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Orthogonal polynomials, associated polynomials and functions of the second kind

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Abstract

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A survey is given of the interaction between orthogonal polynomials, associated polynomials and functions of the second kind with an emphasis on asymptotic results. Various formulas are presented in a unified way in terms of Wronskians of solutions of linear recurrence relations. Some of these formulas are classical and go back to the previous century, but usually they are hard to locate in the literature. Some new formulas are also given, in particular a formula expressing the derivative of an orthogonal polynomial in terms of the orthogonal polynomials and the associated polynomials.

Keywords: Orthogonal polynomials, recurrence relation, asymptotics.

1. Introduction

Let μ be a probability measure on the real line with an infinite number of points of increase and for which all the moments are finite. There exists a unique sequence of orthogonal polynomials $\{p_n(x);\ n=0, 1, 2,\dots\}$ for which

$$\int p_n(x)p_m(x)\,\mathrm{d}\mu(x)=\delta_{m,n}, \quad m, n \geqslant 0, \tag{1.1}$$

with

$$p_n(x)=\gamma_n x^n+\cdots, \quad \gamma_n>0.$$

These polynomials satisfy a three-term recurrence relation

$$xp_n(x)=a_{n+1}p_{n+1}(x)+b_n p_n(x)+a_n p_{n-1}(x), \quad n\geqslant 0, \tag{1.2}$$

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for which the recurrence coefficients satisfy

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} > 0, \qquad b_n = \int x p_n^2(x) \, \mathrm{d}\mu(x) \in \mathbb{R},$$

and with initial values $p_{-1}(x) = 0$ and $p_0(x) = 1$. An interesting result, usually referred to as Favard's theorem [7, Theorem I.4.4], [8, Theorem II.1.5], states that for the polynomial solution of (1.2) with $a_n > 0$ and $b_n \in \mathbb{R}$ and initial conditions $p_{-1}(x) = 0$, $p_0(x) = 1$, there exists a probability measure μ so that the orthogonality (1.1) is satisfied.

Given the sequences $\{a_n > 0;\ n = 1, 2, \dots\}$ and $\{b_n;\ n = 0, 1, 2, \dots\}$, one defines for $k \in \mathbb{N}$ the kth associated polynomials $\{p_n^{(k)}(x);\ n = 0, 1, 2, \dots\}$ by the recurrence relation

$$x p_n^{(k)}(x) = a_{n+k+1} p_{n+1}^{(k)}(x) + b_{n+k} p_n^{(k)}(x) + a_{n+k} p_{n-1}^{(k)}(x), \quad n \geq 0, \tag{1.3}$$

with initial conditions

$$p_{-1}^{(k)}(x) = 0, \qquad p_0^{(k)}(x) = 1.$$

The spectral measure (orthogonality measure) with respect to which these polynomials are orthogonal will be denoted by $\mu^{(k)}$. Notice that for every fixed k the polynomials $\{p_{n-k}^{(k)}(x);\ n = 0, 1, 2, \dots\}$ form a solution of the recurrence relation (1.2).

Functions of the second kind $\{q_n(x);\ n = 0, 1, 2, \dots\}$ are defined by the integral

$$q_n(x) = \int \frac{p_n(y)}{x - y} \, \mathrm{d}\mu(y), \quad x \in \mathbb{C} \setminus \operatorname{supp}(\mu), \tag{1.4}$$

where $\{p_n(x);\ n = 0, 1, 2, \dots\}$ are the orthogonal polynomials with spectral measure μ. These functions are well defined whenever $x \in \mathbb{C} \setminus \operatorname{supp}(\mu)$, where $\operatorname{supp}(\mu)$ — the support of μ — is the smallest closed set containing all the points of increase of μ. A straightforward analysis shows that $\{q_n(x);\ n = 0, 1, 2, \dots\}$ satisfies the recurrence relation (1.2) with initial conditions

$$a_0 q_{-1}(x) = 1, \qquad q_0(x) = \int \frac{\mathrm{d}\mu(y)}{x - y}.$$

Observe that q_0 is the so-called *Stieltjes transform* of the spectral measure μ.

Associated polynomials already appear in Stieltjes' fundamental work [27] and are very natural because they are the numerators for the convergents of certain continued fractions. Some interesting properties may be found in the works of Perron [23] and Geronimus [12]. These properties and some new results have recently been given by Belmehdi [5] without referring to continued fractions. Functions of the second kind and associated polynomials are usually only studied for classical orthogonal polynomials [1,4,7,14,22,25,28,32] but recently more general orthogonal polynomials have also been considered [2,3,6,9,11]. Grosjean has made some very detailed contributions to the analysis of associated polynomials and functions of the second kind [13–15]. In the next section we will show how all these functions are interrelated.

