Robustness of $A$-optimal designs

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Abstract

Suppose that $Y = (Y_i)$ is a normal random vector with mean $Xb$ and covariance $\sigma^2 I_n$, where $b$ is a $p$-dimensional vector $(b_j)$, $X = (X_{ij})$ is an $n \times p$ matrix. $A$-optimal designs $X$ are chosen from the traditional set $\mathcal{D}$ of $A$-optimal designs for $\rho = 0$ such that $X$ is still $A$-optimal in $\mathcal{D}$ when the components $Y_i$ are dependent, i.e., for $i \neq i'$, the covariance of $Y_i, Y_{i'}$ is $\rho$ with $\rho \neq 0$. Such designs depend on the sign of $\rho$. The general results are applied to $X = (X_{ij})$, where $X_{ij} \in \{-1, 1\}$; this corresponds to a factorial design with $-1, 1$ representing low level or high level respectively, or corresponds to a weighing design with $-1, 1$ representing an object $j$ with weight $b_j$ being weighed on the left and right of a chemical balance, respectively.

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1. Introduction

Let $Y = (Y_i)$ be a normal $n$-dimensional random vector with mean $\mu = Xb$, where $X = (X_{ij})$ is an $n \times p$ matrix with $X_{ij}$ in the set $\mathbb{R}$ of real numbers, $b = (b_j)$ is a parameter in the $p$-dimensional Euclidean space $\mathbb{R}^p$, and the covariance of $Y$ is $\sigma^2 I_n$. To simplify our presentation,
we assume that $\sigma = 1$. The precision of the least square estimator $\tilde{b}(Y)$ and confidence ellipsoid of $b$ depends only on the covariance, $\text{cov}(\tilde{b}(Y))$, of $\tilde{b}(Y)$, where
\[
\tilde{b}(Y) = (X'X)^{-1}X'Y
\]
and
\[
\text{cov}(\tilde{b}(Y)) = (X'X)^{-1}. \tag{1.2}
\]
So we wish to choose an $X$ such that among all admissible designs $X$, $(X'X)^{-1}$ is smallest in some way, e.g., to choose $X$ such that $X$ is $A$-optimal, i.e., the trace of $\text{cov}(\tilde{b}(Y))$,
\[
\alpha(X) = \text{tr}(X'X)^{-1} \tag{1.3}
\]
is smallest; the set of all such $X$ will be denoted by $\mathcal{D}$.

Due to random effects or practical circumstance, it may happen that $\text{cov}(Y_i, Y_i') = \rho$ with $\rho \neq 0$; see, e.g. [16,17]. On the other hand, when $\rho = 0$, the traditional set $\mathcal{D}$ of $A$-optimal designs are far from being unique and we may wish to set up a criterion to choose a particular one; see, e.g. [12]. Consideration of $\rho \neq 0$ is such a criterion. Other authors have addressed similar problems (see [1,2,7,10,15]). Now, the least square estimator, $\tilde{b}(Y)$, of $b$ is
\[
\tilde{b}(Y) = (X'G^{-1}X)^{-1}X'G^{-1}Y, \tag{1.4}
\]
where $G = \text{cov}(Y),$
\[
G = (1 - \rho)I_n + \rho J_n, \quad -1/n - 1 < \rho < 1, \tag{1.5}
\]
\[
J_n = e_n e_n', \quad e_n = (1, 1, \ldots, 1)' \in \mathbb{R}^n. \tag{1.6}
\]
Since
\[
\text{cov}(\tilde{b}(Y)) = (X'G^{-1}X)^{-1},
\]
$X$ is $A$-optimal in $\mathcal{D}$ with respect to $\rho$ if it minimizes $\text{tr}((X'G^{-1}X)^{-1})$ for all $X$ in $\mathcal{D}$.

Although all of our main results, Theorems 1–4, are general, we shall merely apply them to the case where $X_{ij} \in \{-1, 1\}$. The models $Y$ considered here correspond to factorial designs with $-1, 1$ representing low level or high level respectively, or correspond to weighing designs with $-1, 1$ representing the object $j$ with weight $b_j$ being weighed on the left and right of a chemical balance, respectively; see, e.g. [3–6,11,12,14,15,18–21]. The results in these papers depend on the existence of a Hadamard matrix, $H = (h_{ij})$, of appropriate order $n$:
\[
H' H = n I_n, \quad \text{all } h_{ij} \in \{1, -1\}. \tag{1.7}
\]
The existence of such a matrix is still a conjecture for $n = 268$ and many other $n > 268$ [13]. This conjecture is referred to as the Hadamard conjecture.

2. Main results

Let $M_{n \times p}$ be the set of all $n \times p$ matrices over $\mathbb{R}$ equipped with the trace norm $\| \cdot \| : \| A \| = (\text{tr}(A'A))^{1/2}$. Note that $M_{n \times p} = \mathbb{R}^n$ if $p = 1$. Let $X \in n \times p$. Recall that
\[
\alpha(X) = \text{tr}(X'X)^{-1} \tag{2.1}
\]
and let
\[
\beta(X) = e_n' X (X'X)^{-1}X'e_n = \| u(X) \|^2 \tag{2.2}
\]
\[ \gamma(X) = e'\!_n X (X'X)^{-2} X'\! e_n. \] (2.3)

**Lemma 1.** Let \( X \in M_{n \times p} \). Then

(a) \( \beta(X) \leq n \).

(b) if \( X'X = nI_p \), then \( \|X'e_n\| \leq n \). Hence if \( e_n \) is a column of \( X \), then \( \|X'e_n\| = n \).

By Lemma 1(a) and (1.5),
\[ 0 < 1 + (n - 1 - \beta(X))\rho. \] (2.4)

**Theorem 1.** Let \( X \in M_{n \times p} \). Then

\[ (X'G^{-1}X)^{-1} = (1 - \rho)(X'X)^{-1} + \frac{(1 - \rho)\rho}{1 + (n - 1 - \beta(X))\rho}(X'X)^{-1} X'e_n e'_n X(X'X)^{-1}. \] (2.5)

**Proof.** By (1.5),
\[ G^{-1} = \frac{1}{1 - \rho} I_n - \frac{\rho}{(1 - \rho)(1 + (n - 1)\rho)} e_n e'_n. \] (2.6)

So
\[ (X'G^{-1}X)^{-1} = (1 - \rho)(X'(I_n - ge_n e'_n)X)^{-1}, \] (2.7)

where
\[ g = \frac{\rho}{1 + (n - 1)\rho}. \] (2.8)

