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Simultaneous bifurcation of limit cycles from two nests of periodic orbits ☆

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Abstract

Let $\dot{z} = f(z)$ be an holomorphic differential equation having a center at p, and consider the following perturbation $\dot{z} = f(z) + \varepsilon R(z, \bar{z})$. We give an integral expression, similar to an Abelian integral, whose zeroes control the limit cycles that bifurcate from the periodic orbits of the period annulus of p. This expression is given in terms of the linearizing map of $\dot{z} = f(z)$ at p. The result is applied to control the simultaneous bifurcation of limit cycles from the two period annuli of $\dot{z} = iz + z^2$, after a polynomial perturbation.

Keywords: Holomorphic vector fields; Bifurcation; Limit cycle

1. Introduction

One of the main open problems in the qualitative theory of planar polynomial vector fields is the celebrated *Hilbert* 16*th problem*. Among other questions, it asks for finding an upper bound for the maximum number of limit cycles of planar polynomial systems in terms of their degrees. This is a very hard problem, still open even when $n = 2$. Thus, some simpler problems have been introduced in order to advance in its solution. One of them is the so-called *weak* (*or tangential, infinitesimal*) *Hilbert* 16*th problem*. Let us recall it.

Consider the family of planar vector fields $X_\varepsilon = X_0 + \varepsilon Y$, with $\varepsilon \in \mathbb{R}$, where Y is a polynomial vector field and $X_0 = (\partial H/\partial y, -\partial H/\partial x)$ a Hamiltonian polynomial vector field having a continuum of periodic orbits. It is well known that to study the periodic orbits of X_ε that remain, for ε small enough, among all the periodic orbits of X_0 it is

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necessary to study the number of isolated zeroes of an Abelian integral, also known as Melnikov function. Precisely, if for $h \in (h_0, h_1)$ the curves $\{H(x, y) = h\}$ are periodic orbits of X_0, the simple zeroes of the *Abelian integral*

$$J(h) = \int\limits_{\{H(x,y)=h\}} Y_2(x, y)\,dx - Y_1(x, y)\,dy, \tag{1}$$

where $Y_i(x, y)$, $i = 1, 2$, denote the two components of $Y(x, y)$, give rise to limit cycles of X_ε, which tend to the corresponding level sets when ε goes to zero. So the weak Hilbert 16th problem asks for an upper bound of the number of zeros of $J(h)$ in terms of the degrees of H and Y, see for instance [5, Part II] for a survey on this subject.

Clearly, instead of taking X_0 as a Hamiltonian system it is possible to consider a similar problem for any X_0 having a continuum of periodic orbits. This situation occurs for instance when X_0 has an integrating factor, it is reversible or when it is an holomorphic vector field. This later case is the one that we will consider in this paper. More concretely, we will find an equivalent expression of (1) when the unperturbed system X_0 instead of being Hamiltonian is an holomorphic system, $\dot z = f(z)$, and we will apply it to study the number of limit cycles of the polynomial perturbations of $\dot z = iz + z^2$.

Let p be a center of an holomorphic equation $\dot z = f(z)$, with period annulus $\mathcal{U}$. It is well known that there exists a conformal map $\phi : \mathcal{U} \to \mathbb{C}$ that linearizes the differential equation, i.e. in the new variable $w = \phi(z)$ it writes as $\dot w = iw$, see for instance [3,7]. For short we will say that ϕ is *a linearizing change* of $\dot z = f(z)$ at p.

Our first result is the following:

Theorem A. *Consider the differential equation*

$$\dot z = f(z) + \varepsilon R(z, \bar z), \tag{2}$$

where $\dot z = f(z)$ has a center at p. Let ϕ be a linearizing change of $\dot z = f(z)$ at p. Then the simple zeros $\tilde c \in (0, c_0)$ of the function

$$I(c) = I^p(c) = -\mathrm{Im}\left(\int\limits_{\gamma_c := \{w\bar w = c\}} \overline{\phi'(z)\, R(z, \bar z)|_{z=\phi^{-1}(w)}}\, dw\right), \tag{3}$$

give rise, for $|\varepsilon|$ small enough, to limit cycles of (2) *that tend when ε goes to zero to $\phi^{-1}(\gamma_{\tilde c})$. Here c_0 is given by the image by ϕ of the boundary of the period annulus of p.*

We will call $I^p(c)$, the *bifurcating function* of (2) at p.

As an application of the above result we study the system

$$\dot z = iz + z^2 + \varepsilon R_m(z, \bar z), \tag{4}$$

where ε is real and small and $R_m(z, \bar z)$ is any polynomial of degree less or equal than m, that is,

$$R_m(z, \bar z) = \sum_{l=0}^{m} \sum_{k=0}^{l} \bar a_{k,l} z^{l-k} \bar z^k, \tag{5}$$

where $a_{k,l} \in \mathbb{C}$ are free parameters. We remark that in the above expression of R we consider its coefficients conjugated to simplify further computations.

Observe that for $\varepsilon = 0$, the phase portrait (as a planar polynomial system) of system (4) consists of two isochronous non-degenerate centers at $z = 0$ and $z = -i$ separated by the invariant straight line at $\mathrm{Im}(z) = 1/2$.

Before stating our main results we briefly discuss the previous work. Although the problem of the simultaneous bifurcation of limit cycles from either continua of periodic orbits, different weak focus, or critical points and infinity, has been also considered in several papers (see for instance [1,6,9] or [2], respectively), we focus here on the works of Chicone and Jacobs [4] and Li, Llibre and Zhang [10]. Both papers deal with simultaneous limit cycles for Eq. (4). In both papers the starting point is to multiply the differential equation by an integrating factor of the unperturbed system. Under this change of time the unperturbed equation becomes Hamiltonian and then it is possible to study the (non-polynomial) Abelian integral given in (1) to determine the limit cycles bifurcating from the periodic orbits of the unperturbed system. A similar approach is applied in [8] to study the polynomial perturbations of $\dot z = iz + z^3$.