2. Wronskians

Consider the second-order recurrence relation

$$x u_n = a_{n+1} u_{n+1} + b_n u_n + a_n u_{n-1}, \quad n \geq 0, \tag{2.1}$$

with $a_{n+1} > 0$, $b_n \in \mathbb{R}$, $n = 0, 1, 2, \ldots$, and $x \in \mathbb{C}$. Let $\{u_n\}$ and $\{v_n\}$ be two solutions of (2.1); then the Wronskian $W(u_n, v_n)$ is given by

$$W(u_n, v_n) = a_{n+1} \begin{vmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{vmatrix} = a_{n+1}(u_n v_{n+1} - u_{n+1} v_n). \tag{2.2}$$

This determinant is sometimes also called the Casorati determinant of the two solutions $\{u_n\}$ and $\{v_n\}$ [18, p.354]. It follows easily that $W(u_n, v_n)$ is independent of n. If the Wronskian of two solutions is different from zero, then the two solutions are said to be linearly independent. From the general theory of linear recurrence relations it follows that every solution of (2.1) is a linear combination of two linearly independent solutions.

Consider $u_n = p_n(x)$ and $v_n = p_{n-1}^{(1)}(x)$ with Wronskian

$$W(p_n, p_{n-1}^{(1)}) = a_{n+1}[p_n(x) p_n^{(1)}(x) - p_{n+1}(x) p_{n-1}^{(1)}(x)] = a_1 > 0. \tag{2.3}$$

It follows that every solution of (2.1) is a linear combination of $p_n(x)$ and $p_{n-1}^{(1)}(x)$. In particular for $k \in \mathbb{N}$,

$$p_{n-k}^{(k)}(x) = A p_n(x) + B p_{n-1}^{(1)}(x).$$

Setting $n = k$ and $n = k - 1$ yields

$$A = -a_k \frac{p_{k-2}^{(1)}(x)}{W(p_n, p_{n-1}^{(1)})} = -\frac{a_k}{a_1} p_{k-2}^{(1)}(x),$$

$$B = a_k \frac{p_{k-1}(x)}{W(p_n, p_{n-1}^{(1)})} = \frac{a_k}{a_1} p_{k-1}(x),$$

which gives the formula

$$a_1 p_{n-k}^{(k)}(x) = a_k [p_{k-1}(x) p_{n-1}^{(1)}(x) - p_{k-2}^{(1)}(x) p_n(x)]. \tag{2.4}$$

An interesting special case is obtained by taking $k = 2$, giving

$$x p_{n-1}^{(1)}(x) = a_1 p_n(x) + b_0 p_{n-1}^{(1)}(x) + \frac{a_1^2}{a_2} p_{n-2}^{(2)}(x).$$

When considering the $(k-1)$st associated version of this formula, we obtain

$$x p_{n-1}^{(k)}(x) = a_k p_n^{(k-1)}(x) + b_{k-1} p_{n-1}^{(k)}(x) + \frac{a_k^2}{a_{k+1}} p_{n-2}^{(k+1)}(x). \tag{2.5}$$

This formula was already given in [9] and is the basis for a perturbation theory for orthogonal polynomials defined by a recurrence relation. If $x \in \mathbb{C} \setminus \operatorname{supp}(\mu)$, then in a similar way

$$q_n(x) = C p_n(x) + D p_{n-1}^{(1)}(x),$$

and setting $n = 0$ and $n = -1$ (observe that $a_0 p_{-2}^{(1)}(x) = -a_1$) gives

$$C = q_0(x), \qquad D = -\frac{1}{a_1},$$

which results in the well-known formula

$$p_{n-1}^{(1)}(x) = a_1 \int \frac{p_n(x) - p_n(y)}{x - y} \, d\mu(y).$$

In general we may consider $\{p_{n-j}^{(j)}(x)\}$ and $\{p_{n-k}^{(k)}(x)\}$ with $j<k$ as two solutions of (2.1). Their Wronskian is given by

$$W\left(p_{n-j}^{(j)}, p_{n-k}^{(k)}\right)=a_{n+1}\left[p_{n-j}^{(j)}(x)p_{n-k+1}^{(k)}(x)-p_{n-j+1}^{(j)}(x)p_{n-k}^{(k)}(x)\right]=a_k p_{k-j-1}^{(j)}(x), \tag{2.6}$$

which means that both solutions are linearly independent if and only if x is not a zero of $p_{k-j-1}^{(j)}$. Finally let us work out the Wronskian of $\{q_n(x)\}$ and $\{p_{n-k}^{(k)}(x)\}$, which is

$$W\left(q_n, p_{n-k}^{(k)}\right)=a_{n+1}\left[q_n(x)p_{n-k+1}^{(k)}(x)-q_{n+1}(x)p_{n-k}^{(k)}(x)\right]=a_k q_{k-1}(x). \tag{2.7}$$

