A Krein-like Formula for Singular Perturbations of Self-Adjoint Operators and Applications

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Given a self-adjoint operator \( A: D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H} \) and a continuous linear operator \( \tau: D(A) \rightarrow X \) with Range \( \tau \cap X' = \{0\} \), \( X \) a Banach space, we explicitly construct a family \( A_\lambda \) of self-adjoint operators such that any \( A_\lambda \) coincides with the original \( A \) on the kernel of \( \tau \). Such a family is obtained by giving a Krein-like formula where the role of the deficiency spaces is played by the dual pair \((\mathcal{H}, X')\); the parameter \( \lambda \) belongs to the space of symmetric operators from \( X \) to \( X \). When \( X = C \) one recovers the “\( \mathcal{H}_{-2} \)” construction of Kiselev and Simon and so, to some extent, our results can be regarded as an extension of it to the infinite rank case. Considering the situation in which \( \mathcal{H} = L^2(\mathbb{R}^n) \) and \( \tau \) is the trace (restriction) operator along some null subset, we give various applications to singular perturbations of non necessarily elliptic pseudo-differential operators, thus unifying and extending previously known results.

1. INTRODUCTION

Let \( A: D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H} \) be a self-adjoint operator on the Hilbert space \( \mathcal{H} \) and suppose that there exists a linear dense set \( N \subseteq D(A) \) which is closed with respect to the graph norm on \( D(A) \). If we denote by \( A_\lambda \) the restriction of \( A \) to \( N \), then \( A_\lambda \) is a closed, densely defined, symmetric operator. Since \( N \neq D(A) \), \( A \) is a non-trivial extension of \( A_\lambda \) and so, by the von Neumann theory on self-adjoint extensions of closed symmetric operators (see [31], [17, §XII.4], [35, §X.1]), we know that the deficiency indices \( n_\pm \), defined as the dimensions of \( K_\pm := \text{Kernel } A_\lambda^\pm \), are equal and strictly positive. The family of self-adjoint extensions of \( A_\lambda \) is then parametrized by the unitary maps from \( K_+ \) onto \( K_- \). When \( A \) is strictly positive, a deeper and more explicit construction of the (positive if \( \dim K = +\infty \)) self-adjoint extensions of \( A_\lambda \) is given by the Birman–Krein–Vishik theory (see [27], [40], [9], [6]). In this case the family of (positive) extensions is parametrized by the (positive) quadratic forms on \( K \).
Any self-adjoint extension $\tilde{A}_N \neq A$ can then be interpreted as a singular perturbation of $A$ since the two operators differ only on $\mathcal{H} \setminus N$, the set $\mathcal{H} \setminus N$ being “thin” since its complement is a linear dense subset of $\mathcal{H}$.

In the case $n_\pm = 1$, Krein obtained, in 1943 (see [25]), a quite explicit formula relating the resolvents of any two self-adjoint extensions of a given symmetric operator. Such a formula was then extended, by Krein himself in 1946 (see [26]), to the case $n_\pm = m < + \infty$. In our setting it states the following: for any $z \in \rho(A) \cap \rho(\tilde{A}_N)$ one has

$$( -\tilde{A}_N + z )^{-1} = ( -A + z )^{-1} + \sum_{j,k=1}^{m} \Gamma(z)_{jk}^{-1} \varphi_j(z) \otimes \varphi_k(z^*) ,$$

where

$$\varphi_k(z) := \varphi_k - (i-z)(-A+z)^{-1} \varphi_k ,$$

$\{ \varphi_k \}_{1}^{m}$ being the set of linear independent solutions of

$$A_N^* \varphi = i\varphi , \quad \varphi \in D(A_N^*),$$

and where the invertible matrix $\Gamma(z)$ satisfies ($\langle \cdot , \cdot \rangle$ denoting the scalar product on $\mathcal{H}$)

$$\langle \Gamma(z)_{jk} - \Gamma(w)_{jk} \rangle = (z-w)\langle \varphi_j(z^*), \varphi_k(z) \rangle .$$

By such a formula, since $N$ is dense, one can then readily define $\tilde{A}_N$ as

$$D(\tilde{A}_N) := \{ \phi \in \mathcal{H} : \phi = \phi_0 + \sum_{j,k=1}^{m} \Gamma(z)_{jk}^{-1} \langle \varphi_j(z), \varphi_* \rangle \varphi_j(z) , \varphi_0 \in D(A) \}$$

$$(-\tilde{A}_N + z) \phi = (-A + z) \phi_0 .$$

Krein’s original papers were written in russian, but his results were popularized in some excellent monographs (see e.g. [1, Chap. VII]). Instead, the analogous formula for the case $n_\pm = + \infty$, which was obtained by Saakjan in 1965 (see [36]), is much less known, since the work is not available in english (see however [18] and references therein). Due probably to this fact, the Krein formula for $n_\pm = + \infty$ (similar considerations also apply to the Birman–Krein–Vishik theory) was rarely used in concrete applications: we are mainly referring to the much studied case of singular perturbations of the Laplacian supported by null sets (see e.g. [4], [3], [10] and references therein). Indeed in situations of this kind other approaches are used: extensions are mainly obtained either as resolvent limits of less singular perturbations or by other constructions often resembling variations of either the Krein formula or the Birman–Krein–Vishik
theory. Usually such approaches rely on the elliptic nature of the Laplacian and are not applicable to the study of singular perturbations of hyperbolic operators (this was the original motivation of our work).

Here we show how, when the (necessarily dense) set $N$ is the kernel of a continuous linear map $\tau : D(A) \to J$ such that $\text{Range } \tau \cap J' = \{0\}$, $J$ a Banach space, one can prove, by almost straightforward arguments, a Krein-like formula for a family $A_N\theta$, $\theta$ a symmetric operator from $J$ to $J$, of self-adjoint extensions of $A_N$, where the role of $K_{\pm}$ is played by the dual pair $(J, J')$ (our construction could be given for $J$ a locally convex space, but we will not strive here for the maximum of generality).

In contrast to other approaches (see e.g. [36], [18], [15], [16] and references therein) the formula given here turns out to be relatively simple being expressed directly in terms of the map $\tau$; moreover we do not need to compute $A_N^*$. In more detail (see Theorem 2.1) one obtains, under a hypothesis which we prove to be satisfied under relatively weak conditions (see Proposition 2.1),

\[ (-A_N^* + z)^{-1} = (-A + z)^{-1} + G(z) \cdot (\Theta + I(z))^{-1} \cdot \tilde{G}(z), \]

where

\[ \tilde{G}(z) := \tau \cdot (-A + z)^{-1}, \quad G(z) := C^{-1} \cdot \tilde{G}(z^*)' \]

($C$ being the canonical isomorphism of $J$ onto $J'$) and the conjugate linear operator $I(z) : D \subseteq J' \to J$ satisfies the equation

\[ \forall \ell \in D, \quad \frac{d}{dz} I(z) \ell = \tilde{G}(z) \cdot G(z) \ell \]

which (see Lemma 2.2) we show to have an explicit (in terms of $\tau$ itself) bounded operator solution. Such a solution plays a fundamental role in finding (see Lemmata 2.3 and 2.4) other nicer (even if unbounded) solutions which we then use in (some of) the examples.

In Section 3, after showing (Example 3.1) how our construction, in the case $J = C$, reproduces the "$\mathcal{K}_-$-construction" given in [24] and how, in the case $A$ is strictly positive, it gives a variation on the Birman–Krein–Vishik theory which comprises the results in [22] (Example 3.2), we use the above Krein-like formula to study singular perturbations of non necessarily elliptic pseudo-differential operators, thus unifying and extending previously known results. More precisely we give the following examples:

- Finitely many point interaction in three dimensions (Example 3.3);
- Infinitely many point interaction in three dimensions (Example 3.4);
• Singular perturbations of the Laplacian in three and four dimensions supported by regular curves (Example 3.5);

• Singular perturbations, supported by null sets with Hausdorff codimension less than 2, of translation invariant pseudo-differential operators with domain $H^s(\mathbb{R}^n)$ (Example 3.6);

• Singular perturbations of the d'Alembertian in four dimensions supported by time-like straight lines (Example 3.7). In order to limit the length of the paper we content ourselves with discussing here only the case of a straight line. A complete study of the case of a generic time-like curve will be the subject of a separate paper. We believe that the detailed study of such a kind of operators will lead to a rigorous framework for the classical and quantum electrodynamics of point particles in the spirit of the results obtained, for the linearized (or dipole) case, in [32]-[34] and [7];

• Singular perturbations, supported by null sets, of translation invariant pseudo-differential operators with domain the Malgrange spaces $H_d(\mathbb{R}^n)$ (Example 3.8).

1.1. Definitions and Notations

• Given a Banach space $\mathcal{X}$ we denote by $\mathcal{X}'$ its strong dual

• $L(\mathcal{X}, \mathcal{Y})$, resp. $\tilde{L}(\mathcal{X}, \mathcal{Y})$, denotes the space of linear, resp. conjugate linear, operators from the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$.

• $B(\mathcal{X}, \mathcal{Y})$, resp. $\tilde{B}(\mathcal{X}, \mathcal{Y})$, denotes the space of bounded, everywhere defined, linear, resp. conjugate linear, operators on the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$. It is a Banach space with the norm $\|A\|_{\mathcal{X}, \mathcal{Y}} := \sup \{ \|Ax\|_\mathcal{Y}, \|x\|_\mathcal{X} = 1 \}$.

• The closed linear operator operator $A'$ and the conjugate linear closed operator $\tilde{A}'$ are the adjoints of the densely defined linear operator $A$ and of the densely defined conjugate linear operator $\tilde{A}$ respectively, i.e.

$$
\forall x \in D(A) \subseteq \mathcal{X}, \quad \forall \ell \in D(A') \subseteq \mathcal{Y}', \quad (A'\ell)(x) = \ell(Ax),
$$

$$
\forall x \in D(\tilde{A}) \subseteq \mathcal{X}, \quad \forall \ell \in D(\tilde{A}') \subseteq \mathcal{Y}', \quad (\tilde{A}'\ell)(x) = (\ell(\tilde{A}x))^*,
$$

where $^*$ denotes complex conjugation.

• $\tilde{S}(\mathcal{X}', \mathcal{X})$ denotes the space of conjugate linear operators $A$ such that

$$
\forall \ell_1, \ell_2 \in D(A), \quad \ell_1(A\ell_2) = (\ell_2(A\ell_1))^*.
$$

• For any $A \in \tilde{S}(\mathcal{X}', \mathcal{X})$ we define

$$
\gamma(A) := \inf \{ \ell(A\ell), \|\ell\|_{\mathcal{Y}} = 1, \ell \in D(A) \}.
$$
\[ J_x \in B(\mathcal{X}, \mathcal{X}^*) \] indicates the injective map (an isomorphism when \( \mathcal{X} \) is reflexive) defined by \( (J_x x)(t) := \ell(x) \).

- If \( \mathcal{H} \) is a complex Hilbert space with scalar product (conjugate linear w.r.t. the first variable) \( \langle \cdot, \cdot \rangle \), then \( C_{\mathcal{H}} \in B(\mathcal{H}, \mathcal{H}^*) \) denotes the isomorphism defined by \( (C_{\mathcal{H}} y)(x) := \langle y, x \rangle \). The Hilbert adjoint of the densely defined linear operator \( A \) is then given by \( A^* = C_{\mathcal{H}}^{-1} \cdot A^* \cdot C_{\mathcal{H}} \).

\( \mathcal{F} \) and \( \ast \) denote Fourier transform and convolution respectively.

- \( H'(\mathbb{R}^s) \), \( s \in \mathbb{R} \), is the usual scale of Sobolev-Hilbert spaces, i.e. \( H'(\mathbb{R}^s) \) is the space of tempered distributions with a Fourier transform which is square integrable w.r.t. the measure with density \( (1 + |x|^2)^s \).

\( c \) denotes a generic strictly positive constant which can change from line to line.

