say, they preserve the upper bounds of ascending ω-chains of labelled precubical sets $K^0 \subset K^1 \subset K^2 \subset \dots$, since they are finitely accessible and since the upper bound is given by the colimit of the chain. The synchronized tensor product is also ω-continuous since it is colimit preserving by [18, Proposition A.2]. Moreover, the condition imposed on $P(x)$ implies that, for all process names Q_1 and Q_2 with $\square[\![Q_1]\!] \subset \square[\![Q_2]\!]$, one has $\square[\![P(Q_1)]\!] \subset \square[\![P(Q_2)]\!]$. Therefore the mapping $P(-)$ is ω-continuous and non-decreasing. Thus, the labelled precubical set

$$\square[\![rec(x)P(x)]\!] := \varinjlim_n \square[\![P^n(nil)]\!] \cong \bigcup_{n\geqslant 0} \square[\![P^n(nil)]\!]$$

will be equal to the least fixed point of $P(-)$. This is a particular case of the Kleene fixed-point theorem on a directed complete partial order.

Interpreting CCS as labelled transverse symmetric precubical sets

Let us give now the new semantics of CCS in terms of labelled transverse symmetric precubical sets.

Definition 4.3.2. A labelled transverse symmetric precubical set $\ell : K \to Sh\mathcal{L}(!\Sigma)$ *decorated by process names* is a labelled transverse symmetric precubical set together with a set map $d : K_0 \to \mathbf{Proc}_\Sigma$ called the *decoration*.

The interpretation of a CCS process name P in terms of a decorated labelled transverse symmetric precubical set $\widehat{\square}[\![P]\!]$ is defined by induction on the syntax of P, as for the case of labelled precubical sets. The only differences with the latter case are the pullback diagram

$$\begin{array}{ccc} \widehat{\square}[\![(\nu a)P]\!] & \longrightarrow & \widehat{\square}[\![P]\!] \\ \downarrow & & \downarrow \\ Sh\mathcal{L}(!(\Sigma\backslash\{a,\overline{a}\})) & \longrightarrow & Sh\mathcal{L}(!\Sigma), \end{array}$$

and the equation $\widehat{\square}[\![P||Q]\!] := \widehat{\square}[\![P]\!] \otimes_\Sigma \widehat{\square}[\![Q]\!]$, where $\otimes_\Sigma$ is now the synchronized tensor product of labelled transverse symmetric precubical sets. Corollary 4.2.5 enables us to construct the least fixed point of $P(-)$ in the same way as in the case of labelled precubical sets.

The two semantics have same geometric realization

Theorem 4.3.3. *For every CCS process name P, there is an isomorphism of labelled transverse symmetric precubical sets $\widehat{\square}[\![P]\!] \cong \mathcal{L}(\square[\![P]\!])$ and an isomorphism of (labelled) flows $|\widehat{\square}[\![P]\!]| \cong |\square[\![P]\!]|$.*

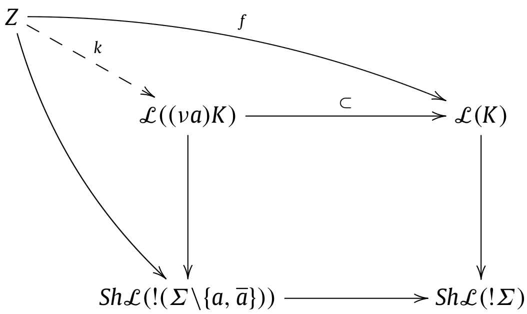
Proof. Let K be a labelled precubical set. Let $a \in \Sigma\backslash\{\tau\}$. Let $(\nu a)K$ be the labelled precubical set defined by the pullback diagram

$$\begin{array}{ccc} (\nu a)K & \xrightarrow{\subset} & K \\ \downarrow & & \downarrow \\ !(\Sigma\backslash\{a,\overline{a}\}) & \longrightarrow & !\Sigma. \end{array}$$