It follows that $\{q_n(x)\}$ and $\{p_{n-k}^{(k)}(x)\}$ are linearly independent whenever $x\in\mathbb{C}\setminus\text{co}(\text{supp}(\mu))$, where co(supp($\mu$)) is the convex hull of supp(μ), i.e., the smallest closed interval containing supp(μ). Indeed, by the representation

$$\begin{aligned} p_n(x)q_n(x) &= \int\frac{p_n(x)-p_n(y)}{x-y}p_n(y)\,\mathrm{d}\mu(y)+\int\frac{p_n^2(y)}{x-y}\,\mathrm{d}\mu(y) \\ &= \int\frac{p_n^2(y)}{x-y}\,\mathrm{d}\mu(y), \end{aligned} \tag{2.8}$$

for which the orthogonality (1.1) was used, it follows that $p_n(x)q_n(x)$ is the Stieltjes transform of the probability measure $p_n^2(y)\,\mathrm{d}\mu(y)$ and it is easy to show that the Stieltjes transform of a probability measure has no zeros outside the convex hull of its support (see, e.g., [16, letter 273] or [26]). None of these Wronskian formulas is new: the equation (2.3) even goes back to the previous century and expresses the well-known relationship between numerators and denominators of the convergents of a Jacobi continued fraction. The Wronskian formulas can also be found in [5].

As a result of all these Wronskian formulas we will now give two formulas closely related to the Christoffel–Darboux formula. We have been unable to find these formulas in the literature and therefore believe them to be new.

Theorem 1. *The following formulas are valid*:

$$\sum_{j=1}^{n}\frac{1}{a_j}p_{j-1}(x)p_{n-j}^{(j)}(y)=\frac{p_n(x)-p_n(y)}{x-y}, \tag{2.9}$$

and its confluent form

$$\sum_{j=1}^{n}\frac{1}{a_j}p_{j-1}(x)p_{n-j}^{(j)}(x)=p_n'(x). \tag{2.10}$$

Proof. Let $k\leqslant n$; then from (2.5) we obtain

$$yp_{n-k}^{(k)}(y)=a_k p_{n-k+1}^{(k-1)}(y)+b_{k-1}p_{n-k}^{(k)}(y)+\frac{a_k^2}{a_{k+1}}p_{n-k-1}^{(k+1)}(y), \tag{2.11}$$

and from (1.2)

$$xp_{k-1}(x)=a_k p_k(x)+b_{k-1}p_{k-1}(x)+a_{k-1}p_{k-2}(x). \tag{2.12}$$

Multiply (2.11) by $p_{k-1}(x)$ and (2.12) by $p_{n-k}^{(k)}(y)$ and subtract the obtained equations to find

$$A_k = A_{k+1} - (x-y)\frac{1}{a_k} p_{k-1}(x) p_{n-k}^{(k)}(y),$$

where we have used the abbreviation

$$A_k = p_{k-1}(x) p_{n-k+1}^{(k-1)}(y) - \frac{a_{k-1}}{a_k} p_{k-2}(x) p_{n-k}^{(k)}(y).$$

By iteration one easily obtains

$$A_k = A_{n+1} - (x-y) \sum_{j=k}^{n} \frac{1}{a_j} p_{j-1}(x) p_{n-j}^{(j)}(y).$$

Setting $k=1$ leads to (2.9). Formula (2.10) is obtained by letting y tend to x. □

Introduce the truncated Jacobi matrix $A_n = (A_{i,j})_{0 \leqslant i,j < n}$ with $A_{i,i} = b_i$ and $A_{i,i-1} = A_{i-1,i} = a_i$; then (2.10) is actually a trace formula since $p_n'(x)/p_n(x)$ is equal to the trace $\operatorname{tr}(xI - A_n)^{-1}$. The left-hand side of (2.10) can then be obtained by computing $(xI - A_n)^{-1}$ using cofactors. As a consequence of Theorem 1 we find

$$p_{n-k}^{(k)}(x) = a_k \int \frac{p_n(x) - p_n(y)}{x-y} p_{k-1}(y) \, \mathrm{d}\mu(y), \tag{2.13}$$

which is obtained by multiplying (2.9) by $p_{k-1}(y)$ and integrating with respect to the measure μ. When $k=1$, the classical formula for $p_{n-1}^{(1)}$ drops out as a special case. Another formula for $p_{n-k}^{(k)}$ is obtained by observing that

$$p_{n-k}^{(k)}(x) = A p_n(x) + B q_n(x),$$

and solving for A and B — using (2.7) — gives

$$\begin{aligned} p_{n-k}^{(k)}(x) &= a_k q_{k-1}(x) p_n(x) - a_k p_{k-1}(x) q_n(x) \\ &= a_k \int \frac{p_n(x) p_{k-1}(x) - p_n(y) p_{k-1}(y)}{x-y} \, \mathrm{d}\mu(y), \end{aligned} \tag{2.14}$$

which could also be verified by comparing with (2.13) and using the orthogonality.