### 2. A KREIN-LIKE FORMULA

Let

\[ A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H} \]

be a self-adjoint operator on the complex Hilbert space \( \mathcal{H} \). \( D(A) \) inherits a Hilbert space structure by introducing the usual scalar product leading to the graph norm \( \| \phi \|_2 := \langle \phi, \phi \rangle + \langle A\phi, A\phi \rangle \). Denoting the resolvent set of \( A \) by \( \rho(A) \) we define, for any \( z \in \rho(A) \),

\[ R(z) := (-A + z)^{-1} : \mathcal{H} \rightarrow D(A), \quad R(z) \in B(\mathcal{H}, D(A)). \]

We consider now a linear operator

\[ \tau : D(A) \rightarrow \mathcal{X}, \quad \tau \in B(D(A), \mathcal{X}), \]

where \( \mathcal{X} \) is a complex Banach space. By means of \( A \) and \( \tau \) we can define, for any \( z \in \rho(A) \), the following operators:

\[ G(z) := C_{\mathcal{H}}^{-1} \cdot \tilde{G}(z^*)' : \mathcal{X}' \rightarrow \mathcal{H}, \quad G(z) \in B(\mathcal{X}', \mathcal{H}) \]

\[ \tilde{G}(z) := \tau \cdot R(z) : \mathcal{X} \rightarrow \mathcal{X}, \quad \tilde{G}(z) \in B(\mathcal{X}, \mathcal{X}), \]

**Remark 2.1.** Being \( R(z) \) surjective, \( R(z)' \) is injective. If \( \tau \) has dense range then \( \tau ' \) is injective. Therefore, when \( \tau \) has dense range, \( \tilde{G}(z) \) has dense range and \( G(z) \) is injective. This implies that the only \( A \in B(\mathcal{X}, \mathcal{X}^*) \) which solves the operator equation \( G(z) \cdot A \cdot \tilde{G}(z) = 0 \) is the zero operator.
Lemma 2.1. For any \(w\) and \(z\) in \(\rho(A)\) one has

\[(z-w) \tilde{G}(w) \cdot R(z) = \tilde{G}(w) - \tilde{G}(z)\]
\[(z-w) R(w) \cdot G(z) = G(w) - G(z).\]

Proof. By first resolvent identity one has

\[(z-w) \tilde{G}(w) \cdot R(z) = \tilde{G}(w) - \tilde{G}(z)\]
\[R(z) = R(w) \cdot G(z).\]

Therefore

\[(z-w) \tilde{G}(w) \cdot R(z) = (z-w) \tilde{G}(w) - \tilde{G}(z)\]
and, by duality (here \(R(z)\) is considered as an element of \(B(\mathcal{H}, \mathcal{H})\)),

\[G(w) - G(z) = C_{\mathcal{H}}^{-1} \cdot (\tilde{G}(w^*) - \tilde{G}(z^*))'\]
\[= C_{\mathcal{H}}^{-1} \cdot ((z^*-w^*) \tilde{G}(z^*) \cdot R(w^*))'\]
\[= C_{\mathcal{H}}^{-1} \cdot ((z^*-w^*) \tilde{G}(z^*) \cdot R(w) \cdot C_{\mathcal{H}}^{-1} \cdot \tilde{G}(z^*))'\]
\[= (z-w) R(w) \cdot G(z).\]

This ends the proof.

Remark 2.2. The second relation in the lemma above shows that
\[\forall w, z \in \rho(A), \quad \text{Range } (G(w) - G(z)) \subseteq D(A).\]

We want now to define a new self-adjoint operator which, when restricted to the kernel of \(\tau\), coincides with the original \(A\). Since, in the case of a bounded perturbation \(V\), for any \(z\) such that \(\|V \cdot R(z)\|_{\mathcal{H}} < 1\) one has

\[-(A + V) + z)^{-1} = R(z) + R(z) \cdot (I - V \cdot R(z))^{-1} \cdot V \cdot R(z),\]

we are lead to write the presumed resolvent as

\[R'(z) = R(z) + B(z) \cdot \tau \cdot R(z) \equiv R(z) + B(z) \cdot \tilde{G}(z),\]

where \(B(z) \in B(\mathcal{H}, \mathcal{H})\) has to be determined.

Self-adjointness requires \(R'(z)^* = R'(z^*)\) or, equivalently,

\[G(z) \cdot B(z^*)' \cdot C_{\mathcal{H}} = B(z) \cdot \tilde{G}(z).\] (1)

Therefore if we put \(B(z) = G(z) \cdot A(z)\), \(A(z) \in \tilde{B}(\mathcal{H}, \mathcal{H}')\), then one can check that (1) is implied by (by Remark 2.1, when \(\tau\) has dense range, is equivalent to)

\[A(z)^* \cdot J_{\mathcal{H}} = A(z^*),\] (2)
We now impose the resolvent identity

\[(z - w) R'(w) R'(z) = R'(w) - R'(z).\]  \tag{3} \]

Since (we make use of Lemma 2.1)

\[(z - w) R'(w) \cdot R'(z) = (z - w) (R(w) \cdot R(z) + R(w) \cdot A(z) \cdot G(z) + G(w) \cdot A(w) \cdot G(w) \cdot R(z)) \]
\[+ G(w) \cdot A(w) \cdot G(w) \cdot G(z) \cdot A(z) \cdot \tilde{G}(z) \]
\[+ (z - w) G(w) \cdot A(w) \cdot \tilde{G}(w) \cdot G(z) \cdot A(z) \cdot \tilde{G}(z) \]
\[= R'(w) - R'(z) + G(w) \cdot A(z) \cdot G(z) - G(z) \cdot A(z) \cdot \tilde{G}(z) \]
\[+ G(w) \cdot A(w) \cdot \tilde{G}(w) - G(w) \cdot A(w) \cdot \tilde{G}(z) \]
\[+ (z - w) G(w) \cdot A(w) \cdot \tilde{G}(w) \cdot G(z) \cdot A(z) \cdot \tilde{G}(z) \]
\[= R'(w) - R'(z) + G(w) \cdot (A(z) - A(w)) \cdot \tilde{G}(z) \]
\[+ (z - w) G(w) \cdot A(w) \cdot \tilde{G}(w) \cdot G(z) \cdot A(z) \cdot \tilde{G}(z), \]

the relation (3) is implied by (by Remark 2.1, when \(\tau\) has dense range, is equivalent to)

\[A(w) - A(z) = (z - w) A(w) \cdot \tilde{G}(w) \cdot G(z) \cdot A(z).\]  \tag{4} \]

Suppose now that there exists a (necessarily closed) operator

\[\Gamma(z) : D \subseteq \mathcal{H} \to \mathcal{H} \]

such that, for some open set \(Z \subseteq \rho(A)\) such that \(z \in Z\) iff \(z^* \in Z\), one has

\[\forall z \in Z, \quad \Gamma(z)^{-1} = A(z).\]

Then we have that (4) forces \(\Gamma(z)\) to satisfy the relation

\[\Gamma(z) - \Gamma(w) = (z - w) \tilde{G}(w) \cdot G(z),\]  \tag{5} \]

which is equivalent to

\[\forall \ell \in D \subseteq \mathcal{H}, \quad \frac{d}{dz} \Gamma(z) \ell = \tilde{G}(z) \cdot G(z) \ell.\]  \tag{6} \]

Regarding the identity (2), suppose that

\[\forall \ell_1, \ell_2 \in D, \quad \ell_2 (\Gamma(z^*) \ell_1) = (\ell_2 (\Gamma(z) \ell_1))^*.\]  \tag{7} \]
This, if \( I(z) \) is densely defined, is equivalent to \( J_\varphi \cdot I(z^*) \subseteq I(z)' \), equality being, in the unbounded case, stronger than (7). In the case \( I(z) \) has a bounded inverse given by \( A(z) \) as we are pretending, (7) implies (2) which, if \( I(z) \) is densely defined, is then equivalent (use e.g. [23, Thm. 5.30, Chap. III]) to

\[
I(z)' = J_\varphi \cdot I(z^*). \tag{7.1}
\]

We will therefore concentrate now on the set of maps

\[
\Gamma: \rho(A) \to \tilde{L}(\mathcal{X}', \mathcal{X})
\]

which satisfy (5) (equivalently (6)) and (7) (we are implicitly supposing that \( D \), the domain of \( I(z) \), is \( z \)-independent).

An explicit representation of the set of such maps is given by the following

**Lemma 2.2.** Given any \( z_0 \in \rho(A) \) the map

\[
\hat{\Gamma}: \rho(A) \to \tilde{B}(\mathcal{X}', \mathcal{X}) \quad \hat{\Gamma}(z) := \tau \cdot \left( \frac{G(z_0) + G(z^*)}{2} - G(z) \right)
\]

satisfies (5) and (7.1).

**Proof.** By Lemma 2.1 one has

\[
(z - w) \cdot \tilde{G}(w) \cdot G(z) = \tau(G(w) - G(z)) = \tau(G(z_0) - G(z)) - \tau(G(z_0) - G(w))
\]

and so \( \tau \cdot (G(z_0) - G(z)) \) solves (5); by linearity also \( \hat{\Gamma}(z) \) is a solution.

As regard (7.1) let us at first note that

\[
J_\varphi \cdot \tilde{G}(z) = \tilde{G}(z^*) \cdot J_{\varphi^*}
\]

and

\[
(C_{\varphi'} \cdot J_{\varphi'}) y(x) = (J_{\varphi'} y(C_{\varphi} x))^* = (C_{\varphi'} y(x))^* = \langle x, y \rangle^* = C_{\varphi'} y(x).
\]

Therefore one has

\[
(J_{\varphi} \cdot \tilde{G}(w'))' = \tilde{G}(z)' \cdot \tilde{G}(w') = \tilde{G}(z^*) \cdot (C_{\varphi'})^{-1} \cdot \tilde{G}(w')
\]

\[
= J_{\varphi} \cdot \tilde{G}(z^*) \cdot J_{\varphi'}^{-1} \cdot (C_{\varphi'})^{-1} \cdot \tilde{G}(w') = J_{\varphi} \cdot \tilde{G}(z^*) \cdot C_{\varphi'}^{-1} \cdot \tilde{G}(w')
\]

\[
= J_{\varphi} \cdot \tilde{G}(z^*) \cdot \tilde{G}(w^*)
\]

which immediately implies that \( \hat{\Gamma}(z) \) satisfies (7.1).
Remark 2.3. Lemma 2.2 shows that the set of maps

$$\Gamma: \rho(A) \to \tilde{L}(\mathcal{X'}, \mathcal{X})$$

which satisfy (6) and (7) can be parametrized by $\tilde{S}(\mathcal{X'}, \mathcal{X})$. Indeed, by (6), any of such maps must differ from $\tilde{F}(z) \in \tilde{B}(\mathcal{X'}, \mathcal{X})$ by a $z$-independent operator in $\tilde{S}(\mathcal{X'}, \mathcal{X})$. Therefore any parametrization is of the kind

$$\Gamma_\Theta: \rho(A) \to \tilde{L}(\mathcal{X'}, \mathcal{X}), \quad \Gamma_\Theta(z) = \Theta + \Gamma(z), \quad \Theta \in \tilde{S}(\mathcal{X'}, \mathcal{X}),$$

(9)

where $\Gamma(z)$ is some map which satisfies (6) and (7).

Lemma 2.2 does not entirely solve the problem of the search of $\Gamma(z)$ since $\tilde{F}(z)$ can give rise to non-local boundary conditions (see Remark 2.7 below); moreover $\Gamma(z)$ explicitly depends on the choice of a particular $z_0 \in \rho(A)$. However the boundedness of $\tilde{F}(z)$ implies a useful criterion for obtaining other maps $\tilde{F}(z)$ which satisfy (6) and (7):

**Lemma 2.3.** Suppose that

$$\tilde{F}(z): D(\tilde{F}) \subseteq \mathcal{X'} \to \mathcal{X}, \quad z \in \rho(A),$$

is a family of conjugate linear, densely defined operators such that

$$\forall \ell_1, \ell_2 \in D(\tilde{F}), \quad \ell_2^*(\tilde{F}(z)^* \ell_1) = (\ell_1(\tilde{F}(z) \ell_2))^*$$

(10)

and

$$\forall \ell \in E, \forall \ell_1 \in D(\tilde{F}), \quad \frac{d}{dz} \langle \tilde{F}(z) \ell_1 \rangle = \ell(G(z)) \ell_1$$

$$\equiv \langle G(z)^* \ell, G(z) \ell_1 \rangle,$$

(11)

where $E \subseteq \mathcal{X'}$ is either a dense subspace or the dual of some Schauder base in $\mathcal{X}$. Then $\tilde{F}(z)$ is closable and its closure satisfies (6) and (7).

**Proof.** By (11) necessarily $\tilde{F}(z)$ differs from (the restriction to $D(\tilde{F})$ of) $\tilde{F}(z)$ by a $z$-independent, densely defined operator $\tilde{\Theta} \in \tilde{S}(\mathcal{X'}, \mathcal{X})$. Being densely defined, $\tilde{\Theta}$ has an adjoint and $J_x \cdot \tilde{\Theta} \equiv \tilde{\Theta}'$. Therefore, being $J_x$ injective, $\tilde{\Theta}$ is closable and so, being $\tilde{F}(z)$ bounded, $\tilde{F}(z) = \tilde{\Theta} + \tilde{F}(z)$ is closable. Denoting by $\tilde{\Theta}$ the closure of $\tilde{\Theta}$, the closure of $\tilde{F}(z)$ is given by $\tilde{\Theta} + \tilde{F}(z)$, which satisfies (6) and (7) by Lemma 2.2. \\
We state now our main result:

**Theorem 2.1.** Let \( \Gamma_\omega(z) \) be as in (9). Under the hypotheses

\[
Z_\omega := \{ z \in \rho(A) : \exists \Gamma_\omega(z)^{-1} \in \mathcal{B}(\mathcal{X}, \mathcal{X}'), \exists \Gamma_\omega(z^*)^{-1} \in \mathcal{B}(\mathcal{X}, \mathcal{X}') \},
\]

\[\text{Range } \tau' \cap \mathcal{H}' = \{0\}, \quad \text{(h2)}\]

the bounded linear operator

\[
R_\omega(z) := R(z) + G(z) \cdot \Gamma_\omega(z)^{-1} \cdot \tilde{G}(z), \quad z \in Z_\omega,
\]

is a resolvent of the self-adjoint operator \( A_\omega' \) which coincides with \( A \) on the kernel of \( \tau \) and which is defined by

\[
D(A_\omega') := \{ \phi \in \mathcal{H} : \phi = \phi_2 + G(z) \cdot \Gamma_\omega(z)^{-1} \cdot \tau \phi_2, \phi_2 \in D(A) \},
\]

\[
(-A_\omega' + z) \phi := (-A + z) \phi_2.
\]

Such a definition is \( z \)-independent and the decomposition of \( \phi \) entering in the definition of the domain is unique.