Note that \( g \) is continuous at \( \rho = 0 \) and is 0 when \( \rho = 0 \). Now by (2.6),
\[ (X'G^{-1}X)^{-1} = (1 - \rho)(X'X)^{-1/2}[I_n - B]^{-1}(X'X)^{-1/2}, \] (2.9)

where
\[ B = g(X'X)^{-1/2} X'e_n e'_n X(X'X)^{-1/2}. \] (2.10)

When \( \rho \) is near 0, \( B \) can be viewed as an element in the Banach algebra of linear self-maps on \( \mathbb{R}^p \) with operator norm. So for some neighborhood of \( B \) at 0, we can expand \((I_n - B)^{-1}:
\[ (I_n - B)^{-1} = I_n + \sum_{k=1}^{\infty} B^k. \] (2.11)

Forming powers \((g\beta(X))^k\) in (2.11), we obtain
\[ (I_n - B)^{-1} = I_n + g(X'X)^{-1/2} e_n \left[ \sum_{k=0}^{\infty} (g\beta(X))^k \right] e'_n X(X'X)^{-1/2}, \]
i.e.,
\[ (I_n - B)^{-1} = I_n + \frac{g}{1 - g\beta(X)} (X'X)^{-1/2} X'e_n e'_n X(X'X)^{-1/2}, \] (2.12)
where by (2.4), \(|g\beta(X)| < 1\). So by (2.9),

\[
(X'G^{-1}X)^{-1} = (1 - \rho) \left[ (X'X)^{-1} + \frac{g}{1 - g\beta(X)} (X'X)^{-1} X'e_n e_n' X (X'X)^{-1} \right],
\]  

(2.13)

where by (2.8),

\[
g \beta(X) = \frac{\rho}{1 + (n - 1 - \beta(X)) \rho}.
\]  

(2.14)

So (2.5) is proved for a neighborhood of \(\rho\) at 0. By analytic continuation, it is valid under the constraints for \(\rho\) in (1.5). □

Formula (2.5) should be useful for finding optimal designs with respect to certain criteria. For \(A\)-optimality, Theorem 1 yields

**Theorem 2.** Let \(X \in M_{n \times p}\). Then

\[
\text{tr}(X'G^{-1}X)^{-1} = (1 - \rho) \alpha(X) + \frac{(1 - \rho)\gamma(X)}{1 + (n - 1 - \beta(X)) \rho},
\]  

(2.15)

where \(\alpha(X), \beta(X), \gamma(X)\) are given by (2.1)–(2.3).

Note that the formula (2.15) for evaluating \(\text{tr}(X'G^{-1}X)^{-1}\) depends on \(X\) merely through \(\alpha(X), \beta(X)\) and \(\gamma(X)\). Thus the finding of an \(A\)-optimal design for a given design problem depends on how \(\alpha(X), \beta(X)\) and \(\gamma(X)\) behave. In this regard, Theorem 2 yields:

**Theorem 3.** For \(\rho > 0\) and \(X \in M_{n \times p}\),

\[
\text{tr}(X'G^{-1}X)^{-1} \text{ increases with } \alpha(X), \beta(X) \text{ and } \gamma(X) \text{ separately.}
\]  

(2.16)

To understand how \(\rho\) affects \(\text{tr}(X'G^{-1}X)^{-1}\), let

\[
f_X(\rho) = \text{tr}(X'G^{-1}X)^{-1}.
\]  

(2.17)

Then we have

\[
f_X'(\rho) = -\alpha(X) + \frac{\gamma(X)[- (n - 1 - \beta(X)) \rho^2 - 2 \rho + 1]}{[1 + (n - 1 - \beta(X)) \rho]^2}
\]  

(2.18)

and

\[
f_X''(\rho) = -\frac{2 \gamma(X)(n - \beta(X))}{[1 + (n - 1 - \beta(X)) \rho]^3}.
\]  

(2.19)

So \(f_X\) is concave on the interval of \(\rho\) defined by (1.5).

With Theorem 3, we need, for a given design problem, to obtain a sharp lower bound for each of \(\alpha(X), \beta(X)\) and \(\gamma(X)\). The search for the sharp lower bound of \(\alpha(X)\) is equivalent to finding \(A\)-optimal designs with respect to \(\rho = 0\). It is much more difficult than finding \(D\)-optimal designs with respect to \(\rho = 0\); see the relevant articles mentioned in Section 1.

For \(X \in M_{n \times p}\), let

\[
X_{*j} = \text{the } j\text{th column sum of } X \text{ and } X_{*} = (X_{*1}, X_{*2}, \ldots, X_{*p}).
\]  

(2.20)

**Theorem 4.** Let \(X \in M_{n \times p}, n \geq p\). Suppose that \(X'X = (a - b)I_p + bJ_p\) for some real numbers \(a, b\) with \(a > b, a + (p - 1)b > 0\). Then
Proof. We shall merely prove (a). Now,

\[
(X'X)^{-1} = \frac{1}{a - b} \left[ I_p - \frac{b}{a + (p - 1)b} J_p \right].
\]

So by (2.2),

\[
\beta(X) = \frac{1}{a - b} \left[ \sum_{j=1}^{p} X^2_{\cdot j} - \frac{b}{a + (p - 1)b} \left( \sum_{j=1}^{p} X_{\cdot j} \right)^2 \right],
\]

i.e.,

\[
\beta(X) = \frac{1}{(a - b)(a + (p - 1)b)} \left[ (a + (p - 1)b) \sum_{j=1}^{p} X^2_{\cdot j} - b \left( \sum_{j=1}^{p} X_{\cdot j} \right)^2 \right].
\]

(2.22)

In order to form nonnegative terms \((X_{\cdot j} + X_{\cdot j'})^2\), we rewrite (2.22) as

\[
\beta(X) = \frac{1}{(a - b)(a + (p - 1)b)} \left[ (a - b + 2(p - 1)b) \sum_{j=1}^{p} X^2_{\cdot j} - b \sum_{j<j'}^{p} (X_{\cdot j} + X_{\cdot j'})^2 \right]
\]

through the identity

\[
\left( \sum_{j=1}^{p} X_{\cdot j} \right)^2 = \sum_{j<j'}^{p} (X_{\cdot j} + X_{\cdot j'})^2 - (p - 2) \sum_{j=1}^{p} X^2_{\cdot j}
\]

and the desired result follows. \(\square\)

Theorem 5. Let \(X \in M_{n \times p}, n \geq p\). Suppose that \(X'X = (a - b)I_p + bJ_p\) for some real numbers \(a, b\) with \(a > b, a + (p - 1)b > 0\). Then