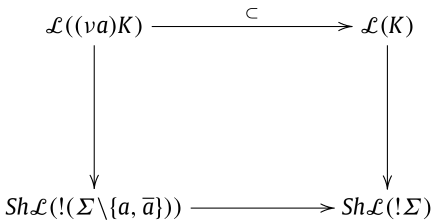
One obtains the commutative diagram of labelled transverse symmetric precubical sets

$$\begin{array}{ccc} \mathcal{L}((\nu a)K) & \xrightarrow{\subset} & \mathcal{L}(K) \\ \downarrow & & \downarrow \\ \mathcal{L}(!(\Sigma\backslash\{a,\overline{a}\})) & \longrightarrow & \mathcal{L}(!\Sigma) \\ \downarrow & & \downarrow \\ Sh\mathcal{L}(!(\Sigma\backslash\{a,\overline{a}\})) & \longrightarrow & Sh\mathcal{L}(!\Sigma). \end{array}$$

The map $(\nu a)K \to K$ is an inclusion of presheaves: the labelled precubical set $(\nu a)K$ is the subobject of K containing the labelled cubes of K not containing a or $\overline{a}$ as label. By Proposition 2.1.14, the transverse symmetric precubical set $\mathcal{L}((\nu a)K)$ is the subobject of $\mathcal{L}(K)$ containing the p-cubes of $\mathcal{L}(K)$ of the form $\mu^*(x)$, where $\mu : [p] \to [p]$ is a map of $\widehat{\square}$ and x is a p-cube of $(\nu a)K$. Therefore the map $\mathcal{L}((\nu a)K) \to \mathcal{L}(K)$ is an inclusion of presheaves as well. Consider now a commutative diagram of labelled transverse symmetric precubical sets



Every p-cube x of Z is taken to a p-cube $f(x)$ of $\mathcal{L}(K)$. By Proposition 4.1.3, $f(x) = \mu^*(y)$ for some p-cube $y \in K$ and for some map $\mu : [p] \to [p]$ of $\widehat{\square}$. By construction, y does not use the labels a or $\overline{a}$. Thus $y \in (\nu a)K$. Therefore $f(x)$ is a p-cube of $\mathcal{L}((\nu a)K)$. Hence k exists and is unique since the map $\mathcal{L}((\nu a)K) \to \mathcal{L}(K)$ is an inclusion of presheaves. Thus, the diagram of labelled transverse symmetric precubical sets



is a pullback. So the isomorphism $\widehat{\square}[\![P]\!] \cong \mathcal{L}(\square[\![P]\!])$ implies the isomorphism $\widehat{\square}[\![(\nu a)P]\!] \cong \mathcal{L}(\square[\![(\nu a)P]\!])$. Therefore, the isomorphism of labelled transverse symmetric precubical sets $\widehat{\square}[\![P]\!] \cong \mathcal{L}(\square[\![P]\!])$ is proved by induction on the syntax of the process name P, using Theorem 4.2.6 and the fact that the functor $\mathcal{L}$ preserves colimits since it is a left adjoint. The isomorphism of labelled flows $|\widehat{\square}[\![P]\!]| \cong |\square[\![P]\!]|$ is a consequence of Proposition 2.2.10. □

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Appendix

A.1. The case of labelled symmetric precubical sets

By Theorem 3.1.15, the category of cubes $\square_S$ is not shell complete. It is interesting anyway for the three following reasons. (1) It is possible to give an explicit description of the symmetric precubical set of labels with Proposition A.1.3. Such a description is still an open problem for the transverse symmetric precubical set of labels (see Conjecture 3.1.21). (2) The category of cubes $\square_S$ is the smallest category of cubes $\mathcal{A}$ such that the labelled cubes $\mathcal{A}[a_{\sigma(1)}, \ldots, a_{\sigma(n)}]$ with $a_1, \ldots, a_n \in \Sigma$ for σ running over the set of permutations of $\{1, \ldots, n\}$ belong to the same isomorphism class. Let us recall that the labelled precubical sets $\square[a_{\sigma(1)}, \ldots, a_{\sigma(n)}]$ and $\square[a_{\sigma'(1)}, \ldots, a_{\sigma'(n)}]$ are not isomorphic as soon as $(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \neq (a_{\sigma'(1)}, \ldots, a_{\sigma'(n)})$. (3) There is a strong link between labelled symmetric precubical sets and higher-dimensional transition systems in the sense of Cattani and Sassone [8]; see [19]. Indeed, it turns out that the category of higher-dimensional transition systems in the sense of Cattani and Sassone is equivalent to a full reflective subcategory of that of labelled symmetric precubical sets.