3. Asymptotic formulas

We will now give some asymptotic formulas for ratios of orthogonal polynomials and their associated polynomials. We will show that the functions of the second kind are very useful in asymptotic analysis.

Let us first discuss some decompositions into partial fractions. We will always denote the zeros of p_n in increasing order by

$$x_{1,n} < x_{2,n} < \cdots < x_{n,n}.$$

It is very well known that the zeros of p_n belong to the convex hull of $\operatorname{supp}(\mu)$ [7, p.29]. The most elementary rational fraction and its partial fractions decomposition is

$$\frac{p_n'(x)}{p_n(x)} = \sum_{j=1}^{n} \frac{1}{x - x_{j,n}}. \tag{3.1}$$

Another rather common fraction is

$$\frac{p_{n-1}^{(1)}(x)}{p_n(x)} = a_1 \sum_{j=1}^{n} \frac{\lambda_{j,n}}{x - x_{j,n}}, \tag{3.2}$$

where

$$\lambda_{j,n} = \frac{p_{n-1}^{(1)}(x_{j,n})}{p_n'(x_{j,n})}.$$

The numerator $p_{n-1}^{(1)}(x_{j,n})$ can be replaced using (2.3) and (1.2) which gives

$$\lambda_{j,n} = \frac{-1}{a_{n+1} p_n'(x_{j,n}) p_{n+1}(x_{j,n})} = \frac{1}{a_n p_n'(x_{j,n}) p_{n-1}(x_{j,n})}. \tag{3.3}$$

The numbers $\{\lambda_{j,n};\ 1 \leqslant j \leqslant n\}$ are all positive and are known as *Christoffel numbers* [28, p.48]. They appear in the Gauss–Jacobi quadrature formula

$$\sum_{j=1}^{n} \lambda_{j,n} P(x_{j,n}) = \int P(x)\, \mathrm{d}\mu(x), \tag{3.4}$$

which is valid for every polynomial P of degree at most $2n-1$. A new decomposition into partial fractions is, $k \leqslant n$,

$$\frac{p_{k-1}(x) p_{n-k}^{(k)}(x)}{p_n(x)} = a_k \sum_{j=1}^{n} \frac{\lambda_{j,n} p_{k-1}^2(x_{j,n})}{x - x_{j,n}}. \tag{3.5}$$

In order to check this formula we need to evaluate the residues of the left-hand side. The residue for $x_{j,n}$ is

$$R_{j,n} = \frac{p_{k-1}(x_{j,n}) p_{n-k}^{(k)}(x_{j,n})}{p_n'(x_{j,n})}.$$

Use (2.6) with $j = 0$ and $x = x_{j,n}$ to find

$$R_{j,n} = -\frac{a_k}{a_{n+1}} \frac{p_{k-1}^2(x_{j,n})}{p_n'(x_{j,n}) p_{n+1}(x_{j,n})},$$

and from (3.3) we see that (3.5) is indeed the desired decomposition. Taking $k = 1$ gives (3.2) and for $k = n$ we find a formula already obtained in [29].

For the next theorem we need some notation. Define

$$Z_N = \{x_{j,n};\ 1 \leqslant j \leqslant n,\ n \geqslant N\},$$

$$X_1 = Z_1' = \{\text{accumulation points of } Z_1\},$$

$$X_2 = \{x \in Z_1;\ p_n(x) = 0 \text{ for infinitely many } n\}.$$

These sets have been introduced in [7]. Note the relations

$$\operatorname{supp}(\mu) \subset X_1 \cup X_2 \subset \operatorname{co}(\operatorname{supp}(\mu)).$$

We can now formulate a generalization of Markov's theorem.

Theorem 2. *Suppose that the moment problem for μ is determined; then for every $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \frac{p_{n-k}^{(k)}(x)}{p_n(x)} = a_k q_{k-1}(x), \tag{3.6}$$

uniformly on compact subsets of $\mathbb{C}\setminus(X_1 \cup X_2)$.

