Generalizations of the Ostrowski–Brauer theorem∗

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Abstract

The main theorem of this paper, which generalizes the Ostrowski–Brauer theorem and
its previous extensions, provides conditions necessary and sufficient for the singularity of an
irreducible matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \) satisfying the conditions

\[ |a_{ii}| |a_{jj}| \geq R_i(A) R_j(A), \]

where

\[ R_k(A) = \sum_{j \neq k} |a_{kj}|, \quad k = 1, \ldots, n, \]

for all \( i \neq j \) such that \( |a_{ij}| + |a_{ji}| \neq 0 \) and implies a new description of the location of matrix

eigenvalues in terms of ovals of Cassini and Gerschgorin circles.

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1. Introduction

Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2, \) be a matrix with complex entries. Throughout the paper, we will use the following common notation:

\[
R_i(A) = \sum_{\substack{j=1 \atop j \neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n.
\]

We will consider generalizations of the following well-known result found by Ostrowski [6] and rediscovered by Brauer [1].

**Theorem 1.1** (The Ostrowski–Brauer Theorem). Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2, \) and assume that

\[
|a_{ii}| |a_{jj}| > R_i(A) R_j(A), \quad i \neq j, \ i, j = 1, \ldots, n. \tag{1.1}
\]

Then \( A \) is nonsingular.

Obviously, the Ostrowski–Brauer theorem can be reformulated in the following equivalent way.

**Theorem 1.2.** Every eigenvalue of a matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2, \) lies in at least one of the \( n(n-1)/2 \) ovals of Cassini

\[
C_{ij}(A) = \{ z \in \mathbb{C} : |a_{ii} - z||a_{jj} - z| \leq R_i(A) R_j(A) \},
\]

\[
i \neq j, \ i, j = 1, \ldots, n. \tag{1.2}
\]

A natural way of generalizing the Ostrowski–Brauer theorem is to pass to irreducible matrices and (in analogy with the famous Taussky generalization of the Gerschgorin circles theorem [7]) to allow the inequalities in (1.1) to be nonstrict. Along this direction, Brauer [2] “proved” the following result.

**Theorem 1.3.** An eigenvalue of an irreducible matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2, \) can be a boundary point of the union of the ovals of Cassini (1.2) only if it is a boundary point of each of them.

Theorem 1.3 is of course equivalent to the following assertion.

**Theorem 1.4.** If an irreducible matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2, \) satisfies the inequalities

\[
|a_{ii}| |a_{jj}| \geq R_i(A) R_j(A), \quad i \neq j, \ i, j = 1, \ldots, n, \tag{1.3}
\]

with at least one strict inequality, then \( A \) is nonsingular.
However, both these theorems are actually not valid, and an irreducible matrix satisfying the nonstrict inequalities in (1.3) can be singular even if \((n - 1)(n - 2)/2, n \geq 3\), of them are strict (see, e.g., [8] or [5]).

The conditions necessary and sufficient for the singularity of an irreducible matrix \(A\) satisfying (1.3) with at least one strict inequality were obtained by Li and Tsatsomeros and are as follows.

**Theorem 1.5** [5]. An irreducible matrix \(A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2\), satisfying (1.3) with at least one strict inequality is singular if and only if for some \(i_0, 1 \leq i_0 \leq n\), the following conditions are fulfilled:

1. \(a_{ij} \neq 0\) if and only if either \(i = j\), or \(i = i_0\), or \(j = i_0\);
2. \(|a_{i0i}| < R_{i0}(A)\);
3. \(|a_{i0j}| = R_{i0}(A)R_j(A),\ j \neq i_0,\ j = 1, \ldots, n\);
4. \(a_{i0i} = \sum_{j \neq i_0} a_{i0j}a_{j0i}/a_{jj}\).

Omitting the requirement that at least one of the inequalities in (1.3) is strict, we arrive at the following generalization of Theorem 1.5.

**Theorem 1.6** [4]. An irreducible matrix \(A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2\), satisfying inequalities (1.3) is singular if and only if either

\[|a_{ii}| = R_i(A), \ i = 1, \ldots, n,\]

and there exists a unitary diagonal matrix \(D \in \mathbb{C}^{n \times n}\) such that

\[D^*(I - D^{-1}_A A)D = |I - D^{-1}_A A|,\]

where \(D_A\) is the diagonal part of the matrix \(A\) and \(|B|\) denotes the matrix whose entries are the moduli of the corresponding entries of \(B\), or conditions (i)–(iv) of Theorem 1.5 are fulfilled.

