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CHESSBOARD DOMINATION PROBLEMS

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A graph may be formed from an $n \times n$ chessboard by taking the squares as the vertices and two vertices are adjacent if a chess piece situated on one square covers the other. In this paper we survey some recent results concerning domination parameters for certain graphs constructed in this way.

1. Introduction

The classical problems of covering chessboards with the minimum number of chess pieces were important in motivating the revival of the study of dominating sets in graphs, which commenced in the early 1970's. These problems certainly date back to de Jaenisch [7] and have been mentioned in the literature frequently since that time (see e.g. [2, 8, 13]).

A graph P_n may be formed from an $n \times n$ chessboard and a chess piece P by taking the n^2 squares of the board as vertices and two vertices are adjacent if piece P situated at one of the squares is able to move directly to the other. For example the Queen's graph Q_n has the n^2 squares as vertices and squares are adjacent if they are on the same line (row, column or diagonal).

In this paper we survey recent results which involve various domination parameters for graphs which are constructed in this way. Outlines of some of the proofs are given, although most appear elsewhere.

2. Domination of the queens' graph

2.1. *An upper bound for the domination number of the queens' graph*

The *domination number* $\gamma(G)$ (*independent domination number* $i(G)$ of a graph $G = (V, E)$ is the smallest cardinality of a subset (independent subset) D of V such that each vertex of $V - D$ is adjacent to at least one vertex of D. Obviously $\gamma(G) \leqslant i(G)$ for any graph G. The determination of $\gamma(Q_n)$, which is the minimum number of queens required to cover the entire $n \times n$ chessboard, is perhaps the best known chessboard covering problem. The following experimental values of $\gamma(Q_n)$ for $n \leqslant 17$ are due mainly to Kraitchik (see [8]). We have corrected (by computer) values for $n = 5$, 6, 7.

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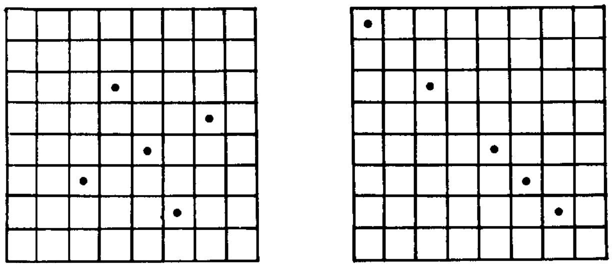

Fig. 1. Minimum dominating sets of Q_8 with 5 queens.

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\gamma(Q_n)$:	1	1	1	2	3	3	4	5	5	5	5	6	7	8	9	9	9

Two optimal queen coverings for the 8×8 chessboard are depicted in Fig. 1. Welch [15] has established an upper bound for $\gamma(Q_n)$.

Theorem 1 (Welch). *Let $n = 3q + r$ where $0 \leqslant r < 3$. Then $\gamma(Q_n) \leqslant 2q + r$.*

Proof. We first describe the placement of queens which shows that $\gamma(Q_n) \leqslant 2n/3$, for the case $n = 3q$. The $n \times n$ board is divided into 9 $q \times q$ sub-boards (numbered 1 through 9 in Fig. 2). Queens are placed on the main diagonal of sub-board 3 and on the diagonal immediately above the main diagonal of sub-board 7. Finally a queen is placed in the bottom left hand corner of sub-board 7. It is easily seen that these $2q$ queens cover the entire $n \times n$ board. If $n = 3q + r$, where $r = 1$ or 2, then consider the configuration of Fig. 2 augmented with r extra rows (cols) added on the bottom (right) and place extra queen(s) at position(s) $\{(3q + i, 3q + i) \mid i = 1, r\}$. This covering by $2q + r$ queens completes the proof. $\square$

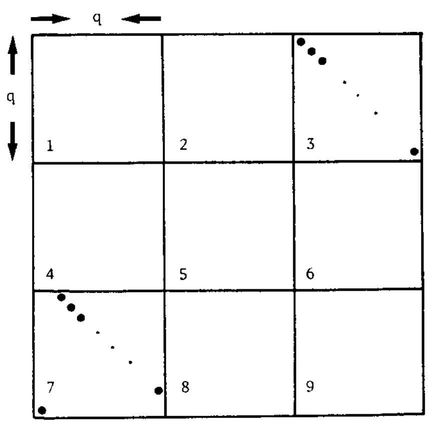

Fig. 2. Example to show $\gamma(Q_{3q}) \leqslant 2q$.