(a)

\[
\gamma(X) = \frac{(a - b)^2 \| e_n' X \|^2 + b(2(a - b) + pb) \left( \sum_{j<j'}^{p} (X_{\cdot j} + X_{\cdot j'})^2 \right)}{(a - b)^2(a + (p - 1)b)^2}.
\]

(b)

\[
\gamma(X) = \frac{(a - b)^2 \| e_n' X \|^2 + b(2(a - b) + pb) \left( \sum_{j<j'}^{p} (X_{\cdot j} - X_{\cdot j'})^2 \right)}{(a - b)^2(a + (p - 1)b)^2}.
\]
**Proof.** We shall merely prove (a). By (2.3),
\[ \gamma(X) = ((X'X)^{-1}X'e_n)'((X'X)^{-1}X'e_n), \]
i.e.,
\[ \gamma(X) = \| (X'X)^{-1}X'e_n \|^2. \]
So by (2.21),
\[ \gamma(X) = \frac{1}{(a - b)^2} \left\| X'e_n - \frac{b}{a + (p - 1)b} J_p X'e_n \right\|^2, \]
i.e.,
\[ \gamma(X) = \frac{1}{(a - b)^2} \sum_{j=1}^{p} \left[ X_{\bullet j} - \frac{b}{a + (p - 1)b} \sum_{j'=1}^{p} X_{\bullet j'} \right]^2. \]
So
\[ (a - b)^2(a + (p - 1)b)^2 \gamma(X) \]
is
\[ \sum_{j=1}^{p} \left[ (a + (p - 1)b)X_{\bullet j} - b \sum_{j'=1}^{p} X_{\bullet j'} \right]^2, \]
whence by expanding squares into sums, we obtain
\[ \sum_{j=1}^{p} \left[ (a + (p - 1)b)^2 X_{\bullet j}^2 - 2b(a + (p - 1)b)X_{\bullet j} \left( \sum_{j'=1}^{p} X_{\bullet j'} \right) + b^2 \left( \sum_{j'=1}^{p} X_{\bullet j'} \right)^2 \right], \]
i.e.,
\[ (a + (p - 1)b)^2 \sum_{j=1}^{p} X_{\bullet j}^2 - 2b(a + (p - 1)b) \left( \sum_{j=1}^{p} X_{\bullet j} \right)^2 + pb^2 \left( \sum_{j=1}^{p} X_{\bullet j} \right)^2, \]
which yields (a) upon forming \( \sum_{j<j'}(X_{\bullet j} + X_{\bullet j'})^2 \) through
\[ \left( \sum_{j=1}^{p} X_{\bullet j} \right)^2 = \sum_{j<j'} (X_{\bullet j} + X_{\bullet j'})^2 - (p - 2) \sum_{j=1}^{p} X_{\bullet j}^2. \]
\[ \Box \]

Theorems 1–3 are proved for general \( X \) and one can apply them to designs \( X = (X_{i,j}) \) without assuming all \( X_{i,j} \in \{-1, 1\} \). Theorems 4 and 5 can be refined according to a given set of design matrices. One example is given in Section 3; see Theorem 6 in the appendix.

### 3. Applications to weighing and factorial designs

In this section, we shall assume that all \( X_{ij} \in \{-1, 1\}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, p; p \leq n \). Designs of this type occur in chemical balance weighing experiments or in factorial experiments. In the weighing experiment, the model used is \( Y \) with \( E(Y) = Xb, \text{cov}(Y) = \sigma^2 I_n \). In
this case, the emphasis is on estimating the vector \( b = (b_j)_{p}^{j=1} \) of weights \( b_j \) of \( p \) objects, where \( Y_j \) is the net weight observed in the \( i \)th trial, \( X_{ij} = -1 \) means that in the \( i \)th trial, the object \( j \) is being weighed on the left side of the given chemical balance and \( X_{ij} = 1 \) means that in the \( i \)th trial, the object \( j \) is being weighed on the right side. An \( A \)-optimal design minimizes the average variance of the least square estimators, \( \tilde{b}_j(Y) \) of \( b_j, j = 1, 2, \ldots, p \). In factorial experiments, we consider the model \( Y \) with no interaction effects: 
\[
E(Y) = W\gamma, \quad \text{cov}(Y) = \sigma^2 I_n,
\]
where \( W = (e_n, X), \gamma' = (\mu, b') \), where \( Y_j \) is the yield of the \( i \)th block, \( X_{ij} = -1 \) means that in the \( i \)th block, the \( j \)th factor is at the low level; \( X_{ij} = 1 \) means that in the \( i \)th block, the \( j \)th factor is at the high level, \( \mu \) is the mean without any factor effect and \( b = (b_j)_{p}^{j=1} \) represents the main effects. The interest here begins with testing the hypothesis \( b = 0 \). In this case, the information matrix for \( b \) is
\[
I(b) = X'X - \frac{1}{n}X'e_n'e_n'X = X'(I_n - \frac{1}{n}J_n)X
\]
(see [16]). Now arguing as in the proof of Theorem 1, we find that \( I(b)^{-1} = (X'X)^{-1} + \frac{1}{n-\beta(X)}(X'X)^{-1}X'e_n'e_n'X(X'X)^{-1} \). Then 
\[
\text{tr}(I(b))^{-1} = \text{tr}(X'X)^{-1} + \frac{\gamma(X)}{n-\beta(X)} \]
Thus it is reasonable to choose a design \( W = (e_n, X) \) as follows: among all designs \( W = (e_n, X) \) for which \( \text{tr}(X'X)^{-1} \) is smallest, select the one for which both \( \beta(X) \) and \( \gamma(X) \) are minimized. The existence of such designs requires the existence of appropriate Hadamard matrices; they are found in cases 1–5 below.

As expected from earlier papers cited in Section 1, we present our findings according to \( n \equiv 0, 1, 2 \) and 3 (mod 4).

Case 1: \( n \equiv 0 \) (mod 4).

In this case, the following family \( D_0 = \{X : X'X = nI_p \} \) of design matrices are known to be \( A \)-optimal for \( \rho = 0 \).

**Proposition 1.** Suppose that \( n \equiv 0 \) (mod 4).

(a) Let \( Z \in D_0 \) such that \( \|Z\|_0 = 0 \). Then for all \( \rho > 0 \), \( Z \) is \( A \)-optimal in \( D_0 \).

(b) Let \( Z \in D_0 \) such that \( \|Z\|_2 = n \). Then for all \( \rho < 0 \), \( Z \) is \( A \)-optimal in \( D_0 \).