It is of importance to observe that, as Theorem 1.5 demonstrates, the singularity/nonsingularity of a matrix can be closely related to its sparsity pattern. This circumstance was explicitly taken into account by Brualdi [3], who established the following generalization of the Ostrowski–Brauer theorem.

**Theorem 1.7** [3]. An irreducible matrix \(A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2\), satisfying the conditions

\[\prod_{i \in \gamma} |a_{ii}| \geq \prod_{i \in \gamma} R_i(A), \ \gamma \in \mathcal{C}(A),\]

with strict inequality for at least one \(\gamma\) is nonsingular.

Here and below, we use the following notation: \(\mathcal{C}(A)\) denotes the set of simple circuits in the directed graph of the matrix \(A\), and if \(\gamma = i_1i_2 \cdots i_ki_{k+1},\ i_{k+1} = i_1,\)
is a simple circuit of length \( k \), then the support of \( \gamma \), i.e., the set \( \{i_1, i_2, \ldots, i_k\} \) is denoted by \( \bar{\gamma} \).

Based on Brualdi’s theorem, it is fairly easy to derive the following more immediate generalization of the Ostrowski–Brauer theorem.

**Theorem 1.8** [8]. Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2 \), be an irreducible matrix and assume that

\[
|a_{ii}| |a_{jj}| \geq R_i(A) R_j(A), \quad i \neq j, \quad i, j \in \bar{\gamma}, \quad \gamma \in \mathcal{C}(A),
\]

and, for some indices \( i_0 \) and \( j_0 \),

\[
|a_{i_0 j_0}| |a_{j_0 i_0}| > R_{i_0}(A) R_{j_0}(A), \quad i_0 \neq j_0, \quad i_0, j_0 \in \gamma_0, \quad \gamma_0 \in \mathcal{C}(A).
\]

Then \( A \) is nonsingular.

The main result of this paper is Theorem 2.1 in Section 2, which solves the singularity/nonsingularity problem for irreducible matrices satisfying the nonstrict inequalities

\[
|a_{ii}| |a_{jj}| \geq R_i(A) R_j(A)
\]

for such \( i \neq j \) that \( |a_{ij}| + |a_{ji}| \neq 0 \). As we will see, this result, being a direct generalization of the Ostrowski–Brauer Theorem 1.1, also simultaneously generalizes Theorems 1.5, 1.6, and 1.8, and involves the matrix sparsity pattern in the simplest possible way.

An almost immediate consequence of Theorem 2.1 (see Corollary 2.2) provides the correct counterpart of Taussky’s theorem, in which one strict inequality ensures the nonsingularity of an irreducible matrix satisfying a set of nonstrict inequalities, and shows how the assumptions of Brauer’s Theorem 1.3 should be modified for the result to hold.

In Section 2, we also establish necessary and sufficient conditions for the singularity of matrices under assumptions weaker than those of Theorem 2.1.

In Section 3, the results obtained are applied in order to describe the location of eigenvalues of irreducible matrices. In particular, it is shown that one actually needs to consider only those ovals of Cassini that correspond to the nonzero entries of a matrix rather than the whole set (1.2) or the reduced set involved in Theorem 2 in [8], stemming from the above Theorem 1.8. Further, it turns out that a boundary point of the domain

\[
\bigcup_{i \neq j} \mathcal{C}_{ij}(A)
\]

for

\[
|a_{ij}| + |a_{ji}| \neq 0
\]

is an eigenvalue of the irreducible matrix \( A \) if and only if it is a common boundary point of either all the Gerschgorin circles or all the Cassini ovals occurring in (1.6) and some additional conditions are fulfilled.
2. New criteria of matrix singularity

The main result of this paper is the following theorem.

**Theorem 2.1.** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \), \( n \geq 2 \), be an irreducible matrix and assume that

\[
|a_{ii}| |a_{jj}| \geq R_i(A) R_j(A), \quad i \neq j.
\]

(2.1)

\[
|a_{ij}| + |a_{ji}| \neq 0, \quad i, j = 1, \ldots, n.
\]

Then \( A \) is singular if and only if there exists a unitary diagonal matrix \( D \in \mathbb{C}^{n \times n} \) such that

\[
D^* (I - D^{-1} A) D = |I - D^{-1} A|
\]

(2.2)

and either

\[
|a_{ii}| = R_i(A), \quad i = 1, \ldots, n,
\]

(2.3)

or the set

\[
S = \{ i, \ 1 \leq i \leq n : |a_{ii}| < R_i(A) \}
\]

is nonempty and the following conditions are fulfilled:

(i) both principal submatrices \( A[S] \) and \( A[\bar{S}] \), where \( \bar{S} = \{1, 2, \ldots, n\}\backslash S \), of the matrix \( A \) are diagonal;