2.2. *An upper bound for the independent domination number of the queens' graph*

In this section we consider $i(Q_n)$, the minimum number of queens which cover the entire $n \times n$ board, with the additional requirement that the queens do not cover each other. Spencer and Cockayne [6] have established the following upper bound for $i(Q_n)$.

Theorem 2 (Spencer and Cockayne). *For any n, $i(Q_n) < 0.705n + 0.895$.*

Idea of Proof. Consider an infinite square chessboard. A queen placed on any square x covers the infinite set of squares which are collinear with x and completely covers the 3×3 board B_1 which surrounds square x (see Fig. 3). The placement procedure is then continued iteratively. For each $n > 1$, a completely covered chessboard B_n is found as follows. Let $A_1 = \{x\}$. Four queens are symmetrically placed on the set X_n of squares. These four squares are not covered by the queens of B_{n-1} and lie immediately outside the board B_{n-1}. The new board B_n is the largest square chessboard symmetrically containing B_{n-1}, which is completely covered by queens placed on the set of squares $A_n = A_{n-1} \cup X_n$. The construction implies that A_n is an independent vertex subset of B_n. The 9×9 board B_2 and the sets X_2 and X_3 are depicted in Fig. 3. In the diagram small dots denote squares covered by $A_2 = \{x\} \cup X_1$.

The size of the board B_n depends on the following pair of recursively defined integer functions:

$$f(1) = g(1) = 1,$$

$$f(n+1) = \text{the least integer greater than } f(n) \text{ which does not equal } f(k) + 2g(k) \text{ for any } k \leqslant n,$$

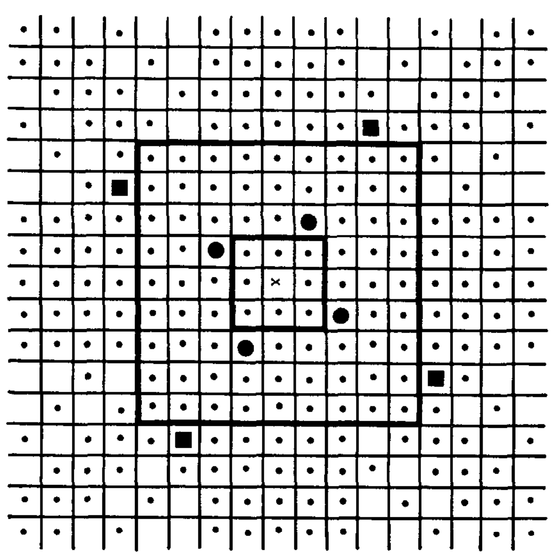

Fig. 3. Illustration of the construction used in proof of Theorem 2.

and

$g(n+1) =$ the least integer greater than $g(n)$ which does not equal $f(k)+g(k)$ for any $k \leqslant n$.

In fact the size of B_n is $2(f(n)+g(n))-1$ and we have

$$i(Q_{2(f(n)+g(n))-1}) \leqslant 4n-3. \tag{1}$$

The remainder of the proof is a lengthy estimation of $f(n)$, $g(n)$ and the details may be found in [6].

2.3. *A lower bound for $\gamma(Q_n)$*

Theorem 3 (Spencer [14]). *For any n, $\gamma(Q_n) \geqslant (n-1)/2$.*

Proof. Consider a covering of the $n \times n$ board using $\gamma = \gamma(Q_n)$ queens. Suppose that the rows and columns are sequentially labelled $1, \ldots, n$ from top to bottom and left to right respectively. A row or column is said to be *occupied* if it contains a queen.

Let column a, (b) be the left most (right most) unoccupied column and let row $c\,(d)$ be the unoccupied row closest to the top (bottom). Further we set $\delta_1 = b - a$ and $\delta_2 = d - c$ and assume without lost of generality that $\delta_1 \geqslant \delta_2$.

Consider the sets S_1 and S_2 of squares in columns a and b respectively, which lie between rows c and $c + \delta_1 - 1$ inclusive and let $S = S_1 \cup S_2$. Since $\delta_1 \geqslant \delta_2$, no diagonal intersects both S_1 and S_2. Hence every queen diagonally dominates at most two squares of S (i.e. at most one per diagonal). Further queens situated above row c or below row $c + \delta_1 - 1$ do not dominate squares of S by row or column.