Proof. We use induction on k. When $k=0$, the result is immediate. For $k=1$ we have Markov's theorem:

$$\lim_{n\to\infty} \frac{p_{n-1}^{(1)}(x)}{p_n(x)} = a_1 \int \frac{d\mu(y)}{x-y},$$

uniformly on every compact subset of $\mathbb{C}\setminus\text{co}(\text{supp}(\mu))$. Markov proved this result for $\text{supp}(\mu) = [a, b]$ with a and b finite [28, p.57], but the result also holds for infinite intervals provided the underlying moment problem is determined. Actually $p_{n-1}^{(1)}(x)/p_n(x)$ is the $[n-1/n]$ Padé approximant of $a_0 q_0$ at ∞. The uniform convergence on compact subsets of $\mathbb{C}\setminus(X_1 \cup X_2)$ can be shown as follows. Let K be a compact subset of $\mathbb{C}\setminus(X_1 \cup X_2)$; then it contains at most a finite number of zeros of the sequence $\{p_n\}$ and none of these zeros belong to X_2. Therefore we can choose an integer N such that $K \cap (X_1 \cup Z_N) = \emptyset$. The distance

$$\delta = \inf\{\, |z-x|;\ z \in K,\ x \in X_1 \cup Z_N \}$$

is strictly positive and by (3.2) we have for $n \geqslant N$,

$$\left| \frac{p_{n-1}^{(1)}(x)}{p_n(x)} \right| \leqslant a_1 \sum_{j=1}^{n} \frac{\lambda_{j,n}}{|x - x_{j,n}|} \leqslant \frac{a_1}{\delta} \sum_{j=1}^{n} \lambda_{j,n} = \frac{a_1}{\delta}.$$

Therefore $p_{n-1}^{(1)}(x)/p_n(x)$ is uniformly bounded on K for $n \geqslant N$ and the result follows from the Stieltjes–Vitali theorem.

Now suppose the result is valid up to k. By (2.5),

$$x p_{n-k}^{(k)}(x) = a_k p_{n-k+1}^{(k-1)}(x) + b_{k-1} p_{n-k}^{(k)}(x) + \frac{a_k^2}{a_{k+1}} p_{n-k-1}^{(k+1)}(x).$$

When we divide each term in this equation by $p_n(x)$, then by the induction hypothesis

$$\lim_{n\to\infty} \frac{p_{n-k-1}^{(k+1)}(x)}{p_n(x)} = \frac{a_{k+1}}{a_k^2}\left[(x-b_{k-1}) a_k q_{k-1}(x) - a_k a_{k-1} q_{k-2}(x)\right] = a_{k+1} q_k(x),$$

uniformly on compact subsets of $\mathbb{C}\setminus(X_1 \cup X_2)$. □

This theorem is very useful in determining the spectral measure $\mu^{(k)}$ for the associated orthogonal polynomials $\{p_n^{(k)}\}$ when the moment problem for μ is determined. Indeed, from

$$\lim_{n\to\infty} \frac{p_{n-1}^{(k+1)}(x)}{p_n^{(k)}(x)} = a_{k+1} \int \frac{d\mu^{(k)}(y)}{x-y}, \quad x \in \mathbb{C}\setminus\mathbb{R},$$

and from

$$\frac{p_{n-1}^{(k+1)}(x)}{p_n^{(k)}(x)} = \frac{p_{n-1}^{(k+1)}(x)}{p_{n+k}(x)} \frac{p_{n+k}(x)}{p_n^{(k)}(x)},$$

we can find the Stieltjes transform of $\mu^{(k)}$ by means of (3.6), giving

$$\int \frac{d\mu^{(k)}(y)}{x-y} = \frac{1}{a_k} \frac{q_k(x)}{q_{k-1}(x)}, \quad x \in \mathbb{C} \setminus \mathbb{R}. \tag{3.7}$$

The measure $\mu^{(k)}$ may now be obtained using the Stieltjes inversion formula [33, pp. 93–96]

$$\tfrac{1}{2}\mu^{(k)}(\{x\}) + \tfrac{1}{2}\mu^{(k)}(\{y\}) + \mu^{(k)}(]x, y[)$$
$$= \frac{1}{2\pi i} \frac{1}{a_k} \lim_{\epsilon \to 0+} \int_x^y \left\{ \frac{q_k(t-i\epsilon)}{q_{k-1}(t-i\epsilon)} - \frac{q_k(t+i\epsilon)}{q_{k-1}(t+i\epsilon)} \right\} dt.$$

The relation (3.7) was already known to Stieltjes for $k=1$ [16, letter 426], [27]. Explicit expressions of the associated measure may be found for classical polynomials [1,4], [7, Chapter VI], [25,32] and Grosjean made a very detailed study of the associated polynomials and functions of the second kind corresponding to the Legendre polynomials [13] and the Jacobi and Gegenbauer polynomials [14].