(ii) there is a constant \( \alpha > 1 \) such that

\[
\frac{R_i(A)}{|a_{ii}|} = \begin{cases} 
\alpha, & i \in S; \\
\alpha^{-1}, & i \in \bar{S}.
\end{cases}
\]

(2.4)

Further, if \( A \) is singular, then the geometric multiplicity of the zero eigenvalue is one, and the corresponding eigenspace is spanned by the vector \( D w \), where \( w = (w_i)_{i=1}^{n} \) and

\[
w_i = \begin{cases} 
\alpha & \text{if } R_i(A) > |a_{ii}|, \\
1 & \text{if } R_i(A) \leq |a_{ii}|,
\end{cases}
\]

where \( \alpha \) is the same as in (2.4) and \( D \) is the same as in (2.2).

The proof of Theorem 2.1 will be based on the following singularity criterion for irreducible matrices with nonstrict generalized diagonal dominance.

**Theorem 2.2** [4]. An irreducible matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \), \( n \geq 2 \), satisfying the conditions

\[
\sum_{j=1, j \neq i}^{n} |a_{ij}| v_j \leq |a_{ii}| v_i, \quad i = 1, \ldots, n,
\]


is singular if and only if all the latter relations hold with equality and there exists a unitary diagonal matrix $D$ such that

$$D^*(I - D^{-1}A)D = |I - D^{-1}A|.$$  

Further, if the matrix $A$ is singular, then the eigenvalue $\lambda = 0$ is of geometric multiplicity one, and the corresponding eigenspace is spanned by the vector $Dv$.

**Proof of Theorem 2.1.** First we note that, under the hypotheses of this theorem, all the diagonal entries of $A$ are nonzero. Indeed, since the matrix $A$ is irreducible,$$R_i(A) > 0, \quad i = 1, \ldots, n,$$and for any $i, 1 \leq i \leq n$, there is an index $j = j(i)$ such that $a_{ij} \neq 0$, whence we have

$$|a_{ii}|a_{jj} \overset{(2.1)}{=} R_i(A)R_j(A) > 0.$$  

**Sufficiency.** The singularity of the matrix $A$ satisfying (2.2) and (2.3) follows directly from Theorem 2.2. Now let $S$ be nonempty and let conditions (2.2) and (i), (ii) be fulfilled. Define the positive vector $w = (w_i)_{i=1}^n$ by setting

$$w_i = \begin{cases} \alpha, & i \in S; \\ 1, & i \in \bar{S}. \end{cases} \quad (2.5)$$

Then we have

$$\sum_{j \neq i} |a_{ij}|w_j \overset{(2.5)}{=} \sum_{j \in S} |a_{ij}| = R_i(A) \overset{(2.4)}{=} |a_{ii}| \alpha \overset{(2.5)}{=} |a_{ii}|w_i, \quad i \in S,$$

and

$$\sum_{j \neq i} |a_{ij}|w_j \overset{(2.5)}{=} \sum_{j \in S} |a_{ij}|\alpha = \alpha R_i(A) \overset{(2.4)}{=} |a_{ii}| \alpha = |a_{ii}|w_i, \quad i \in \bar{S},$$

and again $A$ is singular by Theorem 2.2.

The same theorem also implies that the eigenvalue $\lambda = 0$ of the matrix $A$ is of geometric multiplicity one, and the associated eigenspace is spanned either by the vector $De$, where $e^T = [1, 1, \ldots, 1]$, if $S$ is empty or by the vector $Dw$, where $w$ is defined in (2.5), if $S$ is nonempty, which proves the assertion concerning the null-space of $A$.

**Necessity.** Assume that the matrix $A$ is singular. First we note that conditions (2.1) immediately imply that

$$a_{ij} = 0, \quad i \neq j, \quad i, j \in S,$$  

i.e., the submatrix $A[S]$ is diagonal.

Let us demonstrate that $A$ possesses the property of nonstrict generalized diagonal dominance with the positive vector $u = (u_i)_{i=1}^n$ defined as follows:
\[ u_i = \begin{cases} R_i(A) / |a_{ii}|, & i \in S; \\ 1, & i \in \bar{S}. \end{cases} \] (2.7)

Indeed, if \( i \in S \), then we have
\[ \sum_{j \neq i} |a_{ij}| u_j \overset{\text{(2.6), (2.7)}}{=} \sum_{j \in \bar{S}} |a_{ij}| R_j(A) \overset{\text{(2.7)}}{=} |a_{ii}| u_i, \quad i \in S. \] (2.8)