**Proof.** We have already proven that, under our hypotheses, \( R_\omega(z) \) is a pseudo-resolvent, i.e.

\[
(z - w) R_\omega'(w) R_\omega^{*'}(z) = R_\omega^{*'}(w) - R_\omega'(z). \quad (12)
\]

We proceed now as in the proof of [4, Thm. II.1.1.1]. By [23, Chap. VIII, §1.1] \( R_\omega(z) \), being a pseudo-resolvent, is the resolvent of a closed operator if and only if it is injective. Since \( R_\omega(z) \phi = 0 \) would imply

\[
R(z) \phi = -G(z) \cdot \Gamma_\omega(z)^{-1} \cdot \tilde{G}(z) \phi,
\]

by (h2) we have \( R(z) \phi = 0 \) (see Remark 2.8 below) and so \( \phi = 0 \).

Since, as we have seen before, (7) implies, when \( z \in Z_\omega \),

\[
\Gamma_\omega(z^*)^{-1} = (\Gamma_\omega(z)^{-1})' \cdot J_\mathcal{X},
\]

one has

\[
(G(z) \cdot \Gamma_\omega(z)^{-1} \cdot \tilde{G}(z))^*
\]

\[
= C_\mathcal{X}' \cdot (G(z) \cdot \Gamma_\omega(z)^{-1} \cdot \tilde{G}(z))' \cdot C_\mathcal{X}
\]

\[
= C_\mathcal{X}' \cdot \tilde{G}(z)' \cdot (\Gamma_\omega(z)^{-1})' \cdot \tilde{G}(z^*)'' \cdot (C_\mathcal{X}')^{-1} \cdot C_\mathcal{X}
\]

\[
= G(z^*) \cdot (\Gamma_\omega(z)'')^{-1} \cdot J_\mathcal{X} \cdot \tilde{G}(z^*) \cdot J_\mathcal{X}' \cdot (C_\mathcal{X}')^{-1} \cdot C_\mathcal{X}
\]

\[
= G(z^*) \cdot \Gamma_\omega(z^*)^{-1} \cdot \tilde{G}(z^*),
\]
and so
\[ R_{\varnothing}^*(z)^* = R_{\varnothing}^*(z^*) . \]

This gives the denseness of \( D(A_{\varnothing}^z) := \text{Range } R_{\varnothing}^z(z) \). Indeed \( \psi \perp D(A_{\varnothing}^z) \), which is equivalent to \( \langle R_{\varnothing}^z(z^*) \phi , \psi \rangle = 0 \) for all \( \psi \in \mathcal{H} \), implies \( \phi = 0 \).

Let us now define, on the dense domain \( D(A_{\varnothing}^z) \), the closed operator
\[ A_{\varnothing}^z := R_{\varnothing}^z(z)^{-1} - z \]
which, by the resolvent identity (12), is independent of \( z \); it is self-adjoint since
\[ ((A_{\varnothing}^z)^* + z^*)^{-1} = R_{\varnothing}^z(z)^* = R_{\varnothing}^z(z)^* = (A_{\varnothing}^z + z^*)^{-1}. \]

To conclude, the uniqueness of the decomposition
\[ \phi = \phi_\varnothing + \Gamma_{\varnothing}(z)^{-1} \cdot \tau \phi_z , \quad \phi \in D(A_{\varnothing}^z) , \]
is an immediate consequence of (h2).

**Remark 2.4.** Viewing \( A \) as a bounded operator on \( D(A) \) to \( \mathcal{H} \), we can consider the adjoint \( (-A + z^*)' \), so that
\[ (-A + z^*)' \cdot C_{\mathcal{H}} : \mathcal{H} \to D(A)' , \quad ( -A + z^*)' \cdot C_{\mathcal{H}} |_{D(A)} = C_{\mathcal{H}} \cdot ( -A + z ) \]
and, by the definition of \( G(z) \),
\[ ( -A + z^*)' \cdot C_{\mathcal{H}} \cdot G(z) = \tau'. \]

Therefore, defining \( Q_{\varnothing} := \Gamma_{\varnothing}(z)^{-1} \cdot \tau \phi_z \), one has
\[ C_{\mathcal{H}} \cdot (-A_{\varnothing}^z + z) \phi = ( -A + z^*)' \cdot C_{\mathcal{H}} \phi = ( -A + z^*)' \cdot C_{\mathcal{H}} \phi = \tau' Q_{\varnothing} , \]
i.e.
\[ A_{\varnothing}^z \phi = C_{\mathcal{H}}^{-1} \cdot (A' \cdot C_{\mathcal{H}} \phi + \tau' Q_{\varnothing}) . \]

Formally re-writing the last relation as
\[ A_{\varnothing}^z \phi = A\phi + C_{\mathcal{H}}^{-1} \cdot \tau' Q_{\varnothing} , \]
we can view \( A_{\varnothing}^z \) as a perturbation of \( A \), the perturbation being singular since, by (h2), \( \tau' Q_{\varnothing} \in D(A)' \setminus \mathcal{H}' \).
Remark 2.5. If \( \mathcal{X} \) is reflexive and \( \Gamma_{\phi}(z) \) is densely defined, then, by (7.1), there follows
\[
\Gamma_{\phi}(z)^{-1} \in \tilde{B}(\mathcal{X}, \mathcal{X}^{'}) \Rightarrow \Gamma_{\phi}(z^*)^{-1} \in \tilde{B}(\mathcal{X}, \mathcal{X}^{'}).
\]

Remark 2.6. If \( Z_{\phi} \neq \emptyset \) then \( Z_{\phi} \) is necessarily open. Indeed, by (5),
\[
\Gamma_{\phi}(z + h) = \Gamma_{\phi}(z) + h \tilde{G}(z) \cdot G(z + h),
\]
and so \( \Gamma_{\phi}(z + h)^{-1} \in \tilde{B}(\mathcal{X}^{'}, \mathcal{X}) \) if \( z \in Z_{\phi} \) and \( h \) is sufficiently small.

Remark 2.7. If in the representation (9) there exists \( z_0 \in \rho(A) \) such that \( \hat{f}(z_0) = \hat{f}(z_0^*) = 0 \) (this is certainly true if \( \rho(A) \cap \mathbb{R} \neq \emptyset \) and if one uses representation (8) with \( z_0 \in \mathbb{R} \)) then obviously \( Z_{\phi} \) is non-empty for any invertible \( \Theta \in \mathcal{S}(\mathcal{X}^{'}, \mathcal{X}) \). A more significative criterion leading to (h1) will be given in Proposition 2.1 below.

Remark 2.8. By the definition of \( G(z) \) one has that (h2) is equivalent to
\[
\text{Range } G(z) \cap D(A) = \{0\}.
\]

Remark 2.9. If Kernel \( \tau \) is dense in \( \mathcal{X} \) then (h2) holds true. Indeed the density hypothesis implies, if \( Q \in \mathcal{X}^{'}, \)
\[
\forall \psi \in \text{Kernel } \tau, \quad \langle \phi, \psi \rangle = \hat{Q}(\tau \psi) = \hat{\tau} \hat{Q}(\psi) = 0.
\]
This, by the definition of \( G(z) \), implies
\[
R(z) \phi = G(z) Q \phi = 0,
\]
which gives (h2).

Remark 2.10. If in the above theorem one uses the representation \( \hat{f}_{\phi}(z) := \Theta + \hat{f}(z) \) given by Lemma 2.2 one can readily check that the domain of \( A_{\phi} \) is equivalently characterized in term of "generalized boundary conditions": \( \phi \in D(A_{\phi}) \) if and only if
\[
\exists Q_{\phi} \in D(\Theta) \subseteq \mathcal{X}^{'}, \quad \text{such that } \phi - \frac{G(z_0) + G(z_0^*)}{2} Q_{\phi} \in D(A)
\]
and
\[
\tau \left( \phi - \frac{G(z_0) + G(z_0^*)}{2} Q_{\phi} \right) = \Theta Q_{\phi}.
\]

The following result states that when \( \tau \) is surjective (h1) holds true under relatively weak hypotheses:
**Proposition 2.1.** Let \( \Gamma_\Theta(z) = \Theta + \Gamma(z) \) be closed, densely defined and satisfying (5) and (7.1). If \( \tau \) is surjective then

\[
C \setminus R \cup W^-_\Theta \cup W^+_\Theta \subseteq \mathcal{Z}_\Theta,
\]

where

\[
W^\pm_\Theta := \{ \lambda \in R \cap \rho(A) : \gamma(\pm (\Gamma(z) + \Gamma(z^*))) > -\gamma(\pm \Theta) \}.
\]

If \( \tau \) merely has a dense range then

\[
\overline{W}^-_\Theta \cup \overline{W}^+_\Theta \subseteq \mathcal{Z}_\Theta,
\]

where

\[
\overline{W}^\pm_\Theta := \{ z \in \rho(A) : \frac{1}{\lambda}(\pm (\Gamma(z) + \Gamma(z^*))) > -\gamma(\pm \Theta) \}.
\]

**Proof.** Writing

\[
\Gamma(z) = \frac{i}{2} (\Gamma(z) + \Gamma(z^*)) + \frac{i}{2} (\Gamma(z) - \Gamma(z^*)) \equiv \Gamma_+ + \Gamma_-
\]

by (7) one has

\[
\text{Re} \langle (\Gamma(z) \ell) \rangle = \langle (\Gamma_+ \ell) \rangle, \quad \text{Im} \langle (\Gamma(z) \ell) \rangle = -i\langle (\Gamma_- \ell) \rangle.
\]

Thus by (5) there follows

\[
\text{Im} \langle (\Gamma(z) \ell) \rangle = -\frac{i}{2} \langle (z - z^*)/(G(z^*) \cdot G(z) \ell) \rangle = \text{Im}(z) \| G(z) \ell \|^2_{\mathcal{F}}
\]

and so, since \( \Theta \in \mathcal{S}(\mathcal{A}', \mathcal{A}) \) implies \( \langle \Theta \ell \rangle \in R \), one has

\[
|\langle (\Gamma_\Theta(z) \ell) \rangle|^2 = |\langle (\Theta \ell) \rangle + \langle (\Gamma_+ \ell) \rangle|^2 + \text{Im}(z)^2 \| G(z) \ell \|^4_{\mathcal{F}}.
\]

Injectivity of \( \Gamma_\Theta(z) \) and \( \Gamma_\Theta(z)' \) for any \( z \in C \setminus R \cup W^-_\Theta \cup W^+_\Theta \) then follows by injectivity of \( G(z) \) (see Remark 2.1), (7.1), injectivity of \( J_\mathcal{F} \), and the definitions of \( W^\pm_\Theta \).

Being \( \Gamma_\Theta(z) \) densely defined, one has

\[
(\text{Range} \; \Gamma_\Theta(z))' = \text{Kernel} \; \Gamma_\Theta(z)',
\]

and so injectivity of \( \Gamma_\Theta(z)' \) give denseness of the range of \( \Gamma_\Theta(z) \). Being \( \Gamma_\Theta(z) \) closed, its domain is a Banach space w.r.t. the graph norm and we can apply the open mapping theorem to the continuous map \( \Gamma_\Theta(z) \).
Thus to conclude the proof we need to prove that the range of $\Gamma_\Theta(z)$ is closed. By [23, Thm. 5.2, Chap. IV]

$$\inf\{ \|G(z)\|_{\mathcal{X}}, \|\ell\|_{\mathcal{X}'} = 1 \} > 0,$$

if and only if the range of $G(z)$ is closed; by the closed range theorem (see e.g. [23, Thm. 5.13, Chap. IV]) the range of $G(z)$ is closed if and only if the range of $\tilde{G}(z)$ is closed, and this is equivalent to the range of $\tau$ being closed. Therefore, since

$$\forall \ell \in D(\Gamma_\Theta), \quad \|\ell\|_{\mathcal{X}'} = 1, \quad \|\Gamma_\Theta(z)\ell\|_{\mathcal{X}} \geq |\ell(\Gamma_\Theta(z)\ell)|,$$

when either $z \in \mathbb{C}\setminus\mathbb{R} \cup W^{-}_\Theta \cup W^+_\Theta$ if $\tau$ is surjective, or when $z \in \tilde{W}^{-}_\Theta \cup \tilde{W}^+_\Theta$ if $\tau$ has a dense range, one has

$$\inf\{ \|\Gamma_\Theta(z)\ell\|_{\mathcal{X}}, \|\ell\|_{\mathcal{X}'} = 1 \} > 0,$$

and so, since $\Gamma_\Theta(z)$ is closed, it has a closed range by [23, Thm. 5.2, Chap. IV].

Since $Z_\Theta \subseteq \rho(A_\Theta^*)$, the above proposition immediately implies a semi-boundedness criterion for the extensions $A_\Theta^*$:

**Corollary 2.1.** Let $-A$ be bounded from below and suppose that there exist $\lambda_0 \in \rho(A) \cap \mathbb{R}$ and $\theta_0 \in \mathbb{R}$ such that

$$\forall \lambda \geq \lambda_0 \quad \gamma(\Gamma(\lambda)) > -\theta_0.$$

Then

$$\inf \sigma(-A_\Theta^*) \geq -\lambda_0$$

for any $\Theta \in \tilde{S}(\mathcal{X}', \mathcal{X})$ such that $\gamma(\Theta) \geq \theta_0$.

**Remark 2.11.** By the proposition above, if $\mathcal{X} = \text{Range} \tau$ is finite-dimensional and $\Gamma_\Theta(z)$ is everywhere defined, then (h1) is satisfied with at least $\mathbb{C}\setminus\mathbb{R} \subseteq Z_\Theta$.