Description of the symmetric precubical set of labels

The following combinatorial lemma is well known (see [37] for a survey).

Lemma A.1.1. *Let $p \geqslant 1$. The group of automorphisms of the poset $[p]$ is isomorphic to the symmetric group on $\{1, \ldots, p\}$. In other words, let f be an automorphism of the poset $[p]$. Then there exists a permutation π of the set $\{1, \ldots, p\}$ such that $f(\epsilon_1, \ldots, \epsilon_p) = (\epsilon_{\pi(1)}, \ldots, \epsilon_{\pi(p)})$.*

Proof. Let $I \subset \{1, \ldots, p\}$. Let e_I be the element $(\epsilon_1, \ldots, \epsilon_p)$ of $[p]$ such that $\epsilon_i = 1$ if and only if $i \in I$. Since f is bijective and strictly increasing, it preserves the distance of Proposition 2.1.3. The distance between $e_\varnothing$ and $f(e_{\{i\}})$ is 1. So there exists a permutation π of $\{1, \ldots, p\}$ such that $f(e_{\{i\}}) = e_{\{\pi(i)\}}$. Let $g(\epsilon_1, \ldots, \epsilon_p) = (\epsilon_{\pi^{-1}(1)}, \ldots, \epsilon_{\pi^{-1}(p)})$. Then $g(f(e_{\{i\}})) = g(e_{\{\pi(i)\}}) = e_{\{i\}}$. It then suffices to prove by induction on the cardinality c of I that $g(f(e_I)) = e_I$. Let $c \geqslant 2$ with $c \leqslant p$. Assume that $g(f(e_J)) = e_J$ for all subsets J of $\{1, \ldots, p\}$ of cardinality $c-1$. Let I be a subset of $\{1, \ldots, p\}$ of cardinality c. Then the distance between $g(f(e_{I\backslash\{i\}}))$ and $g(f(e_I))$ is 1 for all $i \in I$. By induction hypothesis, one has $g(f(e_{I\backslash\{i\}})) = e_{I\backslash\{i\}}$. So the only possibility is $g(f(e_I)) = e_I$. □

Proposition A.1.2. *An adjacency-preserving map $f : [m] \to [n]$ belongs to $\square_S$ if and only if f is one-to-one.*

Proof. It is clear that any map of $\square_S$ is one-to-one. Conversely, let $f : [m] \to [n]$ be a one-to-one adjacency-preserving map. Then, by Proposition 3.1.14, f factors uniquely as a composite $[m] \xrightarrow{\psi} [m] \xrightarrow{\phi} [n]$ with $\phi \in \square$ and ψ adjacency preserving one-to-one. A cardinality argument implies that ψ is a bijection. Therefore $f \in \square_S$ by Lemma A.1.1. □

As for precubical sets, let $\partial_i^\alpha = (\delta_i^\alpha)^*$. And let $s_i = (\sigma_i)^*$.

Proposition A.1.3. *The symmetric precubical set of labels $Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma)$ is isomorphic to the following symmetric precubical set, denoted by $!^S\Sigma$:*

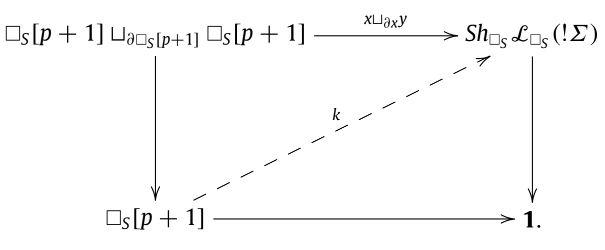
- $(!^S\Sigma)_0 = \{()\}$ *(the empty word)*
- *for* $n \geqslant 1$, $(!^S\Sigma)_n = \Sigma^n$
- $\partial_i^0(a_1, \ldots, a_n) = \partial_i^1(a_1, \ldots, a_n) = (a_1, \ldots, \widehat{a_i}, \ldots, a_n)$, *where the notation $\widehat{a_i}$ means that a_i is removed.*
- $s_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n)$ *for* $1 \leqslant i \leqslant n$.