In the primary paper of Chicone and Jacobs, the authors studied the number of limit cycles bifurcating from any quadratic Hamiltonian system having an isochronous center (at the origin), under homogeneous quadratic perturbations. Their study splits in four cases and, of course, one of them corresponds to Eq. (4) with $m = 2$. They conclude that the number of limit cycles after perturbation is at most 1 (and 1 is possible) in each period annulus. They also consider the problem of the *simultaneous bifurcation* from both continua of periodic orbits, which is the main subject of our paper.

Definition. For a given Eq. (4) we will say that the configuration of limit cycles (u, v), $u \geqslant 0$, $v \geqslant 0$, *is realizable* if, for ε small enough, exactly u (respectively v) limit cycles bifurcate from the periodic orbits surrounding $z = 0$ (respectively $z = -i$).

With this notation, in [4] the authors prove that all configurations (u, v) with $0 \leqslant u, v \leqslant 1$ are realizable for Eq. (4) with $m = 2$. Ten years later, Li, Llibre and Zhang studied in [10] the bifurcation problem when the quadratic unperturbed system has certain specific normal forms, including the case $\dot{z} = iz + z^2$, and the perturbation is of arbitrary degree m. They conclude that for Eq. (4) at most $m - 2$ limit cycles emerge from the center located at the origin, when $m \geqslant 4$ and that this upper bound is attained. Moreover, they also study the simultaneous bifurcation problem by showing that under *symmetric* perturbations the configurations of limit cycles (u, u), with $u \leqslant [(n-1)/2]$ are realizable. Here $[x]$ denotes the integer part of x.

We now state our main result.

Theorem B. *For Eq.* (4)*, the following configurations of limit cycles are realizable*:

(a) (u, v) *with* $0 \leqslant u \leqslant 1$ *and* $0 \leqslant v \leqslant 1$*, if* $m = 1, 2$ *or* 3.
(b) (u, v) *with* $0 \leqslant u \leqslant m - 2$*,* $0 \leqslant v \leqslant m - 2$*, if* $4 \leqslant m \leqslant 8$.
(c) (u, v) *with* $0 \leqslant u \leqslant m - 2$*,* $0 \leqslant v \leqslant m - 2$ *and* $u + v \leqslant m + 4$*, if* $m > 8$.

The key points in proving the above theorem are the following:

(i) The bifurcation function $I^0(c)$ given in (3) is essentially a polynomial in c of degree $m - 2$, see expression (12) in Lemma 3.3.
(ii) The differential equation $\dot{z} = iz + z^2$ remains unchanged when the two centers are permuted. Thus the computations to get the bifurcating function given in Theorem A, $I^0(c)$ can be used to obtain $I^{-i}(c)$. Moreover, the coefficients of the polynomial that controls the zeroes of $I^{-i}(c)$ are given by a linear map of the coefficients of the one of $I^0(c)$, see Proposition 3.4.
(iii) Our proof of statement (b) of Theorem B consists in showing that a linear map that gives the coefficients of $I^0(c)$ and $I^{-i}(c)$ in terms of the coefficients of R has a given rank. Fixing a value m we can obtain the matrix associated to this linear map and prove the desired result. This is done for $4 \leqslant m \leqslant 8$. We could do the same for a bigger fixed m, but we have not been able to do in the general case. From our results it seems natural to believe that statement (b) is true for all $m > 8$, but we have not been able to prove this fact. On the other hand we have obtained the partial results stated in (c).

As a summary in this paper we deal with the problem of perturbing centers in order to better understand the 16th Hilbert problem, on the number of limit cycles of the family of polynomials of certain degree. In the literature this approach is mostly focused when only one Hamiltonian center is perturb by a fixed degree polynomial perturbation. The advantage of perturbing a Hamiltonian center is to be able to use the Abelian integrals in order to control the number of limit cycles bifurcating from the period annulus.

In Theorem A, we show that the limit cycles bifurcating from any holomorphic (non-Hamiltonian) center can also be interpreted in terms of the zeroes of a certain Abelian integral. Moreover, we apply Theorem A to equation $\dot{z} = iz + z^2$. In this setting we are able to obtain an explicit expression for the corresponding Abelian integral for any degree of the polynomial perturbation.

Easily, since equation $\dot{z} = iz + z^2$ has two centers located at $z = 0$ and $z = -i$, using the Abelian integrals method we would be able to know the number of limit cycles bifurcating, after a polynomial perturbation, from each center separately. In contrast, a more interesting problem is to study the number of limit cycles bifurcating simultaneously from both centers. This problem was solved by Chicone and Jacobs [4] for quadratic perturbations, and by Llibre et al. [10] for symmetric perturbations of arbitrary degree. In the light of Theorem B we improve the result in [10] for Eq. (4) in the sense that, with general perturbations (not necessarily symmetric), we obtain any possible configuration of simultaneously limit cycles for $4 \leqslant m \leqslant 8$. Moreover, for each $m > 8$, we generate any pair (u, v) with $u + v \leqslant m + 4$, while following [10] we only get the pairs (u, u) with $2u \leqslant m - 2$.

We notice that the proof of Theorem B points out that to solve the number of limit cycles bifurcating simultaneously for (non-symmetry) perturbations is reduced to a linear algebra problem. Though we are not able to prove it for all m, we believe that choosing a suitable polynomial perturbation of degree $m \geqslant 2$ we could obtain any pair (u, v) with $0 \leqslant u, v \leqslant m - 2$.

We also observe that our study does not take into account the possible limit cycles that could bifurcate from the boundaries of the period annulus formed by the invariant line $\mathrm{Im}(z) = 1/2$ and the equator of the Poincaré compactification.