**Remark 2.12.** By the proposition above, since $\tilde{I}(z)$ is bounded, if one uses the representation $\tilde{I}_\Theta(z)$, with $\Theta \in \tilde{S}(\mathcal{X}', \mathcal{X})$ closed, densely defined and such that $J_\mathcal{X} \cdot \Theta = \Theta'$, then (h1) is satisfied (with at least $\mathbb{C}\setminus\mathbb{R} \subseteq Z_\Theta$) when $\tau$ is surjective.

**Remark 2.13.** If $\mathcal{X}$ is a Hilbert space (with scalar product $\langle \cdot, \cdot \rangle$) we can of course use the map $C_\mathcal{X}$ to identify $\mathcal{X}$ with $\mathcal{X}'$ and re-define $G(z)$ as

$$G(z) := C_\mathcal{X}^{-1} \cdot \tilde{G}(z^*)' \cdot C_{\mathcal{X}'}: \mathcal{X} \to \mathcal{X}.$$
The statements in the above theorem remain then unchanged taking
\[ \Gamma_\Theta : \rho(A) \rightarrow L(\mathcal{X}, \mathcal{Y}), \quad \Gamma_\Theta(z) = \Theta + \Gamma(z) \]
with \( \Theta \) such that
\[ \forall x, y \in D(\Theta), \quad \langle \Theta x, y \rangle = \langle x, \Theta y \rangle \]
and \( \Gamma(z) \) satisfying (6) and
\[ \forall x, y \in D(\Gamma), \quad \langle \Gamma(z) x, y \rangle = \langle x, \Gamma(z^*) y \rangle. \]

**Remark 2.14.** When \( \mathcal{X} \) is a Hilbert space, by theorem 2.1, since \( G(z) \) and \( G(z) \) are bounded, we have that \( R_\Theta^G(z) - R(z) \) is a trace class operator on \( \mathcal{H} \) if and only if \( (\Theta + \Gamma(z))^{-1} \) is a trace class operator on \( \mathcal{X} \) (see e.g. [23, §1.3, Chap. X]). This information can be used (proceeding along the same lines as in [11]) to infer from \( \sigma(A) \) some properties of \( \sigma(A_\Theta) \).

When \( \mathcal{X} \) is a Hilbert space one can give, besides the one appearing in lemma 2.2, another criterion for obtaining the map \( \Gamma_\Theta \). Indeed one has the following

**Lemma 2.4.** Suppose that there exists a densely defined sesquilinear form \( \tilde{E}(z), z \in \rho(A) \), with \( z \)-independent domain \( D(\tilde{E}) \times D(\tilde{E}) \), such that
\[ \forall x, y \in D(\tilde{E}), \quad \tilde{E}(z^*)(x, y) = (\tilde{E}(z)(y, x))^*, \quad (13) \]
\[ \forall x, y \in D(\tilde{E}), \quad \frac{d}{dz} \tilde{E}(z)(x, y) = \langle G(z^*) x, G(z) y \rangle, \quad (14) \]
and such that there exist \( z_0 \in \rho(A), M \in \mathbb{R} \) for which \( \tilde{E}(z_0) \) is closable and
\[ \forall x \in D(\tilde{E}), \quad \text{Re}(\tilde{E}(z_0)(x, x)) \geq M \langle x, x \rangle. \quad (15) \]

Then \( \tilde{E}(z) \) is closable for any \( z \in \rho(A) \) and, denoting by \( \delta(z) \) it closure, there exists a densely defined, closed linear operator \( \Gamma(z) \) with \( z \)-independent domain \( D(\Gamma) \), defined by
\[ \forall x \in D(\tilde{E}), \forall y \in D(\Gamma), \quad \delta(z)(x, y) = \langle x, \Gamma(z) y \rangle, \]
satisfying (6) and the Hilbert space analogue of (7.1), i.e.
\[ \Gamma(z)^* = \Gamma(z^*). \]
Proof. By our hypotheses \( \tilde{\mathcal{E}}(z) \) necessarily differs from (the restriction to \( D(\tilde{\mathcal{E}}) \times D(\tilde{\mathcal{E}}) \)) of the bounded sesquilinear form associated to \( \bar{\mathcal{E}}(z) \) by a \( z \)-independent Hermitean form \( \bar{\mathcal{E}}. \) Therefore
\[
\tilde{\mathcal{E}}(x, y) = \tilde{\mathcal{E}}(z_0)(x, y) - \langle x, \bar{\mathcal{E}}(z_0)y \rangle
\]
is a semi-bounded, densely defined, closable Hermitean form. If \( \Theta \) denotes the unique semi-bounded self-adjoint operator corresponding to the closure of \( \bar{\mathcal{E}} \) (see [23, Thm. 2.6, Chap. VI] for the existence of \( \Theta \)), then the operator \( \bar{\mathcal{E}}(z) := \Theta + \bar{\mathcal{E}}(z) \) gives the thesis.

Remark 2.15. If \( \bar{\mathcal{E}}(z) \) in Lemma 2.3, besides satisfying (10) and (11), is bounded from below in the sense of (15), i.e. if there exist \( z_0 \in \rho(A), M \in \mathbb{R} \) such that
\[
\forall x \in D(\bar{\mathcal{E}}(z_0)), \quad \text{Re}(\langle x, \bar{\mathcal{E}}(z_0)x \rangle) \geq M \langle x, x \rangle,
\]
then, by using both Lemma 2.3 and Lemma 2.4, it is closable and its closure satisfies (5) and (7.1). This is nothing but a variation of Friedrichs extension theorem.

Remark 2.16. The operator \( \bar{\mathcal{E}}(z) \) given by Lemma 2.4 satisfies
\[
\frac{1}{\gamma}(\bar{\mathcal{E}}(z_0) + \bar{\mathcal{E}}(z_0^*)) \geq M
\]
and so, when \( \gamma \) has dense range, by Proposition 2.1 one has that (h1) holds true for \( \bar{\mathcal{E}}(z) \), where \( \Theta \) is any \( \bar{\mathcal{E}}(z) \)-bounded (see [23, Thm. 1.1, Chap. IV]) self-adjoint operator such that \( \gamma(\Theta) > M \). If the constant \( M \) can be made arbitrarily large by letting \( |z_0| \to \infty \), then (h1) is satisfied with any bounded from below self-adjoint operator \( \Theta \).

3. APPLICATIONS

Example 3.1. The \( \mathcal{H}_{\mathcal{E}} \)-construction. Let \( \mathcal{X} = \mathbb{C}, \varphi \in D(A) \setminus \{0\} \) and put \( \tau = \varphi \). Defining
\[
\tilde{\mathcal{E}}(z) := C^{-1} \cdot R(\varphi) \in B(D(A)', \mathcal{H})
\]
one has then
\[
\tilde{\mathcal{G}}(z) : \mathcal{H} \to \mathcal{C}, \quad \tilde{\mathcal{G}}(z) \varphi = \langle \tilde{\mathcal{E}}(z^*), \varphi \rangle
\]
and
\[
\tilde{\mathcal{G}}(z) : \mathcal{C} \to \mathcal{H}, \quad \tilde{\mathcal{G}}(z) \zeta = \zeta \tilde{\mathcal{E}}(z) \varphi.
\]
The hypothesis (h2) is equivalent to the request
\[ \varphi \notin \mathcal{N}, \]
whereas hypothesis (h1) is always satisfied with at least \( \mathcal{R} \subseteq \mathbb{Z}_0 \) since \( \mathcal{X} \) is finite dimensional (see Remark 2.11). Then the self-adjoint operator \( A^*_\alpha \) has resolvent
\[
(-A^*_\alpha + z) = (-A + z)^{-1} + \Gamma_{\alpha}(z)^{-1} \tilde{R}(z) \varphi \otimes \tilde{R}(z^*) \varphi,
\]
where (by Lemma 2.2)
\[
\Gamma_{\alpha}(z) = a + \varphi \left( \frac{\tilde{R}(z_0) \varphi + \tilde{R}(z_0^*) \varphi}{2} - \tilde{R}(z) \varphi \right), \quad \alpha \in \mathbb{R}.
\]
This coincides with the “\( \mathcal{M}_z \)-construction” given in [24] (there only the case \( -A \geq 0, z_0 = 1 \) was considered). For a similar construction also see [5] and references therein.

**Example 3.2. A variation on the Birman-Krein-Vishik theory.** Let \( A \) be a strictly positive self-adjoint operator, so that \( 0 \in \rho(-A) \), and let \( \tau : D(A) \to \mathcal{X} \) satisfy (h2). By Remark 2.7 and Theorem 2.1, for any \( \Theta \in \tilde{S}(\mathcal{X}^*, \mathcal{X}) \) which has a bounded inverse, we can define the (strictly positive when \( \Theta \) is positive, i.e. \( \gamma(\Theta) \geq 0 \)) self-adjoint operator \( A^*_{\alpha} \) by
\[
(A^*_{\alpha})^{-1} = A^{-1} + G \cdot \Theta^{-1} \cdot G,
\]
where \( G := G(0) \) and \( \tilde{G} := \tilde{G}(0) \). Moreover one has
\[
D(A^*_{\alpha}) := \{ \phi \in \mathcal{H} : \phi = \phi_0 + GQ_\phi, \phi_0 \in D(A), \tau \phi_0 = \Theta Q_\phi \}, A^*_{\alpha} \phi = A \phi_0.
\]
This gives a variation of the Birman-Krein-Vishik approach which comprises the result given in [22]. In particular [22, Example 4.1] can be obtained by taking \( \mathcal{H} = L^2(\Omega) \), \( A = -A_\Omega + \lambda, \lambda > 0, \Omega = (0, \pi) \times \mathbb{R}^2 \), \( D(A_\Omega) = H^1_0(\Omega), \tau : H^1_0(\Omega) \to L^2(0, \pi) \) the evaluation along the segment \( \{ (x, 0, x), x \in (0, \pi) \} \), \( \Theta = \bar{A}_0(0, \pi), D(\Theta) = H^1_0(0, \pi) \); [22, Example 4.2] corresponds to \( \mathcal{H} = L^2(\mathbb{R}^3), A = -A + \lambda, \lambda > 0, D(A) = H^2(\mathbb{R}^3) \), whereas \( \tau \) and \( \Theta \) are the same as before.

**Example 3.3. Finitely many point interactions in three dimensions.** We take \( \mathcal{H} = L^2(\mathbb{R}^3), A = A_0, D(A) = H^2(\mathbb{R}^3) \subseteq C_b(\mathbb{R}^3) \). Considering then a finite set \( Y \subseteq \mathbb{R}^3, \# Y = n \), we take as the linear operator \( \tau \) the linear continuous surjective map
\[
\tau : H^2(\mathbb{R}^3) \to C^n \quad \tau \phi := \{ \phi(y) \}_{y \in Y}.
\]
Then one has
\[ \hat{G}(z) : L^2(\mathbb{R}^3) \to \mathbb{C}^n, \quad \hat{G}(z) \phi = \{ \mathcal{G}_y * \phi(y) \}_{y \in Y}, \]
where
\[ \mathcal{G}_y(x) = \frac{e^{-\sqrt{z}|x|}}{4\pi |x|}, \quad \text{Re } \sqrt{z} > 0, \quad \mathcal{G}_y^*(x) := \mathcal{G}_y(x - y), \]
and
\[ G(z) : \mathbb{C}^n \to L^2(\mathbb{R}^3), \quad G(z) \zeta = \sum_{y \in Y} \zeta_y (\mathcal{G}_y \zeta)_y \equiv \mathcal{G}_y * \sum_{y \in Y} \zeta_y \delta_y. \]

A straightforward calculation then gives
\[ (\hat{G}(z) \cdot G(z) \zeta)_y = \sum_{\tilde{y} \neq y} \zeta_{\tilde{y}} (\mathcal{G}_{\tilde{y}} \cdot \mathcal{G}_y^*) \]
\[ = \zeta_y \frac{1}{2\pi} \int_{\mathbb{R}^3} dk \left( \frac{1}{(|k|^2 + |z|)^2} \right) + \sum_{\tilde{y} \neq y} \zeta_{\tilde{y}} \frac{1}{2\pi} \int_{\mathbb{R}^3} dk \left( \frac{e^{-i k \cdot (\tilde{y} - y)}}{(|k|^2 + |z|)^2} \right) \]
\[ = \zeta_y \frac{1}{2\pi} \int_0^\infty \frac{dr}{r^2 + |z|^2} \left( \int_{\mathbb{R}^3} dk \frac{e^{-i k \cdot (\tilde{y} - y)}}{(|k|^2 + |z|)^2} \right) \]
\[ + \sum_{\tilde{y} \neq y} \zeta_{\tilde{y}} \frac{1}{2\pi} \int_0^\infty \frac{dr}{r^2 + |z|^2} \left( \int_{\mathbb{R}^3} dk \frac{e^{-i k \cdot (\tilde{y} - y)}}{(|k|^2 + |z|)^2} \right) \]
\[ = \zeta_y \frac{1}{8\pi |z|^2} + \sum_{\tilde{y} \neq y} \zeta_{\tilde{y}} \frac{e^{-\sqrt{z}|\tilde{y} - y|}}{8\pi \sqrt{z}} \]
\[ = \frac{d}{d\sqrt{z}} \left( \zeta_y \sqrt{z} - \sum_{\tilde{y} \neq y} \zeta_{\tilde{y}} \mathcal{G}_{\tilde{y}} \right), \]
where \( \mathcal{G}_{\tilde{y}} := \mathcal{G}_y(\tilde{y} - y), \tilde{y} \neq y \). Defining
\[ \tilde{\mathcal{G}}_y : \mathbb{C}^n \to \mathbb{C}^n \quad (\tilde{\mathcal{G}}_y \zeta)_y := \sum_{\tilde{y} \neq y} \zeta_{\tilde{y}} \mathcal{G}_{\tilde{y}}, \]
one can take as \( I_\alpha(z) \) the linear operator
\[ I_\alpha(z) = \Theta + \frac{\sqrt{z}}{4\pi} - \tilde{\mathcal{G}}_z, \]
where \( \Theta \) is any Hermitean \( n \times n \) matrix.
Hypothesis (h1) is satisfied with at least $C_R Z^3$ since $X$ is finite dimensional (see Remark 2.11) and hypothesis (h2) is satisfied since $\forall y \in Y$, $G(y)_{H^2(\mathbb{R}^3)}$ for any $y \in Y$. In conclusion, one can define the self-adjoint operator $A_{\beta}^Y$ with resolvent given by

$$(-A_{\beta}^Y + z)^{-1} = (-A + z)^{-1} + \sum_{y, y \neq y \in Y} \left(\Theta + \frac{\gamma}{4\pi} - \delta_{y y}\right)^{-1} \gamma \langle \gamma \rangle \otimes \gamma \langle \gamma \rangle.$$ 

This coincides with the operator constructed in [4, §II.1.1].