**Proof.** (a) By (2.1)–(2.3),
\[
\alpha(X) = \frac{p}{n}, \quad \beta(X) = \frac{\|X\|_2^2}{n}, \quad \gamma(X) = \frac{\|X\|_2^2}{n^2}.
\]
So by (2.16), \( X \) is \( A \)-optimal in \( D_0 \) for all \( \rho > 0 \) if
\[
\|X\|_2^2 = 0.
\]
(b) Suppose that 
\[
\frac{-1}{n-1} < \rho < 0.
\]
By (3.2), (2.17) is reduced to
\[
f_X(\rho) = \frac{(1-\rho)p}{n} + \frac{(1-\rho)\rho\delta}{n^2(1+(n-1-\frac{\delta}{n})\rho)},
\]
where
\[
\delta = \|X\|_2^2.
\]
As means, we extend the domain of \( \delta \) in (3.4) to \((-n, \infty)\) and differentiate \( g(\delta) = f_X(\rho) \) in (3.4) with respect to \( \delta \):
\[
\begin{align*}
g'(\delta) &= \frac{(1 - \rho)\rho(1 + (n - 1)\rho)}{n^2 \left(1 + \left(n - 1 - \frac{\delta}{n}\right)\rho\right)^2}.
\end{align*}
\]

Then \( g'(\delta) < 0 \) for \( \rho \) in (3.3). So (3.4) is minimized when \( \delta \) attains its maximum value which, by Lemma 1(b), is \( n \). \( \square \)

Note that \( Z \) in (a) exists if there is a Hadamard matrix \( H = (h_{ij}) \) of order \( n \):

\[
H = (h_1, h_2, \ldots, h_n), \quad \text{all } h_{ij} \in \{-1, 1\}, \quad H'H = nI_n.
\]

(3.6)

Multiplying \(-1\) to certain rows of \( H \), we may assume that all coordinates of the first column \( h_1 \) of \( H \) are equal to 1. Use \( p \) columns of \( h_2, h_3, \ldots, h_n \) to form a matrix \( Z \). Then \( \frac{p}{2} \) coordinates of each column of \( Z \) are equal to 1 and \( \frac{p}{2} \) coordinates of each column of \( Z \) are equal to \(-1\). Thus \( Z \) is as required.

Let \( Z \) be an \( A \)-optimal design required in (b) above. As suggested by Lemma 1(b), such \( Z \) can be obtained via replacing one column of the above \( Z \) in (a) by \( e_n \).

Now we shall consider

Case 2: \( n \equiv 1 \mod 4 \).

In this case, for \( \rho = 0 \), the class of designs \( \mathcal{D}_1 = \{X : X'X = (n - 1)I_p + J_p\} \) are known to be \( A \)-optimal (see [4]).

**Proposition 2.** Suppose that \( n \equiv 1 \mod 4 \) and let \( Z \in \mathcal{D}_1 \) such that each column has sum 1. Then for all \( \rho > 0 \), \( Z \) is \( A \)-optimal in \( \mathcal{D}_1 \).

**Proof.** Let \( X \in \mathcal{D}_1 \). Since \( n \equiv 1 \mod 4 \), each \( |X_{ij}| \geq 1 \) and therefore \( \|e_n'X\|^2 \geq p \) with \( \|e_n'X\|^2 = p \) if and only if all \( |X_{ij}| = 1 \). Note that all \( Z_{ij} = 1 \). So by Theorems 4 and 5, both \( \beta(X) \) and \( \gamma(X) \) reach their minimum value at \( X = Z \). So by Theorem 3, the desired result follows. Indeed, this conclusion follows whenever there is a design, \( Z \), that minimizes both \( \|e_n'X\|^2 \) and \( \sum_{j < j'}(X_{ij} - X_{ij'})^2 \) in Theorems 4(b) and 5(b) with \( b > 0 \). \( \square \)

For \( n > p \), we can easily use the earlier method for \( n \equiv 0 \mod 4 \) to construct a desired \( Z \) in the above proposition: Suppose that \( n = 1 + 4k \) for some positive integer \( k \) and \( H = (h_1, h_2, \ldots, h_n) \) is a Hadamard matrix of order \( 4k \) such that all entries of the first column \( h_1 \) of \( H \) are equal to 1. Use \( p \) columns of \( h_2, h_3, \ldots, h_n \) to form a matrix and add a row of 1’s to form a matrix \( Z \). Then \( Z \) is a desired matrix.

Like the case where \( \rho = 0 \), the difficulty for finding \( A \)-optimal designs for \( \rho > 0 \) increases with \( n \mod 4 \). Now we shall consider

Case 3: \( n \equiv 2 \mod 4 \), \( p > 1 \).

In this case, the family \( \mathcal{D}_2 \) of designs \( X \) that satisfy \( X'X = \text{diag}[(n - 2)I_r + 2J_r, (n - 2)I_{p-r} + 2J_{p-r}] \), where \( r \) is the integral part of \( \frac{p+1}{2} \), are known to be \( A \)-optimal for \( \rho = 0 \); see [14]. For this case, we need only to apply Theorems 4 and 5 through splitting the underlying sum index \( j \in \{1, 2, \ldots, p\} \) into \( j \in J \) and \( j \in J' \) with \( J = \{1, 2, \ldots, r\} \) and \( J' = \{r + 1, r + 2, \ldots, n\} \). We need more auxiliary results. Lemma 2 and Lemma 3 below investigate the relationship between the column sums of any matrix \( X \) in \( \mathcal{D}_2 \). These column sums, in turn, determine the size of \( \alpha(X) \) and \( \beta(X) \) through \( \|e_n'X\|^2 \) (see Theorems 4 and 5). A proof of Lemma 2 below may be found in [15].
Lemma 2. Let \( n \equiv 2 \pmod{4} \) and \( x, y \) be \( n \)-dimensional vectors with all coordinates \( x_i, y_i \) in \( \{-1, 1\} \). Suppose that \( e'_n x = 0 \) and \( x'y = 0 \). Then \( |e'_n y| \geq 2 \).

The following result follows immediately from Lemma 2:

Lemma 3. Let \( X = [X_1, X_2, \ldots, X_p] \) be a matrix in \( \mathcal{D}_2 \). Suppose that \( e'_n X_j = 0 \) for \( j \leq r (> r) \). Then \( |e'_n X_j| \geq 2 \) for \( j > r \) (respectively \( j \leq r \)).

Proposition 3. (a) Suppose that \( p \) is even. Let \( Z \in \mathcal{D}_2 \) and
\[
e'_n Z_j = 0, \quad j = 1, 2, \ldots, r = \frac{p}{2}; \quad e'_n Z_j = 2, \quad j = r + 1, r + 2, \ldots, p. \tag{3.7}
\]
Then \( Z \) is \( A \)-optimal in \( \mathcal{D}_2 \) for all \( \rho > 0 \).