By definition of c, there are at least $c - 1$ queens above row c. Each row below row d is occupied and $d = c + \delta_2 \leqslant c + \delta_1$. Therefore all the $n - c - \delta_1$ rows below row $c + \delta_1$ are occupied. Hence there are at least $n - c - \delta_1$ queens below row $c + \delta_1 - 1$.

It follows that at least $(c-1) + (n - c - \delta_1) = n - \delta_1 - 1$ queens dominate at most 2 squares of S. The remaining queens of which there are at most $\gamma - (n - \delta_1 - 1)$, may cover at most 4 squares of S. Since all the $2\delta_1$ squares of S must be dominated we have

$$2(n - \delta_1 - 1) + 4(\gamma - (n - \delta_1 - 1)) \geqslant 2\delta_1,$$

which gives $\gamma \geqslant (n-1)/2$ as required. $\square$

2.4. *The diagonal queens' domination problem*

Inspection of Fig. 1 shows that one can cover the 8×8 board with a minimum number of queens by restricting the placement of queens to the main diagonal, hence the following definition:

diag(n) = minimum number of queens which may be placed on the main diagonal of an $n \times n$ chessboard and which dominate the entire board.

Cockayne and Hedetniemi [5] have related diag(n) to the following difficult and well-studied number-theoretic function. Let $r_3(n)$ be the largest cardinality of a subset of $N = \{1, \ldots, n\}$ which contains no 3-term arithmetic progression.

Theorem 4 (Cockayne and Hedetniemi). *For any n,*

$$\operatorname{diag}(n) = n - r_3\left(\left\lceil \frac{n}{2} \right\rceil\right).$$

Indication of Proof. This theorem is proved by way of the following lemma. Define $K \subseteq N$ to be *diagonal dominating* if queens placed in the positions $\{(k, k): k \in K\}$ on the main diagonal cover the entire board. A subset of N is called *midpoint-free* if it contains no 3-term arithmetic progression. Finally a subset of N is called *even-summed* if all its elements have the same parity.

Lemma 1. *$K \subseteq N$ is diagonal dominating if and only if $N - K$ is midpoint-free and even-summed.*

Theorem 3 is easily deduced from this lemma. Several estimates for $r_3(n)$ have appeared in the literature [1, 9–12] and Roth [11] has proved $\lim_{n\to\infty} (r_3(n)/n) = 0$. The latter result implies the following corollary.

Corollary 1.

$$\lim_{n\to\infty} (\operatorname{diag}(n)/n) = 1.$$

Using Theorem 1 we deduce the following:

Corollary 2. *For n sufficiently large, $\gamma(Q_n) < \operatorname{diag}(n)$.*

2.5. *Domination of Q_n by queens in a single column*

Denote by col(n), the minimum number of queens on any single column which are required to dominate the entire $n \times n$ chessboard. (It is easy to see that a column nearest the centre is as good as any other.) Cockayne, Gamble and Shepherd [3] have related col(n) to the same function $r_3(n)$ mentioned in Section 2.4.

Let

$$A(n) = n - r\left(\left\lceil \frac{n}{3} \right\rceil\right) \tag{2}$$

and

$$B(n) = n - \max_{\substack{k+l=\lceil n/2 \rceil \\ k,l \geq 0}} \left[r\left(\left\lceil \frac{k}{2} \right\rceil\right) + r\left(\left\lceil \frac{l}{2} \right\rceil\right) \right]. \tag{3}$$

Theorem 5 (Cockayne, Gamble and Shepherd).

$$\mathrm{col}(n) = \min[A(n), B(n)] \quad (n \geq 2).$$

The proof of this theorem is highly technical and we give no details here. However, one may deduce from the proof:

Corollary 3. *For any n, $\mathrm{col}(n) \geq \mathrm{diag}(n)$.*

2.6. *Unsolved problems concerning queens*

Problem 1. Is $\gamma(Q_n) \leq \gamma(Q_{n+1})$ for all n?