**Example 3.4. Infinitely many point interactions in three dimensions.** We take $\mathcal{H} = L^2(Y \backslash \mathbb{R}^3)$, $A = A_\delta$, $D(A) = H^2(\mathbb{R}^3) \subset C_0(\mathbb{R}^3)$. Considering then an infinite and countable set $Y \subset \mathbb{R}^3$ such that

$$\inf_{y \neq y, y \in Y} |y - \tilde{y}| = d > 0,$$

we take as the linear operator $\tau$ the linear map $\tau_y \phi := \{\phi(y)\}_{y \in Y}$. The hypothesis (16) ensures its surjectivity and (see [4, page 172])

$$\tau_y : H^2(\mathbb{R}^3) \to \ell^2(Y), \quad \tau_y \in B(H^2(\mathbb{R}^3), \ell^2(Y))$$

Proceeding as in the previous example one has then

$$G(z) : L^2(\mathbb{R}^3) \to \ell^2(Y), \quad G(z) \phi = \{\gamma \ast \phi(y)\}_{y \in Y},$$

and

$$G(z) : \ell^2(Y) \to L^2(\mathbb{R}^3)$$

is the unique bounded linear operator which, on the dense subspace

$$\ell^2(Y) := \{\zeta \in \ell^2(Y) : \# \supp(\zeta) < +\infty\},$$

is defined by

$$G(z) \zeta = \sum_{y \in Y} \zeta \gamma \langle \gamma \rangle \gamma \langle \gamma \rangle,$$

i.e.

$$\forall \zeta \in \ell^2(Y), \quad G(z) \zeta = \gamma \ast \tau \zeta(y).$$
where \( \tau_\gamma(\zeta) \in H^{-\frac{1}{2}}(\mathbb{R}^3) \) is the signed Radon measure defined by
\[
\tau_\gamma(\zeta)(\phi) = \langle \zeta^*, \tau_\gamma \phi \rangle.
\]
Taking \( \zeta \in \ell_d(Y) \) one then obtains the proceeding as in Example 3.2,
\[
(G(z) \cdot G(z) \zeta)_y = \frac{1}{8\pi \sqrt{z}} + \sum_{\gamma \neq \gamma'} \frac{\zeta_{\gamma'}}{8\pi \sqrt{z}} e^{-\sqrt{z} | \gamma - \gamma'|} = \frac{d}{dz} \left( \frac{\sqrt{z}}{4\pi} - \sum_{\gamma \neq \gamma'} \zeta_{\gamma'} \phi_{\gamma} \right).
\]
Posing
\[
\tilde{\theta}_z : \ell_d(Y) \to \ell_d(Y), \quad (\tilde{\theta}_z \zeta)_y = \sum_{\gamma \neq \gamma'} \zeta_{\gamma'} \phi_{\gamma}
\]
the operator
\[
\tilde{\tau}_z : \ell_d(Y) \to \ell_d(Y), \quad (\tilde{\tau}_z \zeta)_y = \sum_{\gamma \neq \gamma'} \zeta_{\gamma'} \phi_{\gamma}
\]
the operator
\[
\tilde{\Gamma}(z) := \sqrt{\frac{z}{4\pi}} - \tilde{\theta}_z
\]
satisfies (10) and (11). Therefore, by Lemma 2.3 (with \( E \) the canonical basis of \( \ell_d(Y) \) and \( D(\tilde{\tau}) = \ell_d(Y) \)), \( \tilde{\theta}_z \) is closable and, denoting its closure by the same symbol, the closed and densely defined operator
\[
\Gamma(z) := \sqrt{\frac{z}{4\pi}} - \tilde{\theta}_z
\]
satisfies (5) and (7). Since \( \Gamma(z) + \Gamma(z^*) \) is bounded from below if \( \text{Im}(z) \) is sufficiently large (see [4, page 171]), by Lemma 2.4 it satisfies (7.1). Therefore, considering then \( \Theta + \Gamma(z) \), where \( \Theta \) is any \( \Gamma(z) \)-bounded (see [23, Thm. 1.1. Chap. IV]) self-adjoint operator on \( \ell_d(Y) \), (h1) is satisfied by Proposition 2.1, whereas (h2) is equivalent to \( \tau_\gamma(\zeta) \notin L^2(\mathbb{R}^3) \) for any \( \zeta \neq 0 \), which is always true since the support of \( \tau_\gamma(\zeta) \) is the null set \( Y \). So, by Theorem 2.1, one can define the self-adjoint operator \( A^Y_\theta \) with resolvent given by
\[
(-A^Y_\theta + z)^{-1} = (-A + z)^{-1} + \sum_{\gamma, \gamma' \in Y} \left( \Theta + \sqrt{\frac{z}{4\pi}} \tilde{\theta}_z \right)^{-1}_{\gamma' \gamma} \tilde{\theta}_z \otimes \tilde{\theta}_z^*.
\]
This coincides with the operator constructed (by an approximation method) in [4, §III.1.1].

**Example 3.5.** *Singular perturbations of the Laplacian supported by regular curves.* We take $\mathcal{H} = L^2(\mathbb{R}^n)$, $A = \Delta$, $D(A) = H^2(\mathbb{R}^n)$, $n = 3$ or $n = 4$.

Consider then a $C^3$ curve $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^n$ such that $C := \gamma(I)$ is a one-dimensional embedded submanifold $C \subset \mathbb{R}^n$ which, when unbounded, is, outside some compact set, globally diffeomorphic to a straight line (these hypotheses on $\gamma$ will be weakened in the next example). We will suppose $C$ to be parametrized in such a way that $|\gamma'| = 1$.

We take as linear operator $\tau$ the unique linear map

$$\tau_{\gamma} : H^2(\mathbb{R}^n) \to L^2(I), \quad \tau_{\gamma} \in B(H^2(\mathbb{R}^n), L^2(I))$$

such that

$$\forall \phi \in C^\infty_c(\mathbb{R}^n), \quad \tau_{\gamma} \phi(s) := \phi(\gamma(s)).$$

The existence of such a map is given by combining the results in [8, §10] (straight line) with the ones in [8, §24] (compact manifold). By [8, §25] we have that

$$\text{Range } \tau_{\gamma} = H^s(I), \quad \tau_{\gamma} \in B(H^s(\mathbb{R}^d), H^s(I)), \quad s = 2 - \frac{n-1}{2}$$

and so we could take $\mathcal{H} = H^s(I)$. However, in order to make clearer the connections with the existing literature, we prefer to work with $\mathcal{H} = L^2(I)$ even if with this choice $\tau_{\gamma}$ is not surjective (but has a dense range).

*The case $n = 3$.* One has, proceeding similarly to Examples 3 and 4,

$$\tilde{G}(z) : L^2(\mathbb{R}^3) \to L^2(I), \quad \tilde{G}(z) \phi = \tau_{\gamma}(\mathcal{G}_z * \phi)$$

and

$$G(z) : L^2(I) \to L^2(\mathbb{R}^3), \quad G(z) f = \mathcal{G}_z * \tau'_{\gamma}(f),$$

where $\tau'_{\gamma}(f) \in H^{-2}(\mathbb{R}^3)$ is the signed Radon measure defined by

$$\tau'_{\gamma}(f)(\phi) = \langle f^*, \tau_{\gamma} \phi \rangle.$$
By Fourier transform one has equivalently
\[ \mathcal{F} \cdot G(z) f(k) = \frac{1}{(2\pi)^{3/2}} \frac{1}{|k|^2 + z} \int f(s) e^{-ik \cdot \gamma(s)} \, ds, \quad f \in L^1(I) \cap L^2(I), \]
so that, for any \( f_1, f_2 \in L^1(I) \cap L^2(I) \) one obtains
\[
(z-w) \langle f_1, G(w) \cdot G(z) f_2 \rangle = (z-w) \langle G(w^*) f_1, G(z) f_2 \rangle
\]
\[ = (z-w) \int \frac{d\mathcal{F} f_1(t) f_2(s)}{(2\pi)^3} \frac{dt \, ds}{|\gamma(t) - \gamma(s)|} \int \frac{dk}{(k^2 + w) (|k|^2 + z)}
\]
\[ = \int \int \frac{d\mathcal{F} f_1(t) f_2(s)}{(2\pi)^3} \frac{dt \, ds}{|\gamma(t) - \gamma(s)|} \frac{e^{-\sqrt{|\gamma(t) - \gamma(s)|}}}{4\pi |\gamma(t) - \gamma(s)|} \left( e^{-\sqrt{|\gamma(t) - \gamma(s)|}} - e^{-\sqrt{|\gamma(t) - \gamma(s)|}} \right). \tag{17} \]

Suppose now that, in the case \( I \) is not compact,
\[
\exists \lambda_0 > 0 : \forall \lambda \geq \lambda_0, \sup_{t \in I} \int_I ds \, e^{-\lambda |\gamma(t) - \gamma(s)|} < +\infty. \tag{18} \]

By (17) one can then define a linear operator \( \check{L}_d(z) : L^2_0(I) \to L^2(I) \), \( \epsilon > 0 \), satisfying (5) and (7), by
\[
\check{L}_d(z) f(t) := \int_I ds \, f(s) \left( \frac{\chi_d(t, s)}{4\pi |t-s|} - \frac{e^{-\sqrt{|\gamma(t) - \gamma(s)|}}}{4\pi |\gamma(t) - \gamma(s)|} \right),
\]
where \( \chi_d(t, s) := \chi_{[0, \epsilon]}(|t-s|) \) and \( L^2_0(I) := \{ f \in L^2(I) : f \text{ has compact support} \} \).

When \( f \in C^0_0(I) \) one can then re-write \( \check{L}_d(z) f \) as
\[
\check{L}_d(z) f(t) = \int_I ds \left( f(t) - f(s) \right) \mathcal{G}_d(\gamma(t) - \gamma(s))
\]
\[ + f(t) \int_I ds \frac{\chi_d(t, s)}{4\pi |t-s|} \frac{e^{-\sqrt{|\gamma(t) - \gamma(s)|}}}{4\pi |\gamma(t) - \gamma(s)|}
\]
\[ - \int_I ds \frac{\chi_d(t, s)}{4\pi |t-s|} \frac{f(t) - f(s)}{4\pi |t-s|}. \]
The second term has, as a function of the parameter \( \varepsilon > 0 \), a derivative given by 
\[
(2 \pi) \log(\varepsilon) + \int_I ds \left( f(t) - f(s) \right) \mathcal{H}_\varepsilon(t) - \mathcal{H}_\varepsilon(s)
\]
and the last term is \( z \)-independent. Therefore the operator 
\[
\mathcal{H}(z): C^1_c(I) \to L^2(I),
\]

\[
\mathcal{H}(z) f(t) := \int_I ds \left( f(t) - f(s) \right) \mathcal{H}_\varepsilon(t) - \mathcal{H}_\varepsilon(s)
\]

\[
+ f(t) \left( \frac{1}{2\pi} \log(\varepsilon) + \int_I ds \left( \frac{1}{4\pi |t-s|} + \frac{1}{4\pi |\gamma(t) - \gamma(s)|} \right) \right)
\]

is \( \varepsilon \)-independent and satisfies (10) and (11) with \( E = L^2(I) \) and \( D(\mathcal{H}) = C^1_c(I) \). Moreover, by Lemma 2.3, it is closable and its closure \( \mathcal{H}(z) \) satisfies (6) and (7). Since \( \mathcal{H}(z) + \mathcal{H}(z^*) \) is bounded from below if \( \text{Im}(z) \) is sufficiently large (this is a consequence of (18)), by Remark 2.15 it satisfies (7.1). Moreover (see [38, Lemma 1]) such a bound can be made arbitrarily large by letting \( |z| \to \infty \). Therefore, considering then \( \Theta + \mathcal{H}(z) \), where \( \Theta \) is any \( f(z) \)-bounded self-adjoint operator on \( L^2(I) \), by Remark 2.16 and Proposition 2.1, (h1) is satisfied when \( \Theta \) is bounded from below, whereas (h2) is satisfied since \( \tau(z)(f) \notin L^2(\mathbb{R}^n) \) for any \( f \neq 0 \), being the support of \( \tau(z)(f) \) given by the null set \( C \).