If \( i \in \bar{S} \), then, by the definition of the set \( S \),
\[ |a_{ii}| \geq R_i(A), \] (2.9)
and we derive
\[ \sum_{j \neq i} |a_{ij}| u_j \overset{\text{(2.7)}}{=} \sum_{j \in S} |a_{ij}| R_j(A) / |a_{jj}| + \sum_{j \in \bar{S}} |a_{ij}| \overset{\text{(2.1)}}{\leq} \frac{|a_{ii}|}{R_i(A)} \sum_{j \in S} |a_{ij}| + \sum_{j \in \bar{S}} |a_{ij}| \] (2.10)
\[ \overset{\text{(2.9)}}{\leq} \frac{|a_{ii}|}{R_i(A)} \sum_{j \neq i} |a_{ij}| = |a_{ii}| \overset{\text{(2.7)}}{=} |a_{ii}| u_i, \quad i \in \bar{S}. \]

Applying Theorem 2.2, we arrive at the conclusion that \( A \) satisfies condition (2.2) and the equalities
\[ \sum_{j \neq i} |a_{ij}| u_j = |a_{ii}| u_i, \quad i = 1, \ldots, n. \] (2.11)

Further, the derivation of (2.10) implies that equalities (2.11) hold for every \( i \in \bar{S} \) if and only if either
\[ a_{ij} = 0 \quad \forall \ j \in S \] (2.12)
and
\[ |a_{ii}| = R_i(A), \] (2.13)
or
\[ a_{ij} = 0 \quad \forall \ j \in \bar{S}, \ j \neq i, \] (2.14)
and
\[ |a_{ii}| |a_{jj}| = R_i(A) R_j(A), \quad j \in S, \ a_{ij} \neq 0. \] (2.15)

Denote the subset of indices \( i \in \bar{S} \) such that conditions (2.14) and (2.15) are fulfilled by \( S_1 \) and the subset of indices \( i \in \bar{S} \) for which (2.12) and (2.13) hold by \( S_2 \). Then the matrix \( A \) can be symmetrically permuted to the 3 \times 3 block form
where $D_1 = A[S]$ and $D_2 = A[S_1]$ are diagonal matrices. The block $A[S, S_2]$ in the upper right corner is zero because, by the definition of $S$ and (2.13), we have

$$|a_{ii}| |a_{jj}| < R_i(A)R_j(A), \quad i \in S, \; j \in S_2,$$

whence, taking into account inequalities (2.1), we conclude that

$$a_{ij} = 0, \quad i \in S, \; j \in S_2.$$

Since, by assumption, the matrix $A$ is irreducible, it cannot be permuted to the block triangular form (2.16) unless either both $S$ and $S_1$ are empty or $S_2$ is empty. In the former case, conditions (2.3) are obviously fulfilled, whereas in the latter case $A$ satisfies condition (i).

Our next step is to show that if $A$ satisfies (i), then it satisfies the equalities

$$\sum_{j \neq i} |a_{ij}| v_j = |a_{ii}| v_i, \quad i = 1, \ldots, n, \quad (2.17)$$

where the positive vector $v = (v_i)_{i=1}^n$ is defined as follows:

$$v_i = \begin{cases} 1, & i \in S; \\ R_i(A)/|a_{ii}|, & i \in \bar{S}. \end{cases} \quad (2.18)$$

Indeed, since

$$\sum_{j \neq i} |a_{ij}| v_j \overset{(i), (2.18)}{=} \sum_{j \in S} |a_{ij}| R_j(A) |a_{jj}| \overset{(2.1)}{=} |a_{ii}| R_i(A) \sum_{j \in S} |a_{jj}| \overset{(i)}{=} |a_{ii}| R_i(A) \overset{(2.18)}{=} |a_{ii}| v_i, \quad i \in S,$$

and

$$\sum_{j \neq i} |a_{ij}| v_j \overset{(i), (2.18)}{=} \sum_{j \in S} |a_{ij}| R_j(A) |a_{jj}| \overset{(2.18)}{=} |a_{ii}| v_i, \quad i \in \bar{S},$$

the singular matrix $A$ must satisfy equalities (2.17) by Theorem 2.2.

Finally, since $A$ satisfies the two sets of equalities (2.11) and (2.17), the positive vectors $u$ and $v$ defined in (2.7) and (2.18), respectively, are collinear by Theorem 2.2, i.e.,

$$u = \alpha v, \quad \alpha > 0.$$

As is readily seen, the latter relation amounts to (2.4).

Theorem 2.1 is proved completely. □
The sufficient conditions provided by Theorem 2.1 for a matrix \(A\) with some off-diagonally dominant rows to be singular can be relaxed as follows.