We now refer to equations (2) and (3) of Section 2.5. The computer has determined that $A(n) \geq B(n)$ for $n \leq 150$ and we therefore ask:

Problem 2. Is $A(n) \geq B(n)$ for all n?

Finally, could (3) be simplified by evaluation of the maximum?

Problem 3. Find

$$\max_{\substack{k+l=\lceil n/2\rceil \\ k,l\geq 0}} \left[r\left(\left\lceil\frac{k}{2}\right\rceil\right) + r\left(\left\lceil\frac{l}{2}\right\rceil\right)\right].$$

3. Domination parameters for the bishops' graph

The bishops' graph D_n has the n^2 squares for vertices and two squares are adjacent if they lie on the same diagonal. In [4], Cockayne, Gamble and Shepherd have calculated three parameters for D_n.

3.1. *Domination and independent domination numbers*

Theorem 6 (Cockayne, Gamble and Shepherd). *For any n, $\gamma(D_n) = i(D_n) = n$.*

Indication of Proof. The set of squares of a nearest column to the centre is an independent dominating set of D_n hence

$$\gamma(D_n) \leq i(D_n) \leq n$$

and it remains to show $\gamma(D_n) \geq n$.

The North-West to South-East running diagonals are labelled sequentially $1, \ldots, 2n-1$ in the North-East direction, and w (and b) are the labels of the white (black) diagonal closest to the main diagonal which has no bishop. Without losing generality, one may assume $\{w, b\} \subseteq \{1, \ldots, n\}$. Diagonal w has w squares and these must be dominated. Further by definition of w, there are bishops on each diagonal strictly between w and $2n - w$. Hence n_w the number of white bishops in any dominating set satisfies

$$n_w \geqslant \max(w, n - w - 1). \tag{4}$$

Similarly,

$$n_b \geqslant \max(b, n - b - 1). \tag{5}$$

The result is simply deduced from (4) and (5). □

3.2. *The total domination number*

The *total domination number* $t(G)$ of a graph $G = (V, E)$ is the minimum cardinality of a subset T of vertices, such that each vertex of V is adjacent to at least one vertex of T.

Theorem 7 (Cockayne, Gamble and Shepherd). *For any* $n \geqslant 3$, $t(D_n) = 2\lceil \frac{2}{3}(n - 1)\rceil$.

Outline of Proof. D_n is the disjoint union of the white bishops graph W_n and the black bishops graph B_n. We summarize only the proof that $t(B_n) = \lceil \frac{2}{3}(n - 1)\rceil$ for n even. Notice that a total bishop dominating set of B_n is precisely a total rook dominating set of the diamond shaped chessboard S_n which has n rows and $n - 1$ columns. We exhibit S_8 in Fig. 4. For ease of presentation, we use rooks, rows and columns, rather than bishops and diagonals. □

Lemma 1. *For any n, S_n has a minimum total rook dominating set with the rooks on consecutive rows and columns.*

Proof. See [4]. □

It follows from Lemma 1 that some minimum total rook dominating set of S_n may be used to construct a total rook dominating set of an $m \times p$ rectangular board with property REL, i.e. a *r*ook on *e*very *l*ine (row or column). It is shown

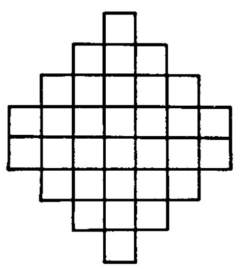

Fig. 4. The diamond-shaped chessboard S_8.

that such a board satisfies $m+p \geqslant n-1$ and hence, if $s(m,p)=$ minimum number of rooks in an REL total dominating set of an $m \times p$ board, we have

$$t(B_n) = \min_{m+p \geqslant n-1} s(m,p). \tag{6}$$

Lemma 2.

$$s(m\,p) = \begin{cases} \lceil \frac{2}{3}(m+p) \rceil & p \leqslant m \leqslant 2p+2, \\ m & m > 2p+2. \end{cases} \tag{7}$$

Proof. By establishing and solving a recurrence for $s(m,p)$. For details see [4]. □

One may deduce from (6) and (7) that $t(B_n) \geqslant \lceil \frac{2}{3}(n-1) \rceil$ and the final part of the proof exhibits a total rook dominating set of S_n with $\lceil \frac{2}{3}(n-1) \rceil$ rooks. This completes the outline of the proof of Theorem 6.

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