The corresponding self-adjoint family given by Theorem 2.1 has resolvents

\[
(-A^0 + z)^{-1} \phi = (-A(z) + z)^{-1} \phi + \mathcal{H}_z \tau(z)(\Theta + \mathcal{H}(z))^{-1} \tau(z)(\phi).
\]

These give singular perturbations of the Laplacian of the same kind obtained (by a quadratic form approach) in [38].

**The case \( n = 4 \).** Proceeding as in the case \( n = 3 \) one obtains

\[
\mathcal{G}(z): \mathcal{L}^2(\mathbb{R}^4) \to \mathcal{L}^2(I), \quad \mathcal{G}(z) \phi = \tau(z)\mathcal{H}_z \phi,
\]

\[
G(z) f = \mathcal{H}_z \tau(z)f, \quad \mathcal{F}\mathcal{H}_z(k) := \frac{1}{|k|^2 + z}, \quad k \in \mathbb{R}^4
\]

and, for any \( f_1, f_2 \in L^1(I) \cap L^2(I) \),

\[
(z-w) \langle f_1, \mathcal{G}(w) \rangle = (z-w) \langle G(w^*) f_1, G(z) f_2 \rangle
\]

\[
= \int_I dt ds f_1^*(t) f_2(s) \left( \mathcal{H}_z(\gamma(t) - \gamma(s)) - \mathcal{H}_z(\gamma(t) - \gamma(s)) \right).
\]
Since

\[ |\mathcal{K}_z(x)| = \frac{1}{4\pi^2 |x|^2} (1 + o(|x|)), \quad |x| \ll 1, \]

\[ |\mathcal{K}_z(x)| = \frac{1}{2} \frac{e^{-2 \Re z |x|}}{(2\pi)^{3/2} |x|^{3/2}} (1 + o(1/|x|)), \quad |x| \gg 1, \]

when \( z \) satisfies (18) the linear operator \( \tilde{T}(z): L^2(I) \to L^2(I) \)

\[ \tilde{T}(z) f(t) = \int_I ds \ f(s) \left( \frac{\mathcal{K}_z(t,s)}{4\pi^2 |t-s|^2} - \mathcal{K}_x(\gamma(t) - \gamma(s)) \right) \]

is well defined and satisfies (10) and (11) with \( E = L^2(I) \) and \( \mathcal{D}(\tilde{T}) = L^2_0(I) \).

In four dimensions, due to the stronger (w.r.t. \( \mathcal{K}_x \)) singularity at the origin of \( \mathcal{K}_z \), it is no more possible to perform the calculations leading to the analogue of the operator \( \tilde{T}(z) \), and one is forced to use sesquilinear forms and to try then to apply Lemma 2.4. Defining for brevity

\[ k_{\varphi}(t,s) := \frac{\mathcal{K}_x(t,s)}{4\pi^2 |t-s|^2}, \quad k_x(t,s) := \mathcal{K}_x(\gamma(t) - \gamma(s)), \]

one can re-write \( \langle f_1, \tilde{T}(z) f_2 \rangle \), when \( f_1, f_2 \in \mathcal{C}_0^1(I) \), as

\[ \langle f_1, \tilde{T}(z) f_2 \rangle = \int_I \int_I ds dt \ f_1^*(t) f_2(s)(k_{\varphi}(t,s) - k_x(t,s)) \]

\[ = \int_I \int_I ds dt \ (f_1^*(t) f_2(s) - f_1^*(t) f_2(t) + f_1^*(t) f_2(s))(k_{\varphi}(t,s) - k_x(t,s)) \]

\[ = \frac{1}{2} \int_I \int_I ds dt \ f_1^*(t) f_2(s)(k_{\varphi}(t,s) - k_x(t,s)) \]

\[ + \int_I \int_I ds \ f_1^*(t) f_2(t)(k_{\varphi}(t,s) - k_x(t,s)) \]

\[ = \frac{1}{2} \int_I \int_I ds \ f_1^*(t) f_2(s)(f_2(t) - f_2(s)) k_x(t,s) \]

\[ + \int_I \int_I ds \ f_1^*(t) f_2(s)(k_{\varphi}(t,s) - k_x(t,s)) \]

\[ - \frac{1}{2} \int_I \int_I ds \ f_1^*(t) f_2(s)(f_2(t) - f_2(s)) k_{\varphi}(t,s). \]
Similarly to the three dimensional case the second term has, as a function of the parameter \( \varepsilon > 0 \), a derivative given by
\[
\frac{(2 \pi^2 \varepsilon^2)^{-1}}{2} \int \int \int \int (f_2(t) - f_2(t)) \left( f_2(t) - f_2(t) \right) k_1(t, s) \ dt \ ds
\]
and the last term is \( z \)-independent. Therefore the sesquilinear form
\[
E(z) := \frac{1}{2 \pi^2 \varepsilon^2} \int \int \int \int \left( f_1(t) - f_1(t) \right) \left( f_2(t) - f_2(t) \right) k_1(t, s) \ dt \ ds
\]
is \( \varepsilon \)-independent and satisfies (13) and (14). It is straightforward to check its closeability (see the proof of Proposition 2 in [38] if you get stuck), whereas (15) is a consequence of (18). Moreover, proceeding as in the case \( n = 3 \), the bound in (15) can be made arbitrarily large by letting \( |z| \uparrow \infty \).

Being (h2) verified by the same argument as in the case \( n = 3 \), by Lemma 2.4, Remark 2.16, Proposition 2.1 and Theorem 2.1, one has a self-adjoint family of self-adjoint operators with resolvents
\[
(\mathcal{A}_\mu^2 + z^{-1}) \phi = (-\mathcal{A} + z)^{-1} \phi + \mathcal{X}_\varepsilon + \mathcal{X}_\varepsilon \cdot \mathcal{T}_1((\Theta + \Gamma(z))^{-1} \cdot \mathcal{T}_2(\mathcal{X}_\varepsilon \ast \phi)),
\]
where \( \Gamma(z) \) is the operator corresponding to the closure of \( \mathcal{A}(z) \) and \( \Theta \) is any bounded from below self-adjoint operator on \( L^2(I) \). This gives singular perturbations of the Laplacian of the same kind obtained in [37].

**Example 3.6. Singular perturbations given by d-sets and d-measures.**

A Borel set \( F \subset \mathbb{R}^n \) is called a d-set, \( d \in (0, n] \), if (see [21, Chap. II]) there exists a Borel measure \( \mu \) in \( \mathbb{R}^n \) such that \( \text{supp} \ (\mu) = F \) and
\[
\exists c_1, c_2 > 0 : \forall x \in F, \forall r \in (0, 1), c_1 r^d \leq \mu(B_r(x) \cap F) \leq c_2 r^d,
\]
where \( B_r(x) \) is the ball of radius \( r \) centered at the point \( x \). By [21, Chap. II, Thm. 1], once \( F \) is a d-set, \( \mu_F \), the d-dimensional Hausdorff measure restricted to \( F \), always satisfies (21) and so \( F \) has Hausdorff dimension \( d \) in the neighborhood of any of its points. From the definition there also follows that a finite union of d-sets which intersect on a set of zero d-dimensional Hausdorff measure is a d-set. Examples of d-sets are d-dimensional Lipschitz manifolds (use Examples 2.1 and 2.4 in [20]) and (when \( d \) is not an integer) self-similar fractals of Hausdorff dimension \( d \) (see [21, Chap. II, Example 2], [39, Thm. 4.7]).
Denoting by \( j_F : F \to \mathbb{R}^n \) the restriction to the \( d \)-set \( F \) of the identity map, we take as the linear operator \( \tau \) the unique continuous map \((0 < n - d < 2s)\)
\[
\tau_F : H^s(\mathbb{R}^n) \to L^2(F), \quad \tau_F \in B(H^s(\mathbb{R}^n), L^2(F))
\]
such that
\[
\forall \phi \in C^0_0(\mathbb{R}^n), \quad \tau_F \phi(x) := \phi(j_F(x)).
\]
Here \( L^2(F) \) denotes the space of (equivalence classes of) functions on \( F \) which are square integrable w.r.t. the measure \( \mu \). For the existence of such a map \( \tau_F \) see the proof of [39, Thm. 18.6]. By [21, Thm. 1, Chap. VII] we have that
\[
\text{Range } \tau_F = H^s(F), \quad \tau_F \in B(H^s(\mathbb{R}^n), H^s(F)), \quad \alpha = s - \frac{n - d}{2},
\]
where the Hilbert space \( H^s(F) \) is a Besov-like space which coincides with the usual Sobolev space when \( F \) is a regular manifold. In the case \( 0 < \alpha < 1 \), \( H^s(F) \) can be defined (see [21, §1.1, Chap. V]) as the set of \( f \in L^2(F) \) having finite norm
\[
\|f\|_{H^s}^2 := \|f\|_{L^2}^2 + \int_{|x - y| < 1} d\mu(x) d\mu(y) \frac{|f(x) - f(y)|^2}{|x - y|^{d + 2s}}.
\]
By Lemma 2.2, Remark 2.12 (taking \( \bar{x} = H^s(F) \) so that \( \tau_F \) is surjective) and Theorem 2.1 (hypothesis (h2) being equivalent to \( \tau_F(f) \notin L^2(\mathbb{R}^n) \), \( f \neq 0 \), which is surely satisfied when \( F \) is a null set) one can then immediately define a family (parametrized by the self-adjoint operators on \( H^s(F) \)) of self-adjoint extensions of \( A|_{\tau_F = 0} \), where \( A \) is any self-adjoint operator on \( L^2(\mathbb{R}^n) \) with domain \( H^s(\mathbb{R}^n) \).

By considering \( d \)-measures one can treat the situation where even more general sets appear. A Borel measure \( \mu \) on \( \mathbb{R}^n \) is said to be a \( d \)-measure, \( d \in (0, n] \), if
\[
\exists c > 0 : \forall x \in \mathbb{R}^n, \forall r \in (0, 1], \quad \mu(B_r(x)) \leq cr^d.
\]
Then, by [21, Lemma 1, Chap. VIII], when
\[
p = \frac{2d}{n - 2s_*}, \quad 0 < s_* \leq s, n - d < 2s_* < n,
\]
and denoting by $L^p(\mu)$ the space of (equivalence classes of) functions which are $p$-integrable w.r.t. $\mu$, the linear operator

$$\tau_\mu: H^s(\mathbb{R}^n) \to L^p(\mu) \quad \tau_\mu \in B(H^s(\mathbb{R}^n), L^p(\mu))$$

$$f(\tau_\mu \phi) := \int_{\mathbb{R}^n} d\mu(x) f(x) \phi(x), \quad f \in L^p(\mu), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

is well defined. Since $\mu_F$, when $F$ is a $d$-set, is a $d$-measure, the previous results tell us that in this case we can take $n - d = 2s < 2s$ (so that $p = 2$) and $\tau_\mu$ coincides with $\tau_F$. An interesting example of a $d$-measure is the one given by the occupation time of Brownian motion: given $\gamma \in C(\mathbb{R}_+, \mathbb{R}^n)$, $n \geq 3$, let us define the Radon measure

$$\mu_\gamma(A) := \int_0^\infty dt \chi_{\gamma(t)}(A).$$

Then, by estimates on Brownian motion occupation times and by a Borel–Cantelli argument (see [13]), one has that, for arbitrarily small positive $\varepsilon$ and almost surely with respect to Wiener measure,

$$\mu_\gamma(B_\varepsilon(x)) \leq c r^{2-\varepsilon};$$

moreover the Hausdorff dimension of the support of $\mu_\gamma$ is equal to two.

Let us now consider the self-adjoint pseudo-differential operator ($s \geq 0$)

$$\psi(D): H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \psi(D) \phi := F^{-1}(\psi \hat{\varphi}),$$

where $\psi$ is a real-valued Borel function such that

$$\frac{1}{c} (1 + |x|^2)^{\nu/2} \leq 1 + |\psi(x)| \leq c (1 + |x|^2)^{\nu/2}.$$  

One has

$$\hat{\psi}(z): L^2(\mathbb{R}^n) \to L^p(\mu), \quad \hat{\psi}(z) \phi := \tau_\mu(\mathcal{K}_z \phi),$$

where

$$\mathcal{K}_z \phi := F^{-1} \left( \frac{1}{\psi + z} \right), \quad \mathcal{K}_z \phi := (2\pi)^{-n/2} F^{-1} \left( \frac{\hat{\varphi}}{\psi + z} \right),$$
and

\[ G(z) : L^q(\mu) \to L^2(\mathbb{R}^n), \quad G(z) f := \mathcal{N}^\psi * \tau_\mu(f^*), \]

where \( \tau_\mu(f) \in H^{-q}(\mathbb{R}^n) \) is the signed measure defined by

\[ \tau_\mu(f)(\phi) = \langle f^*, \tau_\mu \phi \rangle \equiv f(\tau_\mu \phi) \]

and

\[ \mathcal{N}^\psi * \tau_\mu(f) := (2\pi)^{-n/2} \mathcal{F}^{-1}\left( \frac{\mathcal{F} \tau_\mu(f)}{-\psi + z} \right). \]

When one uses the representation \( \hat{f}_\mu(z) \), by Lemma 2.2 one has, if

\[ \mathcal{N}^\psi * v(x) = \int_{\mathbb{R}^n} dv(y) \mathcal{N}^\psi(x-y), \quad v \in H^{-q}(\mathbb{R}^n), \]

and

\[ \mathcal{N}^\psi = \frac{1}{2} (\mathcal{N}^\psi_\mu + \mathcal{N}^\psi_{\mu^*}) \equiv \text{Re}(\mathcal{N}^\psi), \]

\[ \hat{f}(z) : L^q(\mu) \to L^p(\mu), \quad \frac{1}{p} + \frac{1}{q} = 1, \]

\[ f_1(\hat{f}(z) f_2) = \int_{\mathbb{R}^n} \mu(x) \mu(y) f_1(x) f_2(y) (\mathcal{N}^\psi_\mu(x-y) - \mathcal{N}^\psi_{\mu^*}(x-y)). \]

By its definition and by Hahn–Banach theorem we have that \( \tau_\mu \) has dense range when

\[ \left\{ f \in L^q(\mu) : \forall \psi \in H^q(\mathbb{R}^n) \int_{\mathbb{R}^n} \mu(x) f(x) \phi(x) = 0 \right\} = \{0\}. \]

Therefore \( \tau_\mu \) has dense range when the Bessel \( s_* \)-capacity of \( \text{supp}(\mu) \), \( s_* \leq s \), is not zero, and this is true (by Frostman Lemma, see e.g. [29, Thm. 7.1]) when

\[ n - d(\mu) < 2s_*, \quad 0 < s_* \leq s, \]

where \( d(\mu) \) denotes the Hausdorff dimension of \( \text{supp}(\mu) \).