(b) Suppose that \( p \) is odd and \( n > p + 1 \). Let \( Z \in D_2 \) such that
\[
e'_n Z_j = 0, \quad j = 1, 2, \ldots, r = \frac{p+1}{2}; \quad e'_n Z_j = 2, \quad j = r + 1, r + 2, \ldots, p. \tag{3.8}
\]
Then \( Z \) is \( A \)-optimal in \( \mathcal{D}_2 \) for all \( \rho > 0 \).

Proof. Let \( X \in D_2 \). Write \( X = [U, V] \) with \( U = [X_1, X_1, \ldots, X_r], V = [X_{r+1}, X_{r+2}, \ldots, X_p] \).

By Theorem 4(b),
\[
\beta(X) = \frac{\|U' e_n\|^2}{n - 2 + 2r} + \frac{\|V' e_n\|^2}{n - 2 + 2(p - r)}.
\]
So by Lemma 3,
\[
\beta(X) \geq \min \left[ \frac{4r}{n - 2 + 2r}, \frac{4(p - r)}{n - 2 + 2(p - r)} \right],
\]
i.e.,
\[
\beta(X) \geq \frac{4(p - r)}{n - 2 + 2(p - r)}. \tag{3.9}
\]
By Theorem 5,
\[
\gamma(X) = \frac{\|U' e_n\|^2}{(n - 2 + 2r)^2} + \frac{\|V' e_n\|^2}{(n - 2 + 2(p - r))^2}.
\]
So by Lemma 3,
\[
\gamma(X) \geq \min \left[ \frac{4r}{(n - 2 + 2r)^2}, \frac{4(p - r)}{(n - 2 + 2(p - r))^2} \right]. \tag{3.10}
\]
(a) In this case, \( r = \frac{p}{2} \) and (3.9) and (3.10) are reduced, respectively, to
\[
\beta(X) \geq \frac{2p}{n - 2 + p} \tag{3.11}
\]
and
\[
\gamma(X) \geq \frac{2p}{(n - 2 + p)^2}. \tag{3.12}
\]
The above $Z$ attains the lower bounds in (3.11) and (3.12). So by Theorem 3, $Z$ is $A$-optimal in $\mathcal{D}_2$ for all $\rho > 0$.

(b) To decide which of \[ \frac{4r}{(n-2+2r)^2}, \frac{4(p-r)}{(n-2+2(p-r))^2} \] is larger, we note that

\[ \frac{4r}{(n-2+2r)^2} \geq \frac{4(p-r)}{(n-2+2(p-r))^2} \]

if and only if

\[ 4r(n-2+2(p-r))^2 \geq 4(p-r)(n-2+2r)^2 \]

which, upon simplification, is

\[ (n-2)^2 \geq p^2 - 1, \]

i.e.,

\[ n > p + 1. \quad \square \tag{3.13} \]

Note that $Z$ in the above proposition can be constructed as follows: let $H = (h_1, h_2, \ldots, h_{n-2})$ be an Hadamard matrix of order $n-2$ with $e_n$ as its first column. Let

\[ Z_j = (h'_{j+1}, 1, -1)' \quad \text{for} \quad j = 1, 2, \ldots, r, \]

\[ Z_j = (h'_{j+1}, 1, 1)' \quad \text{for} \quad j = r + 1, r + 2, \ldots, p. \]

Then $Z$ is as required.

Now we shall consider

Case 4. $n \equiv 3(\text{mod } 4)$, $p \geq 4, n \geq \left[7p - 16 + ((p - 4)(17p - 36))^{1/2}\right]/4$.

In this case, for $\rho = 0$, the class of designs $\mathcal{D}_3 = \{ X : X'X = (n+1)I_p - J_p \}$ are known to be $A$-optimal, see [18].

In order to find the minimum values of $\alpha(X)$ and $\beta(X)$ in Theorems 4(a) and 5(a), we need to find the minimum value of $(X_{\bullet j} + X_{\bullet j'})^2$, where $X_{\bullet j}$ and $X_{\bullet j'}$ are columns of a matrix $X$ in $\mathcal{D}_3$.

The following lemma serves this purpose. See [15] for its proof.

**Lemma 4.** Let $x, y$ be $n$-dimensional vectors with all coordinates $x_i, y_i$ in $\{-1, 1\}$. Suppose that $n \equiv 3(\text{mod } 4)$ and $xy = -1$. Then $|x_{\bullet} + y_{\bullet}| \geq 2$, where $x_{\bullet}$ ($y_{\bullet}$) is defined in (2.20).

**Proposition 4.** Suppose that $n \equiv 1(\text{mod } 4)$ and let $Z \in \mathcal{D}_3$ such that all $Z_{\bullet j}$ are equal to 1 (or $-1$). Then for all $\rho > 0$, $Z$ is $A$-optimal in $\mathcal{D}_3$.

**Proof.** By Theorem 3, it suffices to show that $\beta(Z)$, $\gamma(Z)$ minimize $\beta(X)$ and $\gamma(X)$ for $X \in \mathcal{D}_3$. Since $n$ is odd, $\|e'_{n}X\|^2 \geq p$. Since $\|e'_{n}Z\|^2 = p$, by Theorems 4(a), 5(a) and Lemma 4, $Z$ is as required. \square

Note that $Z$ in Proposition 4 can be constructed as follows: suppose that $n = 4t - 1$. Let $H = [h_1, h_2, \ldots, h_{n+1}]$ be a Hadamard matrix of order $n+1$. Multiplying $-1$ to appropriate rows and columns of $H$, we may assume that all entries of the first row and column of $H$ are all equal to 1. Delete the first row and column of $H$ to form $W$. Use $p$ columns of $W$ to form $Z$. Then $Z$ is as required.

Case 5. $n \equiv 3(\text{mod } 4)$, $p \geq 4, n < \left[7p - 16 + ((p - 4)(17p - 36))^{1/2}\right]/4$ or $1 < p \leq 3, n \geq p$. 

It is well known that this case is very complicated especially when \( n \) is near \( p \). Consequently, specifying a class of \( A \)-optimal designs is more difficult than our earlier cases. Sathe and She-noy [18] have shown that in this situation, \( A \)-optimal designs can be found among those \( X \) for which \( X'X \) is a block matrix of the following type: \( B = (B_{ij}) \), where the diagonal blocks \( B_{ii} = (n - 3)I_i + 3J_i \) are matrices of two possible sizes \( r_i = r \) or \( r + 1 \) and \( B_{ij} = -J_{r_i r_j} \) for \( i \neq j \). When \( n = [7p - 16 + ((p - 4)(17p - 36))^{1/2}] / 4 \), a design in \( D_3 \) will be \( A \)-optimal but there may be other \( A \)-optimal designs not in \( D_3 \). If \( n < [7p - 16 + ((p - 4)(17p - 36))^{1/2}] / 4 \), a design in \( D_3 \) will not, in general, be \( A \)-optimal. Thus one cannot expect to obtain a result for Case 5 analogous to Proposition 4.