2. Proof of Theorem A

Let ϕ be the linearizing map of $\dot{z} = f(z)$ at p. Recall that it always exits and is well defined in the period annulus of p, see [3,7]. Indeed, this change of variables $w = \phi(z)$ verifies the equation $\phi'(z) f(z) = i\phi(z)$. Applying it to $\dot{z} = f(z) + \varepsilon R(z, \bar{z})$, we obtain

$$\dot{w} = iw + \varepsilon\phi'\big(\phi^{-1}(w)\big) R\big(\phi^{-1}(w), \overline{\phi^{-1}(w)}\big) := iw + \varepsilon L(w, \bar{w}). \tag{6}$$

The above system is a perturbation (in general, non-polynomial) of a Hamiltonian system, with Hamiltonian function $H(w, \bar{w}) = w\bar{w}/2$. It is well known that the Abelian integral (1) can also be applied in this situation to study the number of limit cycles of the perturbed system. If we write $L(w, \bar{w}) = L^R(w, \bar{w}) + iL^I(w, \bar{w})$, the Abelian integral associated to the period annulus of the origin of this equation is given by

$$J(h) = \int\limits_{\{w\bar{w}=2h\}} L^I\,dx - L^R\,dy.$$

On the other hand if $w = x + iy$ we observe that $\overline{L}\,dw = (L^R - iL^I)(dx + i\,dy) = L^R\,dx + L^I\,dy + i(L^R\,dy - L^I\,dx)$, so $-\mathrm{Im}(\overline{L}\,dw) = L^I\,dx - L^R\,dy$. Thus, by introducing the new parameter $c = 2h$ we have that

$$I(c) = J(c/2) = -\int\limits_{\{w\bar{w}=c\}} \mathrm{Im}\big(\overline{L(w, \bar{w})}\big)\,dw = -\mathrm{Im}\bigg(\int\limits_{\{w\bar{w}=c\}} \overline{L(w, \bar{w})}\,dw\bigg).$$

From the above expression the theorem follows.

3. The bifurcating function at $z = 0$ and $z = -i$

We study with detail the case $z = 0$ and later we will see how to reduce the case $z = -i$ to the previous one.

3.1. *The bifurcating function at $z = 0$*

In order to apply Theorem A to (4) at $p = 0$ we need the linearizing change of $\dot{z} = z + iz^2$ and the value c_0. It is easy to check that $c_0 = 1$ and

$$\phi(z) = \frac{z}{i + z}, \qquad \phi'(z) = \frac{i}{(i + z)^2} \quad \text{and} \quad \phi^{-1}(w) = \frac{iw}{1 - w}. \tag{7}$$

By applying Theorem A with $R(z, \bar{z}) = R_m(z, \bar{z})$ given in (5) and $f(z) = iz + z^2$ we obtain that $I_m(c) := I^0(c)$ is

$$I_m(c) = -\sum_{l=0}^{m}\sum_{k=0}^{l} I_{k,l}(c) = \sum_{l=0}^{m} J_l(c),$$

where

$$I_{k,l}(c) = \mathrm{Im}\left(a_{k,l}\int_{\gamma_c} \overline{\phi'\big(\phi^{-1}(w)\big)}\big(\phi^{-1}(w)\big)^k\big(\overline{\phi^{-1}(w)}\big)^{l-k}\,dw\right) \tag{8}$$

and $J_l(c) = \sum_{k=0}^{l} I_{k,l}(c)$. Recall also that γ_c is the circle of radius $0 < \sqrt{c} < 1$ centered at the origin. According with the notation introduced we obtain the following recursive expression for the bifurcating function

$$I_{m+1}(c) = I_m(c) + J_{m+1}(c). \tag{9}$$

In the following Lemmas 3.1, 3.2 and 3.3 we compute in turn $I_{k,l}(c)$, $J_l(c)$, and $I_m(c)$, respectively.

Lemma 3.1. *Fix $l \geqslant 0$ and $0 \leqslant k \leqslant l$. Then*

$$I_{k,l}(c) = c^{l-k}\,\mathrm{Im}\left(a_{k,l}i^{l+1}(-1)^{l-k}\int_{\gamma_c} w^{k-2}(1-w)^{-k}(w-c)^{2+k-l}\,dw\right). \tag{10}$$

Proof. By using (7),

$$\phi'\big(\phi^{-1}(w)\big) = -i(1-w)^2 \quad \text{and} \quad \overline{\phi^{-1}(w)} = \frac{-i\bar{w}}{1-\bar{w}}.$$

Consequently, from Eq. (8), we obtain that

$$\begin{aligned} I_{k,l}(c) &= \mathrm{Im}\left(a_{k,l}\int_{\gamma_c} i(1-\bar{w})^2\left(\frac{iw}{1-w}\right)^k\left(\frac{-i\bar{w}}{1-\bar{w}}\right)^{l-k}dw\right) \\ &= \mathrm{Im}\left(a_{k,l}i^{l+1}(-1)^{l-k}\int_{\gamma_c} w^k\bar{w}^{l-k}\frac{(1-\bar{w})^{2+k-l}}{(1-w)^k}\,dw\right). \end{aligned}$$

Since we have that, in γ_c it is satisfied $w\bar{w} = c$ we can write down the integral in terms only of w, obtaining (10). □

Lemma 3.2.

(a) $J_0(c) = 4\pi\,\mathrm{Im}(a_{0,0})c$.
(b) $J_1(c) = 2\pi(\mathrm{Re}(a_{0,1}) - \mathrm{Re}(a_{1,1})c)c$.
(c) $J_2(c) = 2\pi\,\mathrm{Im}(a_{1,2})c^2$.
(d) $J_3(c) = 2\pi\,\mathrm{Re}(a_{1,3})c^2$.
(e) *For all $l \geqslant 4$, $J_l(c) = \frac{c}{(1-c)^{l-3}}(\alpha_{p_l}c^{p_l} + \cdots + \alpha_{l-2}c^{l-2})$, where $p_l = [l/2]$. Moreover,*

$$\begin{aligned} &\alpha_{l-2} = \alpha_{l-2}(a_{1,l}) = 2\pi\,\mathrm{Im}\big(a_{1,l}i^{l+2}(-1)^{l-1}\big), \\ &\alpha_j = \alpha_j(a_{2,l},\ldots,a_{l-3,l}), \quad \textit{such that } \alpha_j(0,\ldots,0) = 0 \textit{ for all } j = p_l,\ldots,l-3. \end{aligned}$$

Proof. From the expression of $I_{k,l}$ in above lemma, to explicitly compute it, we only need to consider the possible poles inside the integral, and then, apply the Cauchy's Residues Formula. Easily, in the interior of the curve γ_c (notice that $0 < c < 1$), either there are no poles, or there is a unique pole at $z = 0$ or at $z = c$, or there are two poles, one at $z = 0$ and one at $z = c$, depending on l and k. More precisely,

(i) $w = 0$ is a pole if and only if $k < 2$, and
(ii) $w = c$ is a pole if and only if $k < l - 2$.