4. Cesàro summability

In this section we will consider the Cesàro sums

$$\frac{1}{n} \sum_{k=0}^{n-1} p_k(x) q_k(x), \qquad \frac{1}{n} \sum_{k=0}^{n-1} p_k^2(x),$$

the first one outside the spectrum, the latter on the spectrum. The second Cesàro sum is closely related to the Christoffel function $\lambda_n(x)$ which is defined as

$$\lambda_n(x) = \left\{ \sum_{k=0}^{n-1} p_k^2(x) \right\}^{-1}.$$

This Christoffel function plays a crucial role in many investigations concerning orthogonal polynomials [21].

Let us first consider the $(C, 1)$ convergence of $p_n(x) q_n(x)$.

Theorem 3. *Let c_n be a positive sequence such that $(x_{n,n} - x_{1,n})/c_n$ is bounded; then*

$$\frac{p_n'(c_n x)}{p_n(c_n x)} - \sum_{k=0}^{n-1} p_k(c_n x) q_k(c_n x) = O\left(\frac{1}{c_n}\right), \tag{4.1}$$

where the O-term holds uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. If $c_n = 1$ for all n, then the result holds uniformly on compact subsets of $\mathbb{C} \setminus (X_1 \cup X_2)$.

Proof. By Theorem 1 we have

$$\frac{p_n'(c_n x)}{p_n(c_n x)} = \sum_{k=1}^{n} \frac{1}{a_k} \frac{p_{k-1}(c_n x) p_{n-k}^{(k)}(c_n x)}{p_n(c_n x)},$$

and by (2.14),

$$\frac{1}{a_k}\frac{p_{k-1}(c_nx)p_{n-k}^{(k)}(c_nx)}{p_n(c_nx)}=p_{k-1}(c_nx)q_{k-1}(c_nx)-p_{k-1}^2(c_nx)\frac{q_n(c_nx)}{p_n(c_nx)}.$$

Therefore

$$\frac{p_n'(c_nx)}{p_n(c_nx)}-\sum_{k=0}^{n-1}p_k(c_nx)q_k(c_nx)=-\frac{q_n(c_nx)}{p_n(c_nx)}\sum_{k=0}^{n-1}p_k^2(c_nx). \tag{4.2}$$

By means of the confluent form of the Christoffel–Darboux formula [7, p.24] we have

$$\begin{aligned}\frac{q_n(c_nx)}{p_n(c_nx)}\sum_{k=0}^{n-1}p_k^2(c_nx)&=\frac{q_n(c_nx)}{p_n(c_nx)}a_n[p_n'(c_nx)p_{n-1}(c_nx)-p_n(c_nx)p_{n-1}'(c_nx)]\\&=a_np_n(c_nx)q_n(c_nx)\frac{p_{n-1}(c_nx)}{p_n(c_nx)}\left[\frac{p_n'(c_nx)}{p_n(c_nx)}-\frac{p_{n-1}'(c_nx)}{p_{n-1}(c_nx)}\right].\end{aligned}$$

Now let K be a compact set in $\mathbb{C}\setminus\mathbb{R}$ and let δ be the distance between K and $\mathbb{R}$; then for $x\in K$,

$$|p_n(c_nx)q_n(c_nx)|\leqslant\int\frac{p_n^2(y)}{|c_nx-y|}\,\mathrm{d}\mu(y)\leqslant\frac{1}{c_n\delta}$$

and

$$\left|\frac{p_{n-1}(c_nx)}{p_n(c_nx)}\right|\leqslant a_n\sum_{j=1}^{n}\frac{\lambda_{j,n}p_{n-1}^2(x_{j,n})}{|c_nx-x_{j,n}|}\leqslant\frac{a_n}{c_n\delta},$$

where we have used (2.8) and (3.5). Moreover,

$$\begin{aligned}\frac{p_n'(c_nx)}{p_n(c_nx)}-\frac{p_{n-1}'(c_nx)}{p_{n-1}(c_nx)}&=\sum_{j=1}^{n}\frac{1}{c_nx-x_{j,n}}-\sum_{j=1}^{n-1}\frac{1}{c_nx-x_{j,n-1}}\\&=\frac{1}{c_nx-x_{n,n}}-\sum_{j=1}^{n-1}\frac{x_{j,n-1}-x_{j,n}}{(c_nx-x_{j,n})(c_nx-x_{j,n-1})}.\end{aligned}$$

By the interlacing of the zeros $x_{j,n}<x_{j,n-1}<x_{j+1,n}$ this gives

$$\begin{aligned}\left|\frac{p_n'(c_nx)}{p_n(c_nx)}-\frac{p_{n-1}'(c_nx)}{p_{n-1}(c_nx)}\right|&\leqslant\frac{1}{c_n\delta}+\frac{1}{c_n^2\delta^2}\sum_{j=1}^{n-1}(x_{j,n-1}-x_{j,n})\\&\leqslant\frac{1}{c_n\delta}+\frac{1}{c_n^2\delta^2}\sum_{j=1}^{n-1}(x_{j+1,n}-x_{j,n})=\frac{1}{c_n\delta}+\frac{1}{\delta^2}\frac{x_{n,n}-x_{1,n}}{c_n^2}.\end{aligned}$$