To see how difficult these special cases can be, we shall consider the case where \( n = 7 \) and \( p = 6 \). From [18], it is known that, for \( \rho = 0 \), the family \( D_{7,6} \) of designs \( X \) that satisfy (3.14) below (up to the order of columns), are \( A \)-optimal:

\[
X'X = \begin{pmatrix}
+7 & -1 & -1 & -1 & -1 & -1 \\
-1 & +7 & -1 & -1 & -1 \\
-1 & -1 & +7 & +3 & -1 & -1 \\
-1 & -1 & +3 & +7 & -1 & -1 \\
-1 & -1 & -1 & -1 & +3 & +7 \\
-1 & -1 & -1 & -1 & +3 & +7
\end{pmatrix}.
\]  

(3.14)

Note that \( X'X \) in (3.14) is a block matrix of the form \( B = (B_{ij}) \) with \( k = 4 \) blocks of sizes \( r_1 = r_2 = 1, r_3 = r_4 = 2 \). One such \( X \), say \( Z \), is

\[
Z = \begin{pmatrix}
+1 & -1 & +1 & +1 & +1 & +1 \\
-1 & -1 & -1 & -1 & +1 & +1 \\
+1 & +1 & -1 & -1 & -1 & +1 \\
+1 & -1 & -1 & +1 & -1 & -1 \\
+1 & -1 & +1 & -1 & +1 & -1 \\
+1 & -1 & +1 & +1 & -1 & -1
\end{pmatrix}.
\]  

(3.15)

Showing that \( Z \) belongs to \( D_{7,6} \) is important because we do not want to work in an empty set. We now show that for all \( \rho > 0 \), \( Z \) is \( A \)-optimal in \( D_{7,6} \).

**Proposition 5.** \( Z \) in (3.15) is \( A \)-optimal in \( D_{7,6} \) for all \( \rho > 0 \).

**Proof.** By Theorem 3, we need only to show that \( \beta(Z), \gamma(Z) \) minimize \( \beta(X), \gamma(X) \) for all \( X \) in \( D_{7,6} \). The proof of this is very technical. So we will merely present the key ideas, referring, when necessary, to results in the appendix. Such a complication is common when \( n \equiv 3(\mod 4) \) and near \( p \).

By formula (A.6) of Theorem 6 in the appendix with \( k = 4, a = 4, b = 3 \) and \( r_1 = r_2 = 1, r_3 = r_4 = 2 \), \( \beta(X) \) is given by

\[
\frac{1}{8} \sum_{j=1}^{2} X_{*j}^2 + \frac{1}{12} \sum_{j=3}^{6} X_{*j}^2 + \frac{1}{12} [(X_{*3} - X_{*4})^2 + (X_{*5} - X_{*6})^2] \\
+ \frac{12}{5} \left[ \frac{1}{8} \sum_{j=1}^{2} X_{*j} + \frac{1}{12} \sum_{j=3}^{6} X_{*j} \right]^2.
\]  

(3.16)
By (3.16), $\beta(Z) = 1.6$. If $X_{*j} \in \{-5, 3, 7\}$ for at least two values of $j$, then $\beta(X) \geq \frac{2}{8} + \frac{18}{12} > 1.6$. If $X_{*j} \in \{-5, 7\}$ for at least one $j$, then $\beta(X) \geq \frac{26}{12} > 1.6$. If all $X_{*j} = -1$ except for one $j \in \{3, 4, 5, 6\}$, $X_{*j} = 3$, then $\beta(X) \geq \frac{2}{8} + \frac{3}{12} + \frac{9}{12} + \frac{16}{12} > 1.6$. Finally, by Lemma 8 in the appendix, the case where each $X_{*j} = -1$ does not occur. So $\beta(Z)$ minimizes $\beta(X)$ for all $X$ in $\mathcal{D}_{7.6}$.

We now show that $\gamma(Z)$ minimizes $\gamma(X)$ for all $X \in \mathcal{D}_{7.6}$. By formula (A.7) of Theorem 6 in the appendix,

$$\gamma(X) = \frac{t_1^{(2)}}{64} + \frac{t_2^{(2)}}{144} + \frac{(X_{*3} - X_{*4})^2 + (X_{*5} - X_{*6})^2}{36}$$

$$+ \frac{24}{5} \left( \frac{t_1}{64} + \frac{t_2}{144} \right) \left( \frac{t_1}{8} + \frac{t_2}{12} \right) + \frac{17}{50} \left( \frac{t_1}{8} + \frac{t_2}{12} \right)^2,$$

(3.17)

where

$$t_1 = X_{*1} + X_{*2}, \quad t_2 = X_{*3} + X_{*4} + X_{*5} + X_{*6},$$

$$t_1^{(2)} = X_{*1}^2 + X_{*2}^2, \quad t_2^{(2)} = X_{*3}^2 + X_{*4}^2 + X_{*5}^2 + X_{*6}^2.$$  

(3.18)

The fourth term, $\frac{24}{5} \left( \frac{t_1}{64} + \frac{t_2}{144} \right) \left( \frac{t_1}{8} + \frac{t_2}{12} \right)$, of (3.17) is negative if and only if, upon simplification, $(3t_1 + 2t_2)(9t_1 + 4t_2) < 0$, i.e.,

$$t_1 < 0, \quad t_2 > 0, \quad 4t_2 < -9t_1 < 6t_2,$$

(3.19)

or

$$t_1 > 0, \quad t_2 < 0, \quad -4t_2 < 9t_1 < -6t_2.$$  

(3.20)

From (3.19), (3.20) and Lemma 6 in the appendix, we list all the possible values of $X_{*} = (X_{*j})$ that could arise when the 4th term of (3.17) is negative. In each case, $\gamma(X)$ is also evaluated. For example, suppose that $X_{*1} = X_{*2} = -5$. Then $t_1 = -10$. So by (3.19), $15 < t_2 < 22.5$. Thus we find, $X_{*3} = 7, X_{*4} = X_{*5} = X_{*6} = 3, t_2 = 16$ and $\gamma(X) = 1.73778$. Continuing in this way, we obtain the table below.