We prove case by case, so we divide the proof corresponding to each item.

The case $l=0$ ($k=0$).

The only pole inside the integral is $w=0$. Easy computations show that

$$\int_{\gamma_c} \frac{1}{w^2}(w-c)\,dw = 2\pi i(-2c),$$

hence

$$J_0(c) = I_{0,0}(c) = \operatorname{Im}(a_{0,0}i)2\pi i(-2c) = 4\pi \operatorname{Im}(a_{0,0})c,$$

as stated in (a).

The case $l=1$ ($k=0,1$).

The only pole inside the integral, for $k=0$ as well as $k=1$, is $w=0$. Some computations show that the residue of $f_k(w) = w^{k-2}(1-w)^{-k}(w-c)^{1+k}$, $k=0,1$ at $w=0$ is 1 if $k=0$, and c^2 if $k=1$. So, from (10)

$$I_{0,1}(c) = 2\pi \operatorname{Re}(a_{0,1})c, \qquad I_{1,1}(c) = -2\pi \operatorname{Re}(a_{1,1})c^2,$$

and we have $J_1(c) = 2\pi(\operatorname{Re}(a_{0,1}) - \operatorname{Re}(a_{1,1})c)c$.

The case $l=2$ ($k=0,1,2$).

The only pole inside the integral, for $k=0$ as well as $k=1$, is $w=0$, while for $k=2$ there are no poles. So $I_{2,2}(c)=0$. On the other hand, some computations show that the residue of $f_k(w) = w^{k-2}(1-w)^{-k}(w-c)^k$, $k=0,1$ at $w=0$ is 0 if $k=0$, and $-c$ if $k=1$. So, from (10)

$$I_{0,2}(c) = 0, \qquad I_{1,2}(c) = 2\pi \operatorname{Im}(a_{1,2})c^2.$$

So, $J_2(c) = I_{1,2}(c) = 2\pi \operatorname{Im}(a_{1,2})c^2$.

The case $l=3$ ($k=0,1,2,3$).

First we notice that if $k=2$ or $k=3$, there are no poles inside γ_c. Thus, $I_{2,3}(c) = I_{3,3}(c) = 0$. If $k=0$, there are two poles, $w=0$ and $w=c$, inside γ_c. If $k=1$ there is only one pole at $w=0$. Direct computations show that the residue of $f_k(w) = w^{k-2}(1-w)^{-k}(w-c)^{k-1}$, $k=0,1$ at $w=0$ is $-1/c^2$ if $k=0$, and 1 if $k=1$, while the residue of $g(w) = w^{-2}(w-c)^{-1}$ at $w=c$ is $1/c^2$. Because of that, it follows that

$$I_{0,3}(c) = 0, \qquad I_{1,3}(c) = 2\pi \operatorname{Re}(a_{1,3})c^2.$$

So statement (d) follows.

The case $l=4$ ($k=0,1,2,3,4$).

First we notice that if $k=2$, $k=3$ or $k=4$, there are no poles inside γ_c. Thus, $I_{2,4}(c) = I_{3,4}(c) = I_{4,4}(c) = 0$. Moreover, if $k=0$ as well as $k=1$, there are two poles, $w=0$ and $w=c$, inside γ_c. Again, direct computations show that the residue of $f_k(w) = w^{k-2}(1-w)^{-k}(w-c)^{k-1}$, $k=0,1$ at $w=0$ is $-1/c^2$ if $k=0$, and 1 if $k=1$, while the residue at $w=c$ is $-1/c^3$ if $k=0$, and $\frac{1}{c(1-c)}$ if $k=1$. Because of that, it follows that

$$I_{0,4}(c) = 0, \qquad I_{1,4}(c) = 2\pi \operatorname{Im}(a_{1,4})\frac{c}{1-c}c^2.$$

So, $J_4(c) = I_{1,4}(c) = 2\pi \operatorname{Im}(a_{1,4})\frac{c}{1-c}c^2$, and statement (e) follows for $l=4$.

The case $l>4$ ($k=0,1,2,\dots,l$).

Analogous to the previous cases, the problem is to compute the integral, over γ_c, of the function $f_k(w) = w^{k-2}(1-w)^{-k}(w-c)^{k+2-l}$ for all possible choices of $0 \leqslant k \leqslant l$. Clearly, there are only three different cases, depending on $k=0,1$ ($w=0$ and $w=c$ are the only poles of f_k inside γ_c), $2 \leqslant k \leqslant l-3$ ($w=c$ is the unique pole of f_k inside γ_c), and $k=l-2, l-1, l$ (no poles at all).

The residue of $f_k(w)$ at $w=0$ is $(2-l)(-1)^{l-1}c^{1-l}$ if $k=0$, and it is $(-1)^{l-3}c^{3-l}$ if $k=1$.

The computations to get the residue of $f_k(w)$ at $w=c$ are a bit more delicate, since it has to be for any $k \leqslant l-3$. Essentially the way of doing so is to write down the function $f_k(w)$ as follows

$$f_k(w) = \frac{1}{(w-c)^{l-k-2}}\left(\frac{w^{k-2}}{(1-w)^k}\right).$$

In this setting the residue of $f_k(w)$ at $w=c$ is $\frac{1}{(l-k-3)!}(g_k^{(l-k-3)}(w))|_{w=c}$ where

$$g_k(w)=\frac{w^{k-2}}{(1-w)^k}.$$

For our purpose it is necessary to distinguish three cases.