Let us note that, since \( p \in [2, \infty) \) in (19), when \( \mu \) is a finite measure we can view \( \tau_\mu \) as a map into the Hilbert space \( L^2(\mu) \). In this case we can then try to apply Lemma 2.4 in order to find other maps \( \hat{f}(z) \) which satisfy (5)
and (7.1). Supposing that \( \tilde{\psi}(x) = \tilde{\psi}(-x) \), so that \( \mathcal{H}_x^\varphi(x - y) = \mathcal{H}_x^\varphi(y - x) \), and that

\[
\int \mathbb{R}^{2n} \, d\mu(x) \, d\mu(y) \, |x - y|^2 \, |\mathcal{H}_x^\varphi(x - y)| < +\infty
\]

we have, for any \( f_1, f_2 \in C^1_0(\mathbb{R}^n) \), and proceeding similarly to Example 3.5 (case \( n = 4 \)),

\[
\langle f_1, \tilde{\Pi}(z) f_2 \rangle = \int \mathbb{R}^{2n} \, d\mu(x) \, d\mu(y) \, f_1^*(x) \, f_2^*(y) \left( \mathcal{H}_x^\varphi(x - y) - \mathcal{H}_y^\varphi(x - y) \right)
\]

\[
= \frac{1}{2} \int \mathbb{R}^{2n} \, d\mu(x) \, d\mu(y) \, (f_1^*(x) - f_1^*(y))(f_2(x) - f_2(y)) \, \mathcal{H}_x^\varphi(x - y)
\]

\[
+ \int \mathbb{R}^{2n} \, d\mu(x) \, d\mu(y) \, f_1^*(x) \, f_2(x)(\mathcal{H}_x^\varphi(x - y) - \mathcal{H}_y^\varphi(x - y))
\]

\[
- \frac{1}{2} \int \mathbb{R}^{2n} \, d\mu(x) \, d\mu(y) \, (f_2(x) - f_2(y))(f_2(x) - f_2(y)) \, \mathcal{H}_y^\varphi(x - y).
\]

Therefore, being the last term \( z \)-independent, the sequilinear form

\[
\tilde{\mathcal{E}}(z) : C^1_0(\mathbb{R}^n) \times C^1_0(\mathbb{R}^n) \to \mathbb{C}
\]

\[
\tilde{\mathcal{E}}(z)(f_1, f_2)
\]

\[
= \frac{1}{2} \int \mathbb{R}^{2n} \, d\mu(x) \, d\mu(y) \, (f_1^*(x) - f_1^*(y))(f_2(x) - f_2(y)) \, \mathcal{H}_x^\varphi(x - y)
\]

\[
+ \int \mathbb{R}^{2n} \, d\mu(x) \, d\mu(y) \, f_1^*(x) \, f_2(x)(\mathcal{H}_x^\varphi(x - y) - \mathcal{H}_y^\varphi(x - y))
\]

satisfies (13) and (14). In the case \( \tilde{\mathcal{E}}_x^\varphi \geq 0 \) one has \( \text{Re}(\tilde{\mathcal{E}}(z_0)(f, f)) \geq 0 \) and so (15) is satisfied with \( M = 0 \). Moreover \( \tilde{\mathcal{E}}(z_0) \) is readily checked to be closable. So, by Lemma 2.4, Proposition 2.1 and Theorem 2.1, for any strictly positive (i.e. \( \gamma(\Theta) > 0 \)) self-adjoint operator on \( L^2(\mu) \) one obtains a family of self-adjoint extensions \( \psi(D)_\varphi \) with resolvent

\[
(-\psi(D)_\varphi + z)^{-1} \phi
\]

\[
= (-\psi(D) + z)^{-1} \phi + \mathcal{H}_x^\varphi \ast \tau_\mu((\Theta + i\Pi(z))^{-1} \cdot \tau_\mu(\mathcal{H}_x^\varphi \ast \phi))
\]
where $\Gamma(z)$ is the operator corresponding to the closure of $\mathcal{D}(z)$. Such a family, in the particular case $\psi(D) = A$, is the same obtained, by an approximation method, in [3] (also see [12]) and generalizes, although with a different $\Gamma(z)$, the situation discussed in Example 3.5. In this regard suppose that the 1-set $C$ is the range of a Lipschitz path $\gamma : I \subseteq \mathbb{R} \to C \subseteq \mathbb{R}^n$, $n = 3$ or $n = 4$, $|\gamma| = 1$ a.e., so that $\tau_{\gamma(c)}$ has dense range in $L^2(C) \sim L^2(I)$. Under the hypothesis (18) one can again consider, when $n = 3$ the operator $\Gamma(z)$ appearing in (19) and, when $n = 4$ the sesquilinear form $\mathcal{D}(z)$ appearing in (20), the only difference being that now the domain of definition of such objects is $C^1(I \setminus I_\ast)$, with

$$I_\ast := \{ t \in I : \gamma \text{ is not differentiable at } t \} \cup \{ t \in I : \exists s \neq t \text{ s.t. } \gamma(t) = \gamma(s) \}$$

(of course, in order $C^1(I \setminus I_\ast)$ to be still a dense set, one has to suppose that the closure of $I_\ast$ is a null set). However in the case $n = 4$ the problem of the semi-boundedness of $\mathcal{D}(z)$ arises: indeed one can show (see [37]) that $\mathcal{D}(z)$ is unbounded from below in the case $\gamma$ has angle points. This phenomenon is similar to the one related to unboundedness from below of Schrödinger operators describing $n(>2)$ point interacting particles (see [30], [14] and references therein).

**Example 3.7.** Singular perturbations of the d’Alembertian supported by time-like straight lines. We take $\mathcal{H} = L^2(\mathbb{R}^4)$,

$$A = \Box := -A_{(1)} \otimes I + I \otimes A_{(3)} ,$$

$A_{(d)}$ being the Laplacian in $d$ dimensions, and $(h \in \mathbb{R}$, $k \in \mathbb{R}^3$ denoting the variables dual to $t \in \mathbb{R}$, $x \in \mathbb{R}^3$)

$$D(\Box) = \{ \Phi \in L^2(\mathbb{R}^4) : (h^2 - |k|^2) \mathcal{F} \Phi(h, k) \in L^2(\mathbb{R}^4) \} .$$

Let $\ell(s) = y + ws$, $y, w \in \mathbb{R}^4$, be a time-like straight line, i.e.

$$w = (\gamma_x, \gamma_y, v), v \in \mathbb{R}^3, |v| < 1, \gamma_x := (1 - |v|^2)^{-1/2} .$$

Consider now the unique surjective linear operator

$$\tau_0 : D(\Box) \to H^{-1/2}(\mathbb{R}) , \quad \tau_0 \in B(D(\Box), H^{-1/2}(\mathbb{R}))$$

such that

$$\forall \Phi \in C^0_0(\mathbb{R}^4), \quad \tau_0 \Phi(s) := \Phi(s, 0) .$$

For the existence of such a $\tau_0$ see the next Example.
Let then $\Pi_{y,v}$ be the unitary operator which compose any function in $L^2(\mathbb{R}^4)$ with the Lorentz boost corresponding to $v$ and then with the translation by $y$, so that $\Pi_{y,v} \in B(D(\Box), D(\Box))$. Defining

$$\tau_{y,v} := \tau_0 \cdot \Pi_{y,v} : D(\Box) \to H^{-1/2}(\mathbb{R}) \quad \tau_{y,v} \in B(D(\Box), H^{-1/2}(\mathbb{R}))$$

one has

$$\forall \Phi \in C_c^\infty(\mathbb{R}^4), \quad \tau_{y,v} \Phi(x) := \Phi(\tau(x)).$$

We begin studying the self-adjoint extensions given by $\tau_0$. By Fourier transform (here and below $z \in \mathbb{C}\setminus\mathbb{R}$) one obviously has

$$\mathcal{F} \cdot (-\Box + z)^{-1} \Phi(h, k) = \frac{\mathcal{F} \Phi(h, k)}{-h^2 + |k|^2 + z}.$$

So, since, as $R \uparrow \infty$,

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty dh \frac{1}{-h^2 + R^2 + z} \sim \frac{c}{4R},$$

by Hölder inequality and Riemann–Lebesgue Lemma there follows that

$$\forall \Phi = \phi \otimes \varphi \in L^2(\mathbb{R}) \otimes H^s(\mathbb{R}^3), s > 1, \quad (-\Box + z)^{-1} \Phi \in \mathcal{C}_c^\infty(\mathbb{R}^4)$$

and, by Fubini theorem,

$$\left[( -\Box + z)^{-1} \Phi \right](t, x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}^4} dh \frac{1}{\sqrt{-h^2 + |k|^2 + z}} \mathcal{F} \Phi(h, k)$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dh \left( \int_{\mathbb{R}^3} dk e^{ih \cdot \xi} \frac{\mathcal{F} \Phi(h, k)}{-h^2 + |k|^2 + z} \right)$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dh e^{ih \cdot \xi} \Phi(h) \int_{\mathbb{R}^3} d\xi \frac{e^{-\sqrt{-h^2 + |\xi|^2}}}{4\pi |\xi|}$$

$$= \int_{\mathbb{R}^3} d\xi \frac{\varphi(\xi)}{4\pi |\xi|} (e^{-|\xi|^2/4\zeta + z} \Phi)(t).$$
Here $\text{Re} \sqrt{-b^2 + z} > 0$; this choice will be always assumed in the sequel without further specification. The above calculation then gives

$$\dot{G}(z): L^2(\mathbb{R}) \otimes H^s(\mathbb{R}^3) \to C^0(\mathbb{R}), \quad s > 1,$$

$$(\dot{G}(z) \phi \otimes \varphi)(t) := \int_{\mathbb{R}^3} dy \frac{\varphi(y)}{4\pi |y|} \left[ e^{-i|y|/\sqrt{\Delta_{(1)} + z}} \hat{\varphi}(y) \right](t)$$

and

$$G(z): H^{1/2}(\mathbb{R}) \to L^2(\mathbb{R}^4)$$

$$(G(z) \phi)(t, x) := \frac{1}{4\pi |x|} \left[ e^{-|x|/\sqrt{\Delta_{(1)} + z}} \right](t).$$

Let us note that $\dot{G}(z)$ extends to a continuous linear operator from $L^2(\mathbb{R}^4)$ to $H^{-1/2}(\mathbb{R})$ since

$$\| \dot{G}(z) \phi \otimes \varphi \|^2_{L^2} \leq$$

$$\leq \int_{\mathbb{R}} dt \left( \int_{\mathbb{R}^3} dy \left| \frac{\varphi(y)}{4\pi |y|} \right| \left[ \left( -\Delta_{(1)} + 1 \right)^{-1/4} \left[ e^{-i|y|/\sqrt{\Delta_{(1)} + z}} \hat{\varphi}(y) \right] \right]^2 \right)$$

$$\leq \| \varphi \|^2_{L^2} \left( -\Delta_{(1)} + 1 \right)^{-1/2} \left( \int_0^\infty dR e^{-2R \text{Re} \sqrt{\Delta_{(1)} + z}} \phi, \phi \right)$$

$$= \| \phi \|^2_{L^2} \left( -\Delta_{(1)} + 1 \right)^{-1/2} \left( \text{Re}(\sqrt{\Delta_{(1)} + z}) \right)^{-1/2} \| \phi \|^2_{L^2}$$