A combination of all these bounds gives

$$c_n\left|\frac{p_n'(c_nx)}{p_n(c_nx)}-\sum_{k=0}^{n-1}p_k(c_nx)q_k(c_nx)\right|\leqslant\left(\frac{a_n}{c_n}\right)^2\frac{1}{\delta^3}\left(1+\frac{1}{\delta}\frac{x_{n,n}-x_{1,n}}{c_n}\right).$$

If $(x_{n,n} - x_{1,n})/c_n$ is bounded, then also a_n/c_n is bounded because

$$\max(x_{1,n}^2, x_{n,n}^2) \geqslant \max_{0 \leqslant k \leqslant n-1} (a_k^2 + a_{k+1}^2 + b_k^2)$$

[19, p.52]. This concludes the proof. □

A more precise result can be obtained when $\operatorname{supp}(\mu) = [-1, 1]$ [20]. Let us introduce some sequences of probability measures associated with orthogonal polynomials. Define ν_n and ξ_n by

$$\int f(x)\,\mathrm{d}\nu_n(x) = \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{x_{j,n}}{c_n}\right) \tag{4.3}$$

and

$$\int f(x)\,\mathrm{d}\xi_n(x) = \int \frac{1}{n} \sum_{j=0}^{n-1} p_k^2(x) f\left(\frac{x}{c_n}\right) \mathrm{d}\mu(x), \tag{4.4}$$

where f is a bounded continuous function. The behaviour of ν_n as $n \to \infty$ gives the asymptotic distribution of the *contracted* zeros $x_{j,n}/c_n$, $1 \leqslant j \leqslant n$, and the behaviour of ξ_n gives the *weak* $(C, 1)$ behaviour of the polynomials p_n on the spectrum. Recall that a sequence μ_n of probability measures converges weakly to a positive measure μ if and only if

$$\int f(x)\,\mathrm{d}\mu_n(x) \quad \to \quad \int f(x)\,\mathrm{d}\mu(x),$$

for every bounded and continuous function f.

Theorem 4. *If c_n is a positive sequence such that $(x_{n,n} - x_{1,n})/c_n$ is bounded, then the weak limits of ν_n and ξ_n are the same.*

Proof. The Stieltjes transforms of ν_n and ξ_n are

$$\int \frac{\mathrm{d}\nu_n(y)}{x-y} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{x - x_{j,n}/c_n} = \frac{c_n}{n} \frac{p_n'(c_n x)}{p_n(c_n x)}$$

and

$$\int \frac{\mathrm{d}\xi_n(y)}{x-y} = \int \frac{1}{n} \sum_{j=0}^{n-1} p_j^2(y) \frac{\mathrm{d}\mu(y)}{x - y/c_n} = \frac{c_n}{n} \sum_{k=0}^{n-1} p_k(c_n x) q_k(c_n x),$$

and by Theorem 3 we see that the Stieltjes transforms of ν_n and ξ_n have the same limits. The Grommer–Hamburger theorem [33, p.104] states that when a sequence of probability measures μ_n is such that the Stieltjes transforms converge on compact subsets of $\mathbb{C} \setminus \mathbb{R}$ to a function S, then S is the Stieltjes transform of a positive measure μ and μ_n converges weakly to μ. The result thus follows by applying this to our sequences of probability measures. □

This theorem states that the sequence ξ_n and the zero distribution of orthogonal polynomials are closely related. Moreover, the $(C, 1)$ limit of $p_n(x)q_n(x)$ is exactly the Stieltjes transform of the asymptotic zero distribution. This was already known before [10,17], [19, p.49], [30], [31, Chapter 5].

The $(C, 1)$ convergence of $p_n(x)q_n(x)$ can be improved to $(C, 0)$ convergence when the recurrence coefficients behave nicely.

Theorem 5. *Suppose that* $a_n \to a > 0$ *and* $b_n \to b$; *then*

$$\lim_{n\to\infty} p_n(x)q_n(x) = \frac{1}{\sqrt{(x-b)^2-4a^2}},$$

uniformly on compact subsets of $\mathbb{C}\setminus\operatorname{supp}(\mu)$. *The square root is such that*

$$\lim_{x\to\infty} \frac{\sqrt{(x-b)^2-4a^2}}{x} = 1.$$

For every continuous function $f:[b-2a,\ b+2a] \to \mathbb{R}$ *which is bounded outside* $[b-2a,\ b+2a]$ *one has*

$$\lim_{n\to\infty} \int f(y)p_n^2(y)\ \mathrm{d}\mu(y) = \frac{1}{\pi}\int_{b-2a}^{b+2a} \frac{f(y)}{\sqrt{4a^2-(y-b)^2}}\ \mathrm{d}y.$$