(a) If $k=0$,
$$\frac{1}{(l-3)!}\left(\frac{d^{l-3}}{dw^{l-3}}\frac{1}{w^2}\right)\bigg|_{w=c}=(-1)^{l-3}(l-2)\frac{1}{c^{l-1}}.$$
(b) If $k=1$,
$$\frac{1}{(l-4)!}\left(\frac{d^{l-4}}{dw^{l-4}}\frac{1}{w(1-w)}\right)\bigg|_{w=c}=\frac{(-1)^{l-4}(1-c)^{l-3}+c^{l-3}}{(1-c)^{l-3}c^{l-3}}.$$
(c) If $2\leqslant k\leqslant l-3$,
$$\frac{1}{(l-k-3)!}\left(\frac{d^{l-k-3}}{dw^{l-k-3}}\frac{w^{k-2}}{(1-w)^k}\right)\bigg|_{w=c}=\begin{cases}\frac{c^{2k+1-l}P_{l-3-k}(c)}{(1-c)^{l-3}}, & \text{if } \frac{l-1}{2}\leqslant k\leqslant l-3;\\ \frac{\tilde{P}_{k-2}(c)}{(1-c)^{l-3}}, & \text{if } 2\leqslant k\leqslant \frac{l-1}{2},\end{cases}\tag{11}$$
where P_m and $\tilde{P}_m$ are polynomials in c of degree m.

Once we have computed the residues of the integrator at the two poles we might compute the value of $I_{k,l}$ for any $l\geqslant 4$, so the value of the sum with respect to k. The cases $k=0$ and $k=1$ are special since we have to sum the value of the residue at $w=0$ and $w=c$. According to all above computations it is easy to see that

(a) $I_{0,l}(c)=0$ for all l, and
(b) $I_{1,l}(c)=2\pi(-1)^l\,\mathrm{Im}(i^la_{1,l})\frac{c^{l-1}}{(1-c)^{l-3}}$.

Recollecting all the above we are available to claim that

$$J_l(c)=\sum_{k=0}^{l}I_{k,l}(c)=2\pi\frac{c}{(1-c)^{l-3}}\Bigg[\mathrm{Im}\big(a_{1,l}i^l(-1)^lc^{l-2}\big)+\sum_{k=2}^{l-3}\mathrm{Im}\left(a_{k,l}i^l(-1)^{l-k}\begin{cases}c^kP_{l-k-3}(c), & \text{if } \frac{l-1}{2}\leqslant k\leqslant l,\\ c^{l-k-1}\tilde{P}_{k-2}(c), & \text{if } 2\leqslant k\leqslant\frac{l-1}{2}\end{cases}\right)\Bigg].$$

Notice that the above expression can be written expressed as in statement (e) of the present lemma since for all $2\leqslant k\leqslant l$ the maximum degree of $I_{k,l}(c)$ is $l-3$ while the minimum degree is denoted by $p_l=[l/2]$. □

Since we are interested in the number of simple zeros of $I_m(c)$ where $c\in(0,1)$, we adopt the following notation

$$R_l(c)=\frac{1}{c}J_l(c),\quad l=0,1,2,3,$$
$$R_l(c)=\frac{(1-c)^{l-3}}{c}J_l(c)\quad \forall l\geqslant 4.$$

We observe that each $R_l(c)$ is a polynomial on c, $R_0(c)$ has degree 0, $R_1(c)$, $R_2(c)$ and $R_3(c)$ have degree 1 and $R_l(c)$ has degree $l-2$ for all $l\geqslant 4$.

Lemma 3.3. *The following hold*:

(a) $I_0(c)=2\pi c\{2\,\mathrm{Im}(a_{0,0})\}$.
(b) $I_1(c)=2\pi c\{2\,\mathrm{Im}(a_{0,0})+\mathrm{Re}(a_{0,1})-\mathrm{Re}(a_{1,1})c\}$.

(c) $I_2(c) = 2\pi c\{2\,\mathrm{Im}(a_{0,0}) + \mathrm{Re}(a_{0,1}) + [-\mathrm{Re}(a_{1,1}) + \mathrm{Im}(a_{1,2})]c\}$.
(d) $I_3(c) = 2\pi c\{2\,\mathrm{Im}(a_{0,0}) + \mathrm{Re}(a_{0,1}) + [-\mathrm{Re}(a_{1,1}) + \mathrm{Im}(a_{1,2}) + \mathrm{Re}(a_{1,3})]c\}$.
(e) *For all $m \geqslant 4$,*

$$I_m(c) = \frac{c}{(1-c)^{m-3}}\big\{(1-c)S_{m-1}(c) + R_m(c)\big\}, \tag{12}$$

where

$$S_{m-1}(c) = (1-c)^{m-4}\sum_{l=0}^{3} R_l(c) + \sum_{l=4}^{m-1}(1-c)^{m-1-l}R_l(c).$$

Moreover, we have that

$$I_{m-1}(c) = \frac{c}{(1-c)^{m-4}}S_{m-1}(c).$$

Proof. In the previous lemma we have shown the expression of $J_l(c)$ for any fixed $l \geqslant 0$. Thus, to prove this lemma all we have to do is to sum the expressions when l runs from 0 to m, for each $m \geqslant 0$. Concretely, we have

$$\begin{aligned}
I_m(c) &= \sum_{l=0}^{m} J_l(c) = \sum_{l=0}^{3} cR_l(c) + \sum_{l=4}^{m}\frac{c}{(1-c)^{l-3}}R_l(c) \\
&= \frac{c}{(1-c)^{m-3}}\left\{(1-c)^{m-3}\sum_{l=0}^{3}R_l(c) + \sum_{l=4}^{m-1}(1-c)^{m-l}R_l(c) + R_m(c)\right\} \\
&= \frac{c}{(1-c)^{m-3}}\left\{(1-c)\left[(1-c)^{m-1-3}\sum_{l=0}^{3}R_l(c) + \sum_{l=4}^{m-1}(1-c)^{m-1-l}R_l(c)\right] + R_m(c)\right\}.
\end{aligned}$$

From the definition of $S_{m-1}(c)$ in the lemma we end up with (12) as desired. □

3.2. The bifurcating function at $z = -i$

Proposition 3.4. *Let $I_m^0(c)$ and $I_m^{-i}(c)$ be the bifurcation functions of system*

$$\dot z = -iz + z^2 + \varepsilon R_m(z, \bar z),$$

associated to $z = 0$ and $z = -i$, respectively. Write

$$R_m(z, \bar z) = \sum_{l=0}^{m}\sum_{k=0}^{l}\bar a_{k,l}z^{l-k}\bar z^{k}.$$

Then the expression of $I_m^{-i}(c)$ coincides with the expression of $I_m^0(c) = I_m(c)$ given in Lemma 3.3 *where each $a_{k,l}$ is substituted by the corresponding $b_{k,l}$, being*

$$b_{k,l} = (-1)^{l-k}\sum_{p=l}^{m}\sum_{q=k}^{p-(l-k)} a_{q,p}i^{p-l}(-1)^q\binom{q}{k}\binom{p-q}{l-k}. \tag{13}$$