**Corollary 2.1.** An irreducible matrix \(A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2,\) for which the set
\[S = \{i, \ 1 \leq i \leq n : |a_{ii}| < R_i(A)\}\]
is nonempty is singular if the following conditions are fulfilled:

(i) equality (2.2) holds for a unitary diagonal matrix \(D;\)
(ii) both \(A[S]\) and \(A[S]\) are diagonal matrices;
(iii) the equalities
\[|a_{ii}| |a_{jj}| = R_i(A) R_j(A) \quad (2.19)\]
hold either for all \(i \in \hat{S}, \ j \in S\) such that \(a_{ij} \neq 0,\) or for all \(i \in S, \ j \in \hat{S}\) such that \(a_{ij} \neq 0.\)

**Proof.** As is easy to ascertain, conditions (ii) and (iii) imply that either
\[\sum_{j \neq i} |a_{ij}| |u_j| = |a_{ii}| |u_i|, \quad i = 1, \ldots, n,\]
where the vector \(u = (u_i)\) is defined in (2.7), or
\[\sum_{j \neq i} |a_{ij}| |v_j| = |a_{ii}| |v_i|, \quad i = 1, \ldots, n,\]
for the vector \(v = (v_i)\) defined in (2.18). In both cases, taking into account (i) and applying Theorem 2.2, we arrive at the conclusion that \(A\) is singular. □

The following corollary of Theorem 2.1 provides conditions sufficient for an irreducible matrix to be nonsingular and shows how the assumptions of Theorem 1.3 should be modified for this theorem to become valid.

**Corollary 2.2.** If, under the hypotheses of Theorem 2.1, at least one of the inequalities in (2.1) is strict, then \(A\) is nonsingular.

**Proof.** If \(A\) would be singular, then, by Theorem 2.1, either equalities (2.3) would be satisfied, or the set \(S\) would be nonempty, and conditions (i) and (ii) of Theorem 2.1 would be fulfilled. In the former case, the equalities
\[|a_{ii}| |a_{jj}| = R_i(A) R_j(A), \quad i \neq j,\]
would be valid for all \(i, j = 1, \ldots, n,\) whereas in the latter case they would hold for all \(i \in S\) and all \(j \in \hat{S}\). In particular, in both cases, all the inequalities in (2.1) would be equalities, which contradicts the assumption of this corollary. Thus, \(A\) is nonsingular. □
In proving Corollary 2.2, we have actually established the following stronger result.

**Corollary 2.3.** Let a matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2 \), satisfy the assumptions of Theorem 2.1 and has at least one off-diagonally dominant row. If for a pair of indices \( i \neq j \) such that \( i \in S \) and \( j \in \bar{S} \) the strict inequality
\[
|a_{ii}| |a_{jj}| > R_i(A) R_j(A)
\]
is valid, then \( A \) is nonsingular.

**Remark 2.1.** The assertion of Corollary 2.2 readily follows from Brualdi’s Theorem 1.7. Indeed, if \( \gamma = i_1 i_2 \cdots i_k i_{k+1}, \ i_{k+1} = i_1, \ k \geq 2, \) and \( \gamma \in \mathcal{C}(A) \), then
\[
a_{i_j i_{j+1}} \neq 0, \quad j = 1, \ldots, k,
\]
whence, under the conditions of Corollary 2.2, we have
\[
\left( \prod_{i \in \mathcal{P}} |a_{ii}| \right)^2 = \prod_{j=1}^k |a_{i_j i_{j+1}}| \geq \prod_{j=1}^k R_{i_j}(A) R_{i_{j+1}}(A) = \left[ \prod_{i \in \mathcal{P}} R_i(A) \right]^2,
\]
and the nonsingularity of \( A \) will follow if we show that, for at least one \( \gamma \in \mathcal{C}(A) \),
\[
\prod_{i \in \mathcal{P}} |a_{ii}| > \prod_{i \in \mathcal{P}} R_i(A).
\]

Since, by the assumptions of Corollary 2.2,
\[
|a_{ii}| |a_{jj}| > R_i(A) R_j(A)
\]
for some \( i \neq j \) such that \( a_{ij} \neq 0 \), for any \( \gamma \in \mathcal{C}(A) \) going from \( i \) directly to \( j \) the strict inequality (2.20) will obviously hold, and the proof is completed by applying the simple lemma below.