Proof. From (2.7) with $k=0$ one easily obtains

$$a_{n+1}\left[\frac{p_{n+1}(x)}{p_n(x)} - \frac{q_{n+1}(x)}{q_n(x)}\right] = \frac{1}{p_n(x)q_n(x)}.$$

Now $\{p_n\}$ and $\{q_n\}$ both satisfy the same recurrence relation with converging recurrence coefficients. By Poincaré's result it then follows that the ratios $p_{n+1}(x)/p_n(x)$ and $q_{n+1}(x)/q_n(x)$ converge to one of the solutions of the quadratic equation

$$ag^2(x) - (x-b)g(x) + a = 0,$$

provided that $|g(x)| \neq 1$. Clearly,

$$\lim_{x\to\infty} \frac{p_{n+1}(x)}{xp_n(x)} = \frac{\gamma_{n+1}}{\gamma_n}$$

and

$$\lim_{x\to\infty} \frac{q_{n+1}(x)}{q_n(x)}\frac{p_{n+1}(x)}{p_n(x)} = \frac{\lim_{x\to\infty} x\int p_{n+1}^2(y)/(x-y)\ \mathrm{d}\mu(y)}{\lim_{x\to\infty} x\int p_n^2(y)/(x-y)\ \mathrm{d}\mu(y)} = 1,$$

so that

$$\lim_{n\to\infty} \frac{p_{n+1}(x)}{p_n(x)} = \frac{x-b+\sqrt{(x-b)^2-4a^2}}{2a}$$

and

$$\lim_{n\to\infty} \frac{q_{n+1}(x)}{q_n(x)} = \frac{x-b-\sqrt{(x-b)^2-4a^2}}{2a},$$

in the neighborhood of ∞, and in such neighborhoods we thus have

$$\lim_{n\to\infty} p_n(x)q_n(x) = \frac{1}{\sqrt{(x-b)^2-4a^2}}.$$

This convergence is uniform on compact subsets of $\mathbb{C} \setminus \text{supp}(\mu)$ since on such a compact set K we have

$$| p_n(x) q_n(x) | \leqslant \int \frac{p_n^2(y)}{|x-y|} \, d\mu(y) \leqslant \frac{1}{\delta},$$

where δ is the distance between K and $\text{supp}(\mu)$. The uniform convergence thus follows from the Stieltjes–Vitali theorem. The second part of the theorem concerning the weak convergence of the measures ρ_n given by $d\rho_n(y) = p_n^2(y) \, d\mu(y)$ follows, since

$$\frac{1}{\pi} \int_{b-2a}^{b+2a} \frac{1}{\sqrt{4a^2 - (y-b)^2}} \frac{dy}{x-y} = \frac{1}{\sqrt{(x-b)^2 - 4a^2}},$$

so that the Stieltjes transforms of the measures ρ_n converge to the Stieltjes transform of the arcsin measure on $[b-2a, b+2a]$ uniformly on compact sets in $\mathbb{C} \setminus \mathbb{R}$ and we can then apply the theorem of Grommer and Hamburger. □

The previous result was already given in [24] and is an easy consequence of the convergence of the recurrence coefficients. It shows that the functions of the second kind behave like the reciprocal of the orthogonal polynomials outside the spectrum. A stronger result holds when we impose stronger conditions on the measure μ.

Corollary. *Suppose that* $\text{supp}(\mu) = [-1, 1]$ *and that*

$$\int_{-1}^{1} \frac{\log \mu'(y)}{\sqrt{1-y^2}} \, dy > -\infty;$$

then

$$\lim_{n \to \infty} q_n(x) \left(x + \sqrt{x^2 - 1} \right)^n = \sqrt{2\pi} \sqrt{x^2 - 1} \, D\left(x - \sqrt{x^2 - 1} \right),$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$, *where D is the Szegő function*

$$D(z) = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \mu'(\cos \theta) \frac{1 + z \, e^{-i\theta}}{1 - z \, e^{-i\theta}} \, d\theta \right\}, \quad |z| < 1.$$

Proof. This follows immediately since the recurrence coefficients satisfy

$$a_n \to \tfrac{1}{2}, \qquad b_n \to 0,$$

and

$$\lim_{n \to \infty} p_n(x) \left(x + \sqrt{x^2 - 1} \right)^{-n} = \sqrt{2\pi} \, D\left(x - \sqrt{x^2 - 1} \right),$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$ [8, Chapter V], [28, Chapter XII]. □

This result was already given in [3] where a more complicated proof is used.

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