Proof. The change of variables $w = z + i$, transform Eq. (4) into

$$\dot w = -iw + w^2 + \varepsilon R_m(w - i, \bar w + i),$$

which, under the change of variables and time, $\eta(t) = -w(-t)$, writes as

$$\dot\eta = i\eta + \eta^2 + \varepsilon R_m(-\eta - i, -\bar\eta + i). \tag{14}$$

Thus the zeroes of the bifurcating function associated to the center $z=-i$ of Eq. (4), $I_m^{-i}(c)$, are the zeroes of the bifurcating function associated to the origin of Eq. (14). The bifurcating function at the origin of Eq. (14) is given by (12) with the $b_{k,l}$'s as the free parameters. Note that

$$R_m(-\eta - i, -\bar{\eta} + i) = \sum_{p=0}^{m} \sum_{q=0}^{p} \bar{a}_{q,p}(-\bar{\eta}+i)^q(-\eta - i)^{p-q} := \sum_{l=0}^{m} \sum_{k=0}^{l} \bar{b}_{k,l}(\bar{\eta})^k \eta^{l-k}. \tag{15}$$

Hence, if $H(\eta, \bar{\eta}) =: R_m(-\eta - i, -\bar{\eta} + i)$, we have that

$$\bar{b}_{k,l} = \frac{1}{k!(l-k)!} \frac{\partial^l H}{\partial \bar{\eta}^k \partial \eta^{l-k}}(0,0).$$

Computing these partial derivatives from Eq. (15), we obtain the following bijective relation between the coefficients $a_{k,l}$ and $b_{k,l}$,

$$\bar{b}_{k,l} = \sum_{p=l}^{m} \sum_{q=k}^{p-(l-k)} \bar{a}_{q,p} i^{p-l} (-1)^{k-p-q} \binom{q}{k} \binom{p-q}{l-k}.$$

Finally, conjugating both sides of the above equality, we obtain (13). □

Remark 3.5. From the above relation between the $a_{k,l}$'s and the $b_{k,l}$'s we ensure that all coefficients of maximum degree m, satisfy $a_{k,m} = (-1)^m b_{k,m}$. From this equality, and the fact that $R_m^0(c)$ only depends on $a_{k,m}$, $k = 0, \dots, m$, we have that in expression (12), $R_m^0(c) = (-1)^m R_m^{-i}(c)$.

Remark 3.6. From the above proof it also becomes clear that if a configuration (u, v) of limit cycles is realizable for (4), then the configuration (v, u) is also realizable.

4. Proof of Theorem B

Proof of Theorem B(a). The cases $m = 1, 2$ were solved by Chicone and Jacobs in [4] and also follow from Lemma 3.3(a)–(c). We claim that $m = 3$ follows directly from Lemma 3.3(d). To see the claim we notice that, on one hand, $I_3^0(c)$ and $I_3^{-i}(c)$ have degree 1 (after dividing by c), and, on the other hand for a perturbation of degree 3 such that $\mathrm{Re}(a_{1,3}) = 0$ the expressions of $I_3^0(c)$ and $I_3^{-i}(c)$ only depend on the parameters of degree $m = 1, 2$. □

Proof of Theorem B(b). We start by studying the case $m = 4$. As it will become clear the arguments will work similarly for the rest of the cases.

From Lemma 3.3 and Eq. (13) we have that $I_4^0(c) = \frac{2\pi c}{1-c} p(c)$ and $I_4^{-i}(c) = \frac{2\pi c}{1-c} q(c)$ with

$$p(c) = \big[2\,\mathrm{Im}(a_{0,0}) + \mathrm{Re}(a_{0,1})\big](1-c) + \big[-\mathrm{Re}(a_{1,1}) + \mathrm{Im}(a_{1,2}) + \mathrm{Re}(a_{1,3})\big](1-c)c + \mathrm{Im}(a_{1,4})c^2,$$
$$q(c) = \big[2\,\mathrm{Im}(b_{0,0}) + \mathrm{Re}(b_{0,1})\big](1-c) + \big[-\mathrm{Re}(b_{1,1}) + \mathrm{Im}(b_{1,2}) + \mathrm{Re}(b_{1,3})\big](1-c)c + \mathrm{Im}(b_{1,4})c^2.$$

The linear relation between the coefficients of both polynomials is given in Eq. (13). Observe also that by Remark 3.5 it follows that $p(1) = q(1)$.

Let us see that in order to prove the lemma it suffices to show that, by using the free parameters $a_{k,l}$, the polynomials p and q fulfill the linear subspace of dimension 5,

$$\mathcal{L}_4 = \big\{\big(r_1(c), r_2(c)\big),\ r_i(c) \in \mathcal{P}_2[c],\ i = 1, 2 \,\big|\, r_1(1) = r_2(1)\big\},$$

where $\mathcal{P}_k[c]$ stands for the space of real polynomials of degree k. If this is true, then for any two polynomials in $\mathcal{L}_4$ having 0, 1, or 2 simple roots in the interval $(0, 1)$, we can conveniently choose the parameters $\{a_{0,0}, a_{0,1}, a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}\}$ such that the zeroes of the bifurcating functions coincide with the simple zeroes of these two polynomials and so the proof follows for $m = 4$.