<table>
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<tr>
<th>$X_{*1}$</th>
<th>$X_{*2}$</th>
<th>$t_1$</th>
<th>Interval for $t_2$</th>
<th>$X_{*3}$</th>
<th>$X_{*4}$</th>
<th>$X_{*5}$</th>
<th>$X_{*6}$</th>
<th>$t_2$</th>
<th>$\gamma(X)$</th>
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<tr>
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<td>3</td>
<td>3</td>
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<td>-6</td>
<td>$9 &lt; t_2 &lt; 13.5$</td>
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<td>-1</td>
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<td>-1</td>
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<td>-1</td>
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<td>-1</td>
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<td>6</td>
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<td>-5</td>
<td>-5</td>
<td>-1</td>
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<td>-12</td>
<td>0.65111</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>$-13.5 &lt; t_2 &lt; -9$</td>
<td>-5</td>
<td>-5</td>
<td>-1</td>
<td>-1</td>
<td>-12</td>
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</tr>
<tr>
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<td>7</td>
<td>10</td>
<td>$-22.5 &lt; t_2 &lt; -15$</td>
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<td>-5</td>
<td>-5</td>
<td>-1</td>
<td>-16</td>
<td>1.86278</td>
</tr>
</tbody>
</table>
Note that by Lemma 8 in the appendix, a design \( X \in \mathcal{D}_{7,6} \) with \( X_j \) as in line 5 of the above table does not exist and \( \gamma(Z) = .185 \) is less than all other \( \gamma(X) \) in the above table. So now we may assume that the fourth term of (3.17) is \( \geq 0 \). Checking that all \( \gamma(X) \geq \gamma(Z) = .185 \) in this case is similar to the above checking for \( \beta(Z) \), noting that, by Lemma 8, a design \( X \) with \( X_\bullet = (-1, -1, -1, -1, -1) \) does not exist. □

We emphasize here that Theorem 6 in the appendix and the above combinatorial method can be applied to designs well beyond \( \mathcal{D}_{7,6} \).

Appendix

In this appendix, we prove the technical results necessary for the proof of Proposition 5. An expansion of these results can be used to deal with other cases when \( n \equiv 3 \pmod{4} \) and near \( p \).

First, we derive formula for \( \beta(X) \) and \( \gamma(X) \) when \( X \in M_{n \times p} \) and \( X'X \) is a certain type of block matrix. We begin by introducing some notation and stating some known or easily verified results.

Let \( a, b, c \) be real numbers and,

\[
X = [X_1, X_2, \ldots, X_k] \quad \text{with} \quad X_i'X_i = B_i, \quad B_i = aI_{r_i} + bJ_{r_i},
\]

\[
X_i'X_i' = c e_i r_i' e_i' \quad \text{for} \quad i' \neq i;
\]

\[
D = \text{diag}[C_1, C_2, \ldots, C_k], \quad C_i = aI_{r_i} + (b - c)J_{r_i}. \tag{A.2}
\]

Then

\[
C_i^{-1} = \alpha I_{r_i} + \beta_i J_{r_i}, \quad \alpha = \frac{1}{a}, \quad \beta_i = \frac{-(b - c)}{a[a + (b - c)r_i]} \tag{A.3}
\]

and

\[
J_pD^{-1}J_p = \delta J_p, \quad \delta = \text{tr}(J_pD^{-1}) = \alpha p + \sum_{i=1}^k \beta_i r_i^2. \tag{A.4}
\]

Using the fact that \( X'X = D + cJ_p \) together with (A.4), it is easily seen that

\[
(X'X)^{-1} = D^{-1} - \theta D^{-1}J_pD^{-1}, \quad \theta = \frac{c}{1 + cd}. \tag{A.5}
\]

**Theorem 6.** Let \( X \in M_{n \times p} \) be as in (A.1). Then

\[
\beta(X) = \sum_{i=1}^k \frac{1}{a(a + (b - c)r_i)} \left[ a \sum_{m=1}^{r_i} X_{i\bullet m}^2 + (b - c) \sum_{m < m'} (X_{i\bullet m} - X_{i\bullet m'})^2 \right]
\]

\[
- \theta \left( \sum_{i=1}^k \frac{1}{a + (b - c)r_i} \sum_{m=1}^{r_i} X_{i\bullet m} \right)^2. \tag{A.6}
\]
(b)

\[
\gamma(X) = \sum_{i=1}^{k} \left[ (\alpha + \beta_i r_i)^2 \sum_{m=1}^{r_i} X_{i \bullet m}^2 - (2\alpha\beta_i + \beta_i^2 r_i) \sum_{m<m'} X_{i \bullet m} X_{i \bullet m'} \right]
- 2\theta \left[ \sum_{i=1}^{k} \left[ (\alpha + \beta_i r_i)^2 \sum_{m=1}^{r_i} X_{i \bullet m} \right] \right] \left[ \sum_{i=1}^{k} (\alpha + \beta_i r_i) \sum_{m=1}^{r_i} X_{i \bullet m} \right]
+ \theta^2 \left[ \sum_{i=1}^{k} (\alpha + \beta_i r_i)^2 r_i \right] \left[ \sum_{i=1}^{k} (\alpha + \beta_i r_i) \sum_{m=1}^{r_i} X_{i \bullet m} \right]^2.
\]  

(A.7)

**Proof.** We shall prove (b). By (A.5),

\[
(X'X)^{-2} = D^{-2} - \theta[D^{-2} J_p D^{-1} + D^{-1} J_p D^{-2}] + \theta^2 D^{-1} J_p D^{-2} J_p D^{-1}.
\]

So by (2.3),

\[
\gamma(X) = \epsilon'_n XD^{-2} X' e_n - 2\theta \epsilon'_n XD^{-2} J_p D^{-1} X' e_n + \theta^2 \epsilon'_n XD^{-1} J_p D^{-2} J_p D^{-1} X' e_n.
\]  

(A.8)

We shall calculate the three terms in (A.8) separately. By (A.3),

\[
C_i^{-1} = \alpha I_{r_i} + \beta_i J_{r_i}
\]  

(A.9)

and therefore,

\[
C_i^{-2} = \alpha^2 I_{r_i} + (2\alpha\beta_i + \beta_i^2 r_i) J_{r_i}.
\]  

(A.10)

So by (A.2),

\[
\epsilon'_n XD^{-2} X' e_n = \sum_{i=1}^{k} s_i,
\]  

(A.11)

where

\[
s_i = \epsilon'_n X_i C_i^{-2} X'_i e_n.
\]  

(A.12)

Now

\[
s_i = \alpha^2 \sum_{m=1}^{r_i} X_{i \bullet m}^2 + (2\alpha\beta_i + \beta_i^2 r_i) \left[ \sum_{m=1}^{r_i} X_{i \bullet m} \right]^2.
\]  