**Lemma 2.1.** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2 \), be an irreducible matrix and assume that
\[
a_{ij} \neq 0 \quad \text{for some } i \neq j, \quad 1 \leq i, j \leq n.
\]
Then in the directed graph of \( A \) there is a simple circuit \( \gamma = i_1 \cdots i_k i_{k+1}, \ i_{k+1} = i_1, \ k \geq 2, \) such that
\[
i_1 = i, \quad i_2 = j.
\]

**Proof.** Since \( A \) is irreducible, in the directed graph of \( A \) there is a simple path going from \( j \) to \( i \), say,
\[
j_2 \cdots j_k j_{k+1}, \quad j_2 = j, \ j_{k+1} = i.
\]
Then

$$\gamma = ij_1 \cdots j_k j_{k+1}$$

is the required circuit. □

The interrelations between Theorem 2.1 and Theorems 1.5 and 1.6 are described in Remarks 2.2–2.4.

Remark 2.2. The class of singular irreducible matrices satisfying nonstrict inequalities (1.3) with at least one strict inequality coincides with the class of singular irreducible matrices satisfying inequalities (2.1) that have only one off-diagonally dominant row.

Indeed, if $A$ satisfies (1.3) with at least one strict inequality, then, obviously, $A$ satisfies (2.1), and if $A$ is singular and irreducible, then, by Theorem 1.5, it has precisely one off-diagonally dominant row. Conversely, if a singular irreducible matrix $A$ satisfies inequalities (2.1) and, for some $i_0$, $1 \leq i_0 \leq n$, we have

$$|a_{i_0 i_0}| < R_{i_0}(A), \quad (2.21)$$

whereas

$$|a_{jj}| \geq R_j(A), \quad j \neq i_0, \quad j = 1, \ldots, n, \quad (2.22)$$

then, by Theorem 2.1, $A$ is of the form

$$A = \begin{bmatrix}
a_{11} & a_{1i_0} \\
\vdots & \ddots & \vdots \\
0 & \ddots & \vdots & \ddots & \vdots \\
a_{i_01} & \cdots & \cdots & a_{i_0i_0} & \cdots & \cdots & a_{i_0n} \\
\vdots & \ddots & \vdots & 0 & \ddots & \vdots & \ddots & \vdots \\
0 & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & a_{nn}
\end{bmatrix},$$

where, in view of the irreducibility of $A$,

$$a_{i_0 j} \neq 0, \quad a_{j i_0} \neq 0, \quad j = 1, \ldots, n,$$

whence, by (2.4),

$$|a_{i_0 i_0}| |a_{jj}| = R_{i_0}(A) R_j(A), \quad j \neq i_0, \quad j = 1, \ldots, n. \quad (2.23)$$

Relations (2.22) and (2.23) trivially imply that all the nonstrict inequalities in (1.3) are satisfied and, moreover, by (2.21) and (2.23) we have:

$$|a_{ii}| |a_{jj}| = \left( \frac{R_{i_0}(A)}{|a_{i_0 i_0}|} \right)^2 R_i(A) R_j(A) > R_i(A) R_j(A),$$

$$i \neq i_0, \quad j \neq i_0, \quad i, j = 1, \ldots, n.$$
Remark 2.3. Let
\[ S = \{i_0\}, \quad 1 \leq i_0 \leq n, \]
and let the principal submatrix \( A[\bar{S}] \) of a matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2, \) be diagonal. Then equality (2.2), occurring in Theorem 2.1, is equivalent to the conditions
\[
\frac{a_{ii_0}a_{i_0i}}{a_{i_0i_0}} > 0, \quad i \neq i_0, \quad i = 1, \ldots, n. \tag{2.24}
\]
Indeed, in the case considered, relation (2.2) amounts to the equalities
\[
\bar{\epsilon}_i \frac{a_{ii_0}}{a_{i_0i_0}} \epsilon_{i_0} = -\left| \frac{a_{ii_0}}{a_{i_0i_0}} \right|, \quad i \neq i_0, \quad i = 1, \ldots, n, \tag{2.25}
\]
where
\[ D = \text{diag} (\epsilon_1, \ldots, \epsilon_n), \quad |\epsilon_i| = 1, \quad i = 1, \ldots, n, \]
which trivially imply (2.24).