To prove that the polynomials p and q fulfill the space $\mathcal{L}_4$, we first simplify the computations, by using only the free parameters included in the expression of $p(c)$, that is, $\{a_{0,0}, a_{0,1}, a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}\}$ (all the other parameters are taken equal to zero). The polynomials $p(c)$ and $q(c)$ are given by

$$
\begin{aligned}
p(c) &= 2\,\mathrm{Im}(a_{0,0}) + \mathrm{Re}(a_{0,1}) + \big(-2\,\mathrm{Im}(a_{0,0}) - \mathrm{Re}(a_{0,1}) - \mathrm{Re}(a_{1,1}) + \mathrm{Im}(a_{1,2}) + \mathrm{Re}(a_{1,3})\big)c \\
&\quad + \big(\mathrm{Re}(a_{1,1}) - \mathrm{Im}(a_{1,2}) - \mathrm{Re}(a_{1,3}) + \mathrm{Im}(a_{1,4})\big)c^2, \\
q(c) &= 2\,\mathrm{Im}(a_{0,0}) + \mathrm{Re}(a_{0,1}) - 2\,\mathrm{Re}(a_{1,1}) + 2\,\mathrm{Im}(a_{1,2}) - 2\,\mathrm{Im}(a_{1,4}) \\
&\quad + \big(-2\,\mathrm{Im}(a_{0,0}) - \mathrm{Re}(a_{0,1}) + 3\,\mathrm{Re}(a_{1,1}) - \mathrm{Im}(a_{1,2}) - \mathrm{Im}(a_{1,4})\big)c \\
&\quad + \big(-\mathrm{Re}(a_{1,1}) - \mathrm{Im}\big(a_{1,2} + 4\,\mathrm{Im}(a_{1,4})\big) + \mathrm{Im}(a_{1,4})\big)c^2.
\end{aligned}
$$

Notice again that $p(1) = q(1)$ and so, $(p, q) \in \mathcal{L}_4$. Since the rank of the matrix

$$
A_4 = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 \\
-2 & -1 & -1 & 1 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 \\
2 & 1 & -2 & 2 & 0 & -2 \\
-2 & -1 & 3 & -1 & 0 & -1 \\
0 & 0 & -1 & -1 & 0 & 4
\end{pmatrix}
$$

is 5 the result follows.

Now we show that the same arguments works for $m \leqslant 8$. From Lemma 3.3 we have that $I_m^0(c) = \frac{2\pi c}{(1-c)^{m-3}} p_m(c)$ and $I_m^{-i}(c) = \frac{2\pi c}{(1-c)^{m-3}} q_m(c)$. The coefficients of the polynomial p_m depend on $a_{k,l}$ while the coefficients of the polynomial q_m depend on $b_{k,l}$. Equation (13) defines the linear relation between the coefficients of the two polynomials and $p_m(1) = (-1)^m q_m(1)$.

To finish the proof it suffices to prove that, by using the parameters $a_{k,l}$, the polynomials p and q fulfill the linear subspace of dimension $2m - 3$,

$$
\mathcal{L}_m = \big\{\big(r_1(c), r_2(c)\big),\ r_i(c) \in \mathcal{P}_{m-2}[c],\ i = 1, 2 \,\big|\, r_1(1) = (-1)^m r_2(1)\big\}.
$$

We claim this is the case for $m = 5, 6, 7, 8$. To see the claim, it is enough to see that the corresponding matrix A_m (by taking only the parameters $a_{k,l}$ involved in the polynomial p_m) has rank $2m - 3$. As a second example we give the matrix when $m = 6$, for which there are eleven parameters $a_{k,l}$ in p_m and which has rank 9:

$$
A_6 = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & -3 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 3 & 3 & -3 & -3 & 1 & 0 & 1 & 0 & 0 & 0 \\
-2 & -1 & -3 & 3 & 3 & -2 & -1 & -1 & 0 & -2 & -1 \\
0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 & -1 & 0 & 0 \\
2 & 1 & -2 & 2 & 0 & -2 & 2 & -1 & 2 & -2 & 2 \\
-6 & -3 & 7 & -5 & 0 & 3 & -3 & 1 & -1 & -2 & 3 \\
6 & 3 & -9 & 3 & 0 & 4 & 1 & 0 & -19 & 26 & -24 \\
-2 & -1 & 5 & 1 & 0 & -9 & 0 & 0 & 33 & -40 & 30 \\
0 & 0 & -1 & -1 & 0 & 4 & 0 & 0 & -16 & 16 & -12
\end{pmatrix}.
$$

For instance, for $m = 8$ the matrix A_8 has 14 rows, 20 columns and rank 13. □

To prove part (c) we will use a different approach. We will assume that there is a perturbation of degree m such that the polynomials $p_m(c)$ and $q_m(c)$ associated to $I_m^0(c)$ and $I_m^{-i}(c)$ have exactly u and v simple reals roots in $(0, 1)$, respectively and we will build a $m + 1$ perturbation (by adding a term $\bar{a}\bar{z}z^m$ for a suitable value of a) so that the new corresponding polynomials $p_{m+1}(c)$ and $q_{m+1}(c)$ have exactly either $u + 1$ and v simple reals roots in $(0, 1)$, or u and $v + 1$ simple reals roots in $(0, 1)$, respectively, and both possibilities can be realized. We will need some preliminary results.

Lemma 4.1. *Consider the family of polynomials*

$$
p(\delta, c) = \beta_0(\delta) + \beta_1(\delta)c + \cdots + \beta_n(\delta)c^n,
$$

where $\beta_j(\delta)$, $j = 0, \ldots, n$, are real differentiable functions at the origin and $\delta \in \mathbb{R}$. Assume that $p(0, c)$ has exactly $0 \leqslant k \leqslant n$ roots in $(0, 1)$, all of them being simple, and that $p(0, 0)p(0, 1) \neq 0$. Then, when $s\delta p(0, 1) < 0$ (respectively

$s\delta p(0,1) \geqslant 0$) and $|\delta|$ is small enough, the new polynomial $\tilde{p}(\delta, c) := (1-c)p(\delta, c) + s\delta c^{n+1}$ has exactly $k+1$ (respectively k) roots in $(0,1)$, all of them being simple. Moreover, for $s\delta \neq 0$, $\tilde{p}(\delta, 0)\tilde{p}(\delta, 1) \neq 0$.