Proof. The category of cubes $\square_S$ is the small category freely generated by the δ_i^α and σ_i operators and by the cocubical relations, the algebraic relations of Proposition 3.1.9, and the Moore relations for symmetry operators $\sigma_i\sigma_i = Id$, $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $i = j - 1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i < j - 1$ by [21, Theorem 8.1].

It is easy to prove that the s_i and ∂_i^α operators of $!^S\Sigma$ satisfy the dual of these algebraic relations. So $!^S\Sigma$ together with the ∂_i^α and s_i operators is a well-defined symmetric precubical set.

The identity of $!\Sigma$ yields a map of precubical sets $!\Sigma \to \omega_{\square_S}(!^S\Sigma)$. Hence, by adjunction, one obtains a map $\mathcal{L}_{\square_S}(!\Sigma) \to !^S\Sigma$. The symmetric precubical set $!^S\Sigma$ is orthogonal to the set of morphisms $\{\square_S[p] \sqcup_{\partial\square_S[p]} \square_S[p] \to \square_S[p], p \geqslant 2\}$ for the same reason as $!\Sigma$ is orthogonal to the set of morphisms $\{\square[p] \sqcup_{\partial\square[p]} \square[p] \to \square[p], p \geqslant 2\}$. Hence, by adjunction, one obtains a map of symmetric precubical sets $f : Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma) \to !^S\Sigma$ which is clearly onto: an inverse image of $(a_1, \ldots, a_n) \in (!^S\Sigma)_n$ for $n \geqslant 1$ is given by the image of $(a_1, \ldots, a_n) \in \mathcal{L}_{\square_S}(!\Sigma)_n$ by the canonical map $\mathcal{L}_{\square_S}(!\Sigma) \to Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma)$.

Let us prove by induction on $p \geqslant 1$ that the map $f_{\leqslant p} : (Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma))_{\leqslant p} \to (!^S\Sigma)_{\leqslant p}$ is one-to-one. The map induces the isomorphism $f_{\leqslant 1} : (Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma))_{\leqslant 1} \to (!^S\Sigma)_{\leqslant 1}$ by Propositions 2.1.18 and 2.1.24. Hence the proof is complete for $p = 1$. Let us suppose that the map $f_{\leqslant p} : (Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma))_{\leqslant p} \to (!^S\Sigma)_{\leqslant p}$ is an isomorphism for $p \geqslant 1$. Let $x, y \in (Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma))_{p+1}$ be two $(p+1)$-cubes having the same image in $!^S\Sigma$. Then they have the same boundary in $(!^S\Sigma)_{\leqslant p}$, and therefore x and y have the same boundary $\partial x = \partial y$ by induction hypothesis. One obtains a commutative square of solid arrows



The lift k exists and is unique. So $x = k = y$. The induction hypothesis is therefore proved for $p + 1$. □

The labelled directed symmetric coskeleton construction

The following proposition is similar to Proposition 4.1.3.

Proposition A.1.4. *Let K be a precubical set. For any p-cube x of $\omega_{\square_S}\mathcal{L}_{\square_S}(K)$ with $p \geqslant 0$, there exists a p-cube y of $K \subset \omega_{\square_S}\mathcal{L}_{\square_S}(K)$ and a map $\mu \in \square_S([p], [p])$ such that $x = \mu^*(y)$, where $\mu^* : \mathcal{L}_{\square_S}(K)_p \to \mathcal{L}_{\square_S}(K)_p$ is the image of μ by the presheaf $\mathcal{L}_{\square_S}(K) \in \square_S^{op}\mathbf{Set}$.*

Note that, as in Proposition 4.1.3, the decomposition is actually unique.