Conversely, setting
\[
\frac{a_{ii_0}}{a_{i_0i_0}} = -\beta_i \left| \frac{a_{ii_0}}{a_{i_0i_0}} \right|, \quad \frac{a_{ii_0}}{a_{i_0i_0}} = -\alpha_i \left| \frac{a_{ii_0}}{a_{i_0i_0}} \right|, \quad |\alpha_i| = |\beta_i| = 1, \quad i \neq i_0,
\]
from (2.24) we derive the relations
\[
\beta_i = \bar{\alpha}_i, \quad i \neq i_0,
\]
implying that
\[
\frac{a_{ii_0}}{a_{i_0i_0}} \alpha_i = -\left| \frac{a_{ii_0}}{a_{i_0i_0}} \right|, \quad \bar{\alpha}_i \frac{a_{ii_0}}{a_{i_0i_0}} = -\left| \frac{a_{ii_0}}{a_{i_0i_0}} \right|, \quad i \neq i_0.
\]
The latter equalities prove (2.25) with
\[
\epsilon_i = \begin{cases} 
1, & i = i_0; \\
\bar{\alpha}_i, & i \neq i_0.
\end{cases}
\]

Remark 2.4. If \( S = \{i_0\}, \ 1 \leq i_0 \leq n, \) and the principal submatrix \( A[\bar{S}] \) of the matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, \ n \geq 2, \) is diagonal, then equality (2.2) together with condition (ii) of Theorem 2.1 imply the relation
\[
a_{i_0i_0} = \sum_{j \neq i_0} a_{i_0j}a_{j_0i}/a_{jj},
\]
which occurs as condition (iv) in Theorem 1.5 and means that the Schur complement \( A/A[\bar{S}] \) of \( A \) relative to \( A[\bar{S}] \) is zero.
Indeed, in view of Remark 2.3, we have

\[ \sum_{j \neq i} \frac{a_{ii}a_{jj}}{a_{ii0}a_{jj}} = a_{ii0} \sum_{j \neq i} \frac{a_{ij0}a_{jj0}}{a_{ii0}a_{jj0}} \]

\[ \equiv a_{ii0} \sum_{j \neq i} \frac{|a_{ij0}|}{|a_{jj0}|} |a_{jj0}| 
\]

\[ = a_{ii0} \sum_{j \neq i} \frac{|a_{ij0}|}{|a_{jj0}|} R_j(A) \]

\[ \equiv (ii) \]

\[ = a_{ii0} R_i(A) \]

As Remark 2.4 shows, Theorem 2.1 extends Theorems 1.5 and 1.6 to the case of matrices that may have more than one off-diagonally dominant row. Now we will demonstrate that Theorem 2.1 generalizes Theorem 1.8 as well.

Indeed, if \( a_{ij} \neq 0 \) for some \( i \neq j \), then, by Lemma 2.1, \( i, j \in \bar{\gamma} \) for some \( \gamma \in C(A) \). Therefore, under the assumptions of Theorem 1.8, the nonstrict inequalities

\[ |a_{ii}| |a_{jj}| \geq R_i(A) R_j(A) \]

hold for all \( i \neq j \) such that \( a_{ij} \neq 0 \), i.e., the hypotheses of Theorem 2.1 are satisfied. The strict inequality (1.5) trivially implies that condition (2.3) is violated.

Now let \( S \) be nonempty. From inequalities (1.4) it follows that, for any \( \gamma \in C(A) \), \( \gamma = i_1 \cdots i_k i_{k+1}, \ i_{k+1} = i_1, \ k \geq 2 \), only one of the indices \( i_1, \ldots, i_k \in \bar{\gamma} \) can belong to \( S \). Thus, if \( k \geq 3 \), then \( A[S] \) is not a diagonal matrix, and, by Theorem 2.1, \( A \) is nonsingular if \( \gamma(A) \) contains a circuit of length \( k \geq 3 \).

Assume now that all circuits from \( \gamma(A) \) are of length two. In this case, from (1.5) it follows that

\[ |a_{ii0}| |a_{j0,j0}| > R_{i0}(A) R_{j0}(A) \]

for some \( i_0 \in S, \ j_0 \in S \) such that \( a_{i_0,j_0} \neq 0 \), which shows that condition (ii) of Theorem 2.1 is not fulfilled, whence \( A \) is nonsingular.

We conclude this section by presenting the necessary and sufficient conditions for the singularity of an irreducible matrix with off-diagonally dominant rows that satisfies only a part of inequalities (2.1). As we will see, in addition to strictly diagonally and off-diagonally dominant rows, such singular matrices may also have “neutral” rows satisfying the equality

\[ |a_{ii}| = R_i(A). \]