Proof. The polynomial $(1-c)p(\delta, c)$ has exactly $k+1$ simple roots in the interval $[0,1]$, say $c_0^* < c_2^* < \cdots < c_k^* = 1$. By the Implicit Function Theorem, for $|\delta|$ small enough, the polynomial $\tilde{p}(\delta, c)$ has also exactly $k+1$ roots, all of them being simple, in a neighborhood of $[0,1]$, say $c_0^*(\delta) < c_2^*(\delta) < \cdots < c_k^*(\delta)$. Furthermore, when δ goes to zero, then each $c_j^*(\delta)$ goes to c_j^*, for $j = 0, 1, \ldots, k$. By implicit derivation we obtain that $c_k^*(\delta) = 1 + s\delta/p(0,1) + o(\delta)$. Thus the biggest root belongs to the interval $(0,1)$ if and only if $s\delta p(0,1) < 0$, as we wanted to prove. Notice that when $s\delta = 0$, indeed $c_k^*(\delta) \equiv 1$. □

Proposition 4.2. *Let $\mathcal{R}_m(\bar{z}, z)$ be a polynomial perturbation of degree $m \geqslant 4$ of Eq. (4) and let $p_m(c)$ and $q_m(c)$ be the two polynomials of degree $m-2$ such that $I_m^0(c) = \frac{c}{(1-c)^{m-3}} p_m(c)$ and $I_m^{-i}(c) = \frac{c}{(1-c)^{m-3}} q_m(c)$. Assume that p_m and q_m have exactly u and v zeroes in the interval $(0,1)$, respectively, all of them being simple and that $p_m(0)p_m(1)q_m(0)q_m(1) \neq 0$. Then, there exist values of $a \in \mathbb{C}$ such that the perturbation of degree $m+1$,*

$$\mathcal{R}_m(\bar{z}, z) + \bar{a}\bar{z}z^m, \tag{16}$$

verifies

$$I_{m+1}^0(c) = \frac{c}{(1-c)^{m-2}} p_{m+1}(c) \quad \text{and} \quad I_{m+1}^{-i}(c) = \frac{c}{(1-c)^{m-2}} q_{m+1}(c),$$

where p_{m+1} and q_{m+1} are polynomials of degree $m-1$ and either p_{m+1} has exactly $u+1$ zeroes and q_{m+1} has exactly v zeroes in the interval $(0,1)$, or p_{m+1} has exactly u zeroes and q_{m+1} has exactly $v+1$ zeroes in the interval $(0,1)$, being both possibilities realizable. Moreover, all zeros of p_{m+1} and q_{m+1} are simple and $p_{m+1}(0)p_{m+1}(1)q_{m+1}(0)q_{m+1}(1) \neq 0$.

Proof. Consider the perturbation of degree $m+1$, given in (16) with $a \in \mathbb{C} \setminus \{0\}$. By Lemmas 3.2 and 3.3,

$$J_{m+1}^0(c) = \frac{c}{(1-c)^{m-2}} \delta c^{m-1},$$

$$I_{m+1}^0(c) = \frac{c}{(1-c)^{m-2}} \left\{ p_m(c)(1-c) + \delta c^{m-1} \right\},$$

where $\delta = 2\pi \operatorname{Im}(i^{m+3}(-1)^m a)$. Moreover, choose a such that $\delta \neq 0$.

Now, we turn our attention to obtain $I_{m+1}^{-i}(c)$. From Proposition 3.4 and (13) we have that

$$I_{m+1}^{-i}(c) = \frac{c}{(1-c)^{m-2}} \left(q_m(\delta, c)(1-c) + (-1)^{m-1}\delta c^{m-1} \right),$$

where

$$q_m(\delta, c) = \beta_0(\delta) + \beta_1(\delta)c + \cdots + \beta_{m-2}(\delta)c^{m-2},$$

being $\beta_j(\delta)$, $j = 0, \ldots, m-2$, linear functions in δ, and such that $q_m(\delta, c)$ is $q_m(c)$ when $a = 0$. Let $p_{m+1}(\delta, c) = p_m(c)(1-c) + \delta c^{m-1}$ and $q_{m+1}(\delta, c) = q_m(\delta, c)(1-c) + (-1)^{m-1}\delta c^{m-1}$. So, both polynomials have degree $m-1$. Moreover, from Remark 3.5 we know that $p_m(1) = (-1)^m q_m(1)$.

Thus by applying Lemma 4.1 to $p_{m+1}(\delta, c)$ with $s = 1$ we get that the condition for having exactly $u+1$ (respectively u) roots in $(0,1)$ is $\delta p_m(1) < 0$ (respectively $\delta p_m(1) \geqslant 0$). Similarly, for $q_{m+1}(\delta, c)$ taking $s = (-1)^{m-1}$ we get that the condition for having exactly $v+1$ (respectively v) roots in $(0,1)$ is $(-1)^{m-1}\delta q_m(1) < 0$ (respectively $(-1)^{m-1}\delta g_m(1) \geqslant 0$). Since $p_m(1) = (-1)^m q_m(1)$, we get that $\delta p_m(1)(-1)^{m-1}\delta q_m(1) = (-1)^{2m-1}\delta^2 p_m^2(1) < 0$. Then, either the number of zeroes of $p_{m+1}(\delta, c)$ and $q_{m+1}(\delta, c)$ are $u+1$ and v, or u and $v+1$, respectively, according with the sign of δ. From all the above results the proposition follows. □

Proof of Theorem B(c). By part (b), there are suitable perturbations of degree 8 having any configuration of limit cycles (u, v) with $u, v \leqslant 6$. Moreover, the corresponding polynomials $p_8(c)$ and $q_8(c)$ associated to $I^0(c)$ and $I^{-1}(c)$ can be taken such that $p_8(0)p_8(1)q_8(0)q_8(1) \neq 0$.

Consider $m = 9$, and fix any configuration (u^*, v^*) with $u^*, v^* \leqslant 6$. By applying Proposition 4.2 to Eq. (4) with R_8 being as above, we can find out suitable new perturbations of degree 9 having the configurations $(u^* + 1, v^*)$ and $(u^*, v^* + 1)$. So, for $m = 9$, all the configurations (u, v), with $u, v \leqslant 7$ and $u + v \leqslant 13$ can be realized. Given any $m > 8$, by repeating the same argument $m - 8$ times we end up with a perturbation of degree m with any configuration (u, v) so that $u, v \leqslant m - 2$ and $u + v \leqslant m + 4$, as desired. □

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