Similarly $G(z)$ is a continuous linear operator from $H^{1/2}(\mathbb{R})$ to $L^2(\mathbb{R}^4)$ since

$$\| G(z) \phi \|^2_{L^2} = \int_0^\infty dR \left\| e^{-R \sqrt{\Delta_{(1)} + z}} \phi \right\|^2_{L^2}$$

$$= \left( \int_0^\infty dR e^{-2R \text{Re} \sqrt{\Delta_{(1)} + z}} \phi, \phi \right)$$

$$= \frac{1}{2} \left( \text{Re}(\sqrt{\Delta_{(1)} + z}) \right)^{-1} \| \phi \|^2_{L^2}$$

$$\| G(z) \phi \|^2_{L^2} \leq \frac{1}{2} \| \phi \|^2_{L^2} \left( \text{Re}(\sqrt{\Delta_{(1)} + z}) \right)^{-1/2} \| \phi \|^2_{H^{1/2}}.$$
We now look for the map $\Gamma(z)$. Since

$$
(z-w) \mathcal{F} \cdot \hat{G}(w) \cdot G(z) \phi(h)
$$

$$= z - w \left( \frac{1}{2\pi} \int_{\mathbb{C}^2} \frac{1}{(-h^2 + |k|^2 + w)(-h^2 + |k|^2 + z)} dk \right)
$$

$$= z - w \left( \frac{1}{2\pi} \int_{0}^{\infty} \frac{r^2}{(-h^2 + r^2 + w)(-h^2 + r^2 + z)} dr \right)
$$

$$= \frac{1}{4\pi} \left( \sqrt{-h^2 + z} - \sqrt{-h^2 + w} \right) \hat{F} \phi^*(h)
$$

one defines

$$
\Gamma(z): H^{1/2}(\mathbb{R}) \rightarrow H^{-1/2}(\mathbb{R})
$$

$$\Gamma(z) \phi := \frac{1}{4\pi} \sqrt{A(z) + z} \phi^*.
$$

Of course we can view $\Gamma(z)$ as a (unbounded) closed and densely defined linear operator on the Hilbert space $H^{-1/2}(\mathbb{R})$; evidently $\Gamma(z)$ satisfies (7.1). Therefore, by Proposition 2.1, $\Gamma_{\theta}(z)$ satisfies (h1) (with $Z_{\theta} = \rho(\Box \chi)$) for any self-adjoint operator $\theta$ on $H^{-1/2}(\mathbb{R})$ which is $\Gamma(z)$-bounded. It is immediate, by Fourier transform, to check the validity of (h2). Therefore the trace $\tau_{\theta}$ gives rise to the family of self-adjoint extensions $\Box_{\theta}^0$ with resolvent

$$
(-\Box_{\theta}^0 + z)^{-1} = (-\Box + z)^{-1} + G(z) \cdot \left( \Theta + \frac{1}{4\pi} \sqrt{A(z) + z} \right)^{-1} \cdot \hat{G}(z)
$$

(here, since they annihilates between themselves, we did not put the complex conjugations appearing in both the definitions of $G(z)$ and $\Gamma_{\theta}(z)$). By our definition of $\tau_{\gamma, z}$ we have, since $\Pi_{\gamma, z}$ commutes with $\Box$,

$$
\hat{G}_{\gamma, z}(z) := \tau_{\gamma, z} \cdot R(z) = \hat{G}(z) \cdot \Pi_{\gamma, z}
$$

and

$$
C_{L}^{-1} \cdot \hat{G}_{\gamma, z}(z^*) = \Pi_{\gamma, z}^* \cdot G(z).
$$

This immediately implies that one can use the same $\Gamma_{\theta}(z)$ as before and so the trace $\tau_{\gamma, z}$ gives rise to the family of self-adjoint extensions $\Box_{\theta}^0$ with resolvent

$$
(-\Box_{\theta}^0 + z)^{-1} = (-\Box + z)^{-1} + \Pi_{\gamma, z}^* \cdot G(z) \cdot \left( \Theta + \frac{1}{4\pi} \sqrt{A(z) + z} \right)^{-1} \cdot \hat{G}(z) \cdot \Pi_{\gamma, z}.
$$

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Moreover the following kind of Poincaré-invariance holds:

$$D(\square^v g) = \Pi^v_\gamma, D(\square^v g)$$ and $$\square^v g = \Pi^v_\gamma \cdot \square^v \cdot \Pi^v_\gamma.$$ 

Let us remark that, even if the operator $\Gamma_\phi(z)$ appearing in the resolvent above coincides with the one used in the case $v = 0$, it is applied to functions which depend on different variables: when $v = 0$ it acts on functions of the relative time whereas it acts on functions of the proper time when $v \neq 0$. Therefore if in the case $v \neq 0$ one uses relative time, then $\Gamma_\phi(z)$ becomes a velocity-dependent operator.

**Example 3.8.** Singular perturbation given by traces on Malgrance spaces. Given any continuous functions $\phi > 0$ on $\mathbb{R}^n$, $\phi \in \mathcal{H}$ will mean that there exists a polynomial $P$ such that

$$\forall x \in \mathbb{R}^n, \quad \frac{1}{|P(x)|} \leq \phi(x) \leq |P(x)|.$$ 

Then we define the Hilbert space $H_\phi(\mathbb{R}^n)$, $\phi \in \mathcal{H}$, as the set of tempered distribution $f$ such that $\mathcal{F}f$ is a functions and

$$\|f\|_2^2 := \int_{\mathbb{R}^n} |\phi(k) \mathcal{F}f(k)|^2 dk < +\infty.$$ 

Such a class of function spaces were introduced by Malgrange in [28]. In connection with the previous examples note that

$$\phi(x) = (1 + |x|^2)^{s/2}, s \in \mathbb{R} \Rightarrow H_\phi(\mathbb{R}^n) = H^s(\mathbb{R}^n)$$

and

$$\phi(t, x) = (1 + (t^2 + |x|^2)^{1/2}, t \in \mathbb{R}, x \in \mathbb{R}^3 \Rightarrow H_\phi(\mathbb{R}^4) = D(\square).$$

We list now some properties of the spaces $H_\phi(\mathbb{R}^n)$ following [28, §1], [19, §II.2.2] and [41]. Let us remark here that the definition of $H_\phi(\mathbb{R}^n)$ given in [19] and [41] is different: it corresponds to the case in which $\phi$ belongs to the narrower class $\mathcal{K}$ defined by

$$\phi \in \mathcal{K} \iff \exists c, N > 0 : \forall x, y \in \mathbb{R}^n, \quad \phi(x + y) \leq (1 + c |x|)^N \phi(y).$$

The choice $\phi \in \mathcal{K}$ ensures that $H_\phi(\mathbb{R}^n)$ is a module over $C_0^\infty(\mathbb{R}^n)$. However the results we will quote from [19, §II.2.2] and [41] hold true also for the more general case in which $\phi \in \mathcal{H}$ (see [41, Remark 2.3]).
The dual space of $H_\varphi$ can be explicitly characterized (see [28, §1.1], [19, Thm. 2.2.9], [41, §2.1]) as

$$H_\varphi(\mathbb{R}^n)' \simeq H_{\varphi'}(\mathbb{R}^n).$$

As regards the relation between different spaces, by [19, Thm. 2.2.2] one has

$$(1 + |x|)^{\delta}/\varphi(x) \in L^2(\mathbb{R}^n) = \varphi_2(x) \mapsto \int_{\mathbb{R}^n} \frac{1}{\varphi(x)} \frac{1}{\varphi'(x)} d\tilde{x},$$

the embedding being continuous. Therefore, for any $\varphi \in \mathcal{M}$ such that $\varphi \geq c > 0$, one has $H_\varphi(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$; $H_\varphi(\mathbb{R}^n)$ is then dense in $L^2(\mathbb{R}^n)$ since $C^\infty_0(\mathbb{R}^n)$ is dense in $H_\varphi(\mathbb{R}^n)$ (see [28, §1.1], [19, Thm. 2.2.1], [41, §2.1]). The regularity of elements in $H_\varphi(\mathbb{R}^n)$ is given by [19, Thm. 2.2.7]:

$$(1 + |x|)^{\delta}/\varphi(x) \in L^2(\mathbb{R}^n) = \varphi_2(x) \mapsto \int_{\mathbb{R}^n} \frac{1}{\varphi(x)} \frac{1}{\varphi'(x)} d\tilde{x},$$

the embedding being continuous.

Let us now come to the trace operator on $H_\varphi(\mathbb{R}^n)$ (see [19, Thm. 2.2.8], [41, §6]). We write $\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^{n-d}$, $1 \leq d \leq n - 1$, $x = (\tilde{x}, \hat{x})$, $\tilde{x} \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{n-d}$. Suppose that

$$\left( \int_{\mathbb{R}^{n-d}} \frac{1}{\varphi(x)} d\tilde{x} \right)^{-1/2} < +\infty.$$ 

Then there exists an unique surjective linear operator $\tau_{(d)}$

$$\tau_{(d)}: H_\varphi(\mathbb{R}^n) \rightarrow H_\varphi(\mathbb{R}^d), \quad \tau_{(d)} \in B(H_\varphi(\mathbb{R}^n), H_\varphi(\mathbb{R}^d)),$$

such that

$$\forall \phi \in C^\infty_0(\mathbb{R}^n), \quad \tau_{(d)}(\phi)(\tilde{x}) = \phi(\tilde{x}, 0).$$

The reader can check that the case $\phi(t, x) = (1 + (t^2 + |x|^2)^2)^{1/2}$, $d = 1$, reproduces the trace $\tau_0$ given in the previous example.

The trace $\tau_{(d)}$ can be generalized to cover the case of non-linear subsets in the following way: let $\mu \in H_\varphi'(\mathbb{R}^n)$, $\phi \in \mathcal{X}$ (for example $\mu$ could be the Hausdorff measure of some subset of $\mathbb{R}^n$ but more general distributions are allowed), for which there exists $\overline{\phi} \in \mathcal{X}$ such that

$$\int_{\mathbb{R}^n} \frac{\phi^2(\tilde{x}, \hat{x})}{\varphi(\tilde{x}) \varphi'(\hat{x})} d\tilde{x} < c < +\infty.$$
Then, by \([41, \S 2.4]\),
\[
\forall f \in \mathcal{H}_d(\mathbb{R}^n), \quad f\mu \in \mathcal{H}_d(\mathbb{R}^n),
\]
where
\[
f\mu := (2\pi)^{-n/2} \mathcal{F}^{-1}(\mathcal{F}f * \mathcal{F}\mu).
\]
So we can define
\[
\tilde{\tau}_\mu : \mathcal{H}_d(\mathbb{R}^n) \to \mathcal{H}_d(\mathbb{R}^n), \quad \tilde{\tau}_\mu(f) := f\mu,
\]
and then we have a trace generalizing \(\tau_{(\mu)}\) by
\[
\tau_\mu : \mathcal{H}_d(\mathbb{R}^n) \to \mathcal{H}_d(\mathbb{R}^n), \quad \tau_\mu := \tilde{\tau}_\mu \cdot J_{H^s}.
\]
Let us now consider the self-adjoint pseudo-differential operator (here \(\varphi \geq c > 0\))
\[
\psi(D) : \mathcal{H}_d(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \psi(D) \Phi := \mathcal{F}^{-1}(\psi \mathcal{F}\Phi),
\]
where \(\psi\) is a real-valued Borel function such that
\[
\frac{1}{c} \varphi(x) \leq 1 + |\varphi(x)| \leq c\varphi(x).
\]
By Fourier transform one has, if \(\tau_\mu\) is defined as above,
\[
G(z) f = G^\mu(z) f := \frac{1}{(2\pi)^n} \mathcal{F}^{-1} \left( \frac{\mathcal{F}f^* \ast \mathcal{F}\mu}{-\varphi + z^*} \right).
\]
Therefore (h2) is equivalent to \(\mathcal{F}f \ast \mathcal{F}\mu \notin L^2(\mathbb{R}^n)\), i.e. \(f\mu \notin L^2(\mathbb{R}^n)\). This condition is surely satisfied when the support of \(\mu\) is a set of zero Lebesgue measure.

By Lemma 2.2 we have then, for any \(f_1, f_2 \in \mathcal{H}_d(\mathbb{R}^n)\),
\[
\hat{f}(z) f_1 f_2 = f_2 \mu((\hat{G}^\mu - G^\mu(z)) f_1), \quad (22)
\]
where
\[
\hat{G}^\mu := \frac{G^\mu(z_0) + G^\mu(z_0^*)}{2}, \quad z_0 \in \rho(\psi(D)).
\]
In the case
\[ \varphi \in \mathcal{K}, \quad \int_{\mathbb{R}^n} \frac{\phi^2(x - y)}{\phi^2(x) \phi^2(y)} \, dy < \epsilon < +\infty, \]
by [41, §2.4] as above, we have
\[ \forall \Phi \in H_{\mu}(\mathbb{R}^n), \quad \Phi \mu \in H'_{\mu}(\mathbb{R}^n), \]
and so, by (22), \( \hat{\Phi}(z) = \hat{\Phi}_{\mu}(z) \), where
\[ \hat{\Phi}_{\mu}(z) : H_{\mu}(\mathbb{R}^n) \simeq H_{\mu}(\mathbb{R}^n) \to H'_{\mu}(\mathbb{R}^n), \quad \hat{\Phi}_{\mu}(z) f := (\hat{\mathcal{G}} + \hat{\mathcal{G}}_{\mu}(z)) f \mu. \]
In the case \( \tau_{\mu} \) is surjective, by Remark 2.12, Proposition 2.1 and Theorem 2.1, \( \tau_{\mu} \) gives rise to the family of self-adjoint operators \( \psi(D)_{\mu} \) with resolvents
\[ (-\psi(D)_{\mu} + z)^{-1} = (-\psi(D) + z)^{-1} + G(z) \cdot (\Theta + \hat{\Phi}_{\mu}(z))^{-1} \cdot \hat{\mathcal{G}}_{\mu}(z), \]
where \( \Theta \) is any operator from \( H_{\mu}(\mathbb{R}^n) \) to \( H'_{\mu}(\mathbb{R}^n) \) such that \( \Theta = \Theta' \).

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REFERENCES

26. M. G. Krein, Resolvents of Hermitian operators with defect index $(m, m)$, Dokl. Akad. Nauk SSSR 52 (1946), 657–660. [In Russian.]
27. M. G. Krein, The theory of self-adjoint extensions of half-bounded Hermitian operators and their applications, Mat. Sb. 69(2) (1947), 431–459. [In Russian.]