Proof. With the notations of the proof of Proposition 4.1.3. By Propositions 3.1.14 and A.1.2, the set map $\overline{x} : [p] \to [n]$ factors as a composite $[p] \xrightarrow{\mu} [p] \xrightarrow{\phi} [n]$ with $\mu \in \square_S$ and $\phi \in \square$. □

By Proposition 2.3.4, the truncation functor

$$\square_S^{op}\mathbf{Set}{\downarrow}Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma) \to (\square_S)_n^{op}\mathbf{Set}{\downarrow}Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma)$$

has a right adjoint

$$cosk_n^{\square_S,\Sigma} : (\square_S)_n^{op}\mathbf{Set}{\downarrow}Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma) \to \square_S^{op}\mathbf{Set}{\downarrow}Sh_{\square_S}\mathcal{L}_{\square_S}(!\Sigma).$$

Definition A.1.5. (Compare with Definition 3.1.22.) Let K be a 1-dimensional labelled symmetric precubical set with $K_0 = [p]$ for some $p \geqslant 0$. The *labelled symmetric directed coskeleton* of K is the labelled precubical set $\overrightarrow{cosk}_S^{\Sigma}(K)$ defined as the subobject of $cosk_1^{\square_S,\Sigma}(K)$ such that

- $\overrightarrow{cosk}_S^{\Sigma}(K)_{\leqslant 1} = cosk_1^{\square_S,\Sigma}(K)_{\leqslant 1}$,
- for every $n \geqslant 2$, $x \in cosk_1^{\square_S,\Sigma}(K)_n$ is an n-cube of $\overrightarrow{cosk}_S^{\Sigma}(K)$ if and only if the set map $x_0 : [n] \to [p]$ is *non-twisted*, i.e. $x_0 : [n] \to [p]$ is a composite[9]

  $$x_0 : [n] \xrightarrow{\phi} [q] \xrightarrow{\psi} [p],$$

  where ψ is a morphism of the small category $\square$ and where ϕ is of the form

  $$(\epsilon_1, \ldots, \epsilon_n) \mapsto (\epsilon_{i_1}, \ldots, \epsilon_{i_q})$$

  such that $\{1, \ldots, n\} \subset \{i_1, \ldots, i_q\}$.

  The link with labelled precubical sets is as follows.

Proposition A.1.6. *Let K be a 1-dimensional labelled (symmetric) precubical set with $K_0 = [p]$ for some $p \geqslant 0$. Then there is the isomorphism of labelled symmetric precubical sets*

$$\mathcal{L}_{\square_S}(\overrightarrow{cosk}^{\Sigma}(K)) \cong \overrightarrow{cosk}_S^{\Sigma}(K).$$

Proof. By a proof similar to that of Theorem 4.1.5, one obtains the inclusion of presheaves

$$\mathcal{L}_{\square_S}(\overrightarrow{cosk}^{\Sigma}(K)) \subset cosk_1^{\square_S,\Sigma}(K).$$

It is clear that the inclusion above factors as the composite of inclusions

$$\mathcal{L}_{\square_S}(\overrightarrow{cosk}^{\Sigma}(K)) \subset \overrightarrow{cosk}_S^{\Sigma}(K) \subset cosk_1^{\square_S,\Sigma}(K).$$

The left-hand inclusion is an equality by Proposition A.1.4. □

Thanks to Proposition 2.2.10, one obtains the isomorphism of flows

$$|\overrightarrow{cosk}^{\Sigma}(K)| \cong |\overrightarrow{cosk}_S^{\Sigma}(K)|.$$

Interpreting CCS as labelled symmetric precubical sets

Definition A.1.7. Let K and L be two labelled symmetric precubical sets. The *tensor product with synchronization* (or *synchronized tensor product*) of K and L is

$$K \otimes_\Sigma L := \varinjlim_{\square_S[m]\to K} \varinjlim_{\square_S[n]\to L} \overrightarrow{cosk}_S^{\Sigma}(\square_S[m]_{\leqslant 1} \times_\Sigma \square_S[n]_{\leqslant 1}).$$

One can then easily adapt the semantics of CCS to the case of labelled symmetric precubical sets. The interest of this setting is that it is simpler than that of transverse symmetric precubical sets, and that, as in Proposition 4.2.8, there is an isomorphism of labelled symmetric precubical sets $K \otimes_\Sigma L \cong L \otimes_\Sigma K$ for all labelled symmetric precubical sets K and L. The synchronized tensor product of symmetric precubical sets is also colimit preserving by Proposition 4.2.4, and therefore associative.

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