**Theorem 2.3.** Let \( A = (a_{ij}) \in C^{n \times n}, \ n \geq 2 \), be an irreducible matrix and let the set

\[ S = \{i, \ 1 \leq i \leq n : |a_{ii}| < R_i(A)\} \]
be nonempty. Assume that the principal submatrix \( A[S] \) is diagonal and the inequalities
\[
|a_{ii}||a_{jj}| \geq R_i(A)R_j(A)
\]
are satisfied for all \( i \in \bar{S} \) and all \( j \in S \) such that \( a_{ij} \neq 0 \). Then \( A \) is singular if and only if there exists a unitary diagonal matrix \( D \) such that
\[
D^*(I - D^{-1}A)D = |I - D^{-1}A|,
\]
and for any \( i \in \bar{S} \) either
\[
a_{ij} = 0 \quad \forall j \in S
\]
and
\[
|a_{ii}| = R_i(A),
\]
or
\[
a_{ij} = 0 \quad \forall j \in \bar{S}, \quad j \neq i,
\]
and
\[
|a_{ii}||a_{jj}| = R_i(A)R_j(A) \quad \forall j \in S \text{ such that } a_{ij} \neq 0.
\]

**Proof.** As in the proof of Theorem 2.1, we define the positive vector \( u = (u_i)_{i=1}^n \) by setting
\[
u_i = \begin{cases} R_i(A)/|a_{ii}|, & i \in S; \\ 1, & i \in \bar{S}, \end{cases}
\]
and ascertain that (see (2.8) and (2.10))
\[
\sum_{j \neq i} |a_{ij}|u_j = |a_{ii}|u_i, \quad i \in S,
\]
and
\[
\sum_{j \neq i} |a_{ij}|u_j \leq |a_{ii}|u_i, \quad i \in \bar{S}.
\]
The application of Theorem 2.2 then leads us to the conclusion that \( A \) is singular if and only if there exists a unitary diagonal matrix \( D \) such that (2.26) holds true and all the inequalities in (2.27) are equalities. The conditions necessary and sufficient for (2.27) to be satisfied with equalities are provided by relations (2.12)–(2.15), which proves the assertion of Theorem 2.3. □

3. Application to eigenvalue location

The application of Theorem 2.1 to the shifted matrix \( A - \lambda I, \lambda \in \mathbb{C} \), immediately leads to the following result, which improves Theorem 1.2 by reducing the set of the
Cassini ovals, corrects the wrong Theorem 1.3, and provides conditions necessary and sufficient for a boundary point of a proper union of the ovals of Cassini to be an eigenvalue of the irreducible matrix $A$.

**Theorem 3.1.** Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $n \geq 2$, be an irreducible matrix.

(i) All eigenvalues of $A$ lie in the union

$$
\hat{C}(A) = \bigcup \nolimits_{|a_{ij}| + |a_{ji}| \neq 0} C_{ij}(A)
$$

of the ovals of Cassini associated with those $i \neq j$ for which $a_{ij} \neq 0$ and/or $a_{ji} \neq 0$.

(ii) A boundary point $\lambda$ of the domain (3.1) is an eigenvalue of $A$ if and only if there is a unitary diagonal matrix $D \in \mathbb{C}^{n \times n}$ such that

$$
D^* [I - D^{-1}A^{-1}(A - \lambda I)] D = |I - D^{-1}A^{-1}(A - \lambda I)|,
$$

and either

$$
|a_{ii} - \lambda| = R_i(A), \quad i = 1, \ldots, n,
$$

i.e., $\lambda$ is a common boundary point of all the Gerschgorin circles

$$
G_i(A) = \{z \in \mathbb{C} : |a_{ii} - z| \leq R_i(A)\}, \quad i = 1, \ldots, n,
$$

or there is a nonempty proper subset $S$ of the index set $\{1, 2, \ldots, n\}$ such that the following conditions are fulfilled:

(a) both principal submatrices $A[S]$ and $A[\bar{S}]$ are diagonal;

(b) $|a_{ii} - \lambda| < R_i(A)$, $i \in S$, i.e., $\lambda$ is a common interior point of the Gerschgorin circles $G_i(A)$, $i \in S$;

(c) $\lambda$ is a common boundary point of all the ovals of Cassini $C_{ij}(A)$, where $i \in S$ and $j \in \bar{S}$, i.e.,

$$
|a_{ii} - \lambda||a_{jj} - \lambda| = R_i(A)R_j(A), \quad i \in S, \quad j \in \bar{S}.
$$

(iii) An eigenvalue $\lambda$, that is a boundary point of the domain (3.1) is of geometric multiplicity one, and the corresponding eigenspace is spanned by a vector $z = (z_i) \in \mathbb{C}^n$ such that

$$
|z_i| = \begin{cases} 
\alpha & \text{if } R_i(A) > |a_{ii} - \lambda|, \\
1 & \text{if } R_i(A) \leq |a_{ii} - \lambda|,
\end{cases}
$$

where

$$
\alpha = \max_{1 \leq i \leq n} R_i(A)/|a_{ii} - \lambda|.
$$
References