(A.13)

In order to form \((X_{i \bullet m} - X_{i \bullet m'})^2\), we use

\[
\left( \sum_{m=1}^{r_i} X_{i \bullet m} \right)^2 = r_i \left[ \sum_{m=1}^{r_i} X_{i \bullet m}^2 \right] - \sum_{m<m'} \sum_{m} X_{i \bullet m} X_{i \bullet m'}
\]

to transform (A.12):

\[
s_i = (\alpha + \beta_i r_i)^2 \sum_{m=1}^{r_i} X_{i \bullet m}^2 - (2\alpha\beta_i + \beta_i^2 r_i) \sum_{m<m'} (X_{i \bullet m} X_{i \bullet m'})^2.
\]  

(A.14)

Now by (A.2),

\[
\epsilon'_n XD^{-2} J_p D^{-1} X' e_n = \sum_{i=1}^{k} \sum_{i'=1}^{k} t_{i,i'}.
\]

(A.15)
where
\[ t_{i,i'} = e'_n X_i C^{-2} e_{r_i} e'_{r_{i'}} C^{-1} X_{i'} e_n. \] (A.16)

So
\[ t_{i,i'} = e'_n X_i [(\alpha^2 I_{r_i} + (2\alpha \beta_i + \beta_i^2 r_{i'}) J_{r_i}) e_{r_i} e'_{r_{i'}} (\alpha I_{r_{i'}} + \beta_i J_{r_{i'}})] X_{i'} e_n, \]
i.e.,
\[ t_{i,i'} = (\alpha + \beta_i r_i)^2 (\alpha + \beta_{i'} r_{i'}) e'_n X_i e_{r_i} e'_{r_{i'}} X_{i'} e_n, \]
i.e.,
\[ t_{i,i'} = (\alpha + \beta_i r_i)^2 (\alpha + \beta_{i'} r_{i'}) \sum_{m,m'=1}^{r_i,r_{i'}} X_{i*m} X_{i'*m'}. \] (A.17)

Note that
\[ D^{-1} J_p D^{-2} J_p D^{-1} = (D^{-1} J_p D^{-1})^2. \]

So in block form,
\[ D^{-1} J_p D^{-2} J_p D^{-1} = [C^{-2} e_{r_i} e'_{r_{i'}} C^{-1}]^2, \]
and therefore, via block multiplication,
\[ D^{-1} J_p D^{-2} J_p D^{-1} = \left[ \sum_{u=1}^{k} C^{-1} e_{r_u} e'_{r_u} C^{-2} e_{r_u} e'_{r_{u'}} C^{-1} \right], \]
i.e.,
\[ D^{-1} J_p D^{-2} J_p D^{-1} = \sum_{u=1}^{k} (\alpha + \beta_u r_u)^2 r_u (\alpha + \beta_{i'} r_{i'}) e_{r_u} e'_{r_{u'}}. \] (A.18)

So
\[ e'_n X [D^{-1} J_p D^{-2} J_p D^{-1} e'_n] X' e_n = \sum_{i,i'=1}^{k} u(i,i'), \] (A.19)
where
\[ u(i,i') = \sum_{u=1}^{k} (\alpha + \beta_u r_u)^2 r_u (\alpha + \beta_{i'} r_{i'}) e'_n X_i e_{r_i} e'_{r_{i'}} X_{i'} e_n. \] (A.20)

Now
\[ u(i,i') = \sum_{u=1}^{k} (\alpha + \beta_u r_u)^2 r_u (\alpha + \beta_{i'} r_{i'}) \sum_{m,m'=1}^{r_i,r_{i'}} X_{i*m} X_{i'*m'}. \] (A.21)

By (A.10), (A.13), (A.14), (A.16), (A.18) and (A.20), we obtain (A.7) as required. \(\square\)

We shall now identify certain potential designs that do not exist under the present contraints. For this kind of work, one may consult [8,9] and other references on optimality mentioned in Section 1.

**Lemma 5.** Let \( X = (X_{ij}) \) be an \( n \times p \) matrix in \([-1, 1]\) with \( n \geq p \) and \( n \equiv 3 (\text{mod} 4) \). Let \( Z = EXF \), where \( E \) is a matrix obtained via interchanging certain rows of \( I_n \) and \( F \) is a diagonal matrix of size \( p \) with all diagonal elements equal to \(-1\) or \(1\). Then \( \beta(Z) = \beta(X) \).
Lemma 6. Suppose that \( n \equiv 3 \pmod{4} \) and \( a, b \) are \( n \)-dimensional vectors with entries in \( \{-1, 1\} \). Then \( a' b \equiv 3 \pmod{4} \) if and only if both \( a \) and \( b \) have an even number of negative coordinates or an odd number of negative coordinates; in the even case, the sum of the coordinates in \( a \) (and \( b \)) is \( 3 \pmod{4} \). Hence in Lemma 5, \( Z \) can be so chosen that all \( Z_j = 3 \pmod{4} \) and all \( Z_j Z_j' \equiv 3 \pmod{4} \), i.e.,
\[
Z_j \in \{-5, -1, 3, 7\}, \quad Z_j Z_j' = 7 \quad \text{and} \quad Z_j Z_j' \in \{-5, -1, 3, 7\} \quad \text{for} \ j \neq j'. \tag{A.22}
\]

Proof. This is just a simple modification of Lemma 2.2(c) of [14]. (Note that in (b) of this lemma, \( p_1 \) before the word “and” is missing; in (c), mod 6 should be mod 4.) \( \square \)

The following result can be proved through Lemma 5 and grouping all possibilities into cases.

Lemma 7. Let \( u = (u_i), v = (v_i) \in \{-1, 1\}^7 \) such that \( u_\bullet = 3 = v_\bullet \) and \( u \cdot v = 3 \). Suppose that \( a, b, c \) are distinct solutions of the equations
\[
u' x = v' x = x_\bullet \tag{A.23}
\]
in \( x = (x_i) \in \{-1, 1\}^7 \) and \( a' b = -1 \). Then \( a' c \in \{-5, 3\} \) or \( b' c \in \{-5, 3\} \).

As a consequence of Lemma 7, we have

Lemma 8. For any \( X \) in \( D_{7,6} \),
\[
X_\bullet \notin \{(-1, -1, -1, -1, 3, 3), (-1, -1, -1, -1, -1, -1)\}.
\]

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References

[9] Ehlich, Hartmut Determinantenabschätzung für binäre Matrizen mit \( n \equiv 3 \pmod{4} \) (German), Math. Z. 84 (1967) 438–447.