Simultaneous real diagonalization of rectangular quaternionic matrix pairs and its algorithm

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The simultaneous real diagonalization(SRD) of a pair of rectangular quaternionic matrices is introduced, and some necessary and sufficient conditions for the existence of the SRD are derived. Based on one of the necessary and sufficient condition, an algorithm for gaining the SRD of a given matrix pair is presented.

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1. Introduction

A rectangular matrix \( A = (a_{ij})_{m \times n} \) is said to be diagonal if \( a_{ij} = 0 \) for all \( i \neq j \) holds. In this article, we undertake the problem of simultaneous real diagonalization(SRD) of a pair of rectangular quaternionic matrices, and for a given quaternionic matrix pair, we present an algorithm to compute its SRD. Ever since the theory of quaternionic matrix was successfully applied in quantum mechanics and quantum field [1,2], many authors in different academic fields have studied quaternionic matrices [3–10]. Zhang [3] studied the canonical forms, determinants, rank and decompositions, including the singular value decomposition(SVD) of quaternionic matrices. He demonstrated the existence of the SVD of a quaternionic matrix by using the isomorphism between a quaternionic matrix and its complex representation matrix, which is called the complex adjoint matrix of the quaternionic matrix. Further, in [4,5], computing the SVD of quaternionic matrices by using the complex adjoint matrix can be found. A more effectively method for computing the SVD of quaternionic matrices was given in [6] by using the quaternion Householder transformation. Other quaternionic matrix decompositions also have been studied. For instance, Bunse-Gerstner et al. [7] studied a quaternion QR algorithm to compute the Schur decomposition of quaternionic matrices, and Zhuang [8] studied the simultaneous polar decompositions of quaternionic matrix pairs.

This article is organized as follows. In Section 2, we introduce a concept, simultaneous real diagonalization(SRD) of a pair of rectangular quaternionic matrices. We present some necessary and sufficient conditions for the existence of SRD of a given quaternionic matrix pair. Based on this condition, an algorithm for computing the SRD of two given quaternionic matrices is presented in Section 3. Finally, We give a summary of the paper in Section 4.

Throughout the paper we use \( \mathbb{R} \), \( \mathbb{C} \) and \( \mathbb{Q} \) to denote sets of real numbers, complex numbers and quaternions, while \( \mathbb{R}^{m \times n} \), \( \mathbb{C}^{m \times n} \) and \( \mathbb{Q}^{m \times n} \) denote sets of all \( m \times n \) matrices over real field, complex field and quaternion skew-field, respectively; \( U_n(\mathbb{Q}) \) is the set of \( n \times n \) unitary quaternionic matrices; \( SC_n(\mathbb{Q}) \) and \( SC_n^+(\mathbb{Q}) \) are sets of \( n \times n \) hermitian and positive definite quaternionic matrices respectively; \( I_n \) is the unite matrix of order \( n \); rank \( A \) and \( A^* \) are the rank and conjugate transpose of a quaternionic

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matrix $A$, respectively; $A \oplus B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$ is the direct sum of quaternionic matrices $A$ and $B$; $A^+$ is the Moore–Penrose inverse of $A$ which satisfies the following conditions:

\begin{align}
AA^+A &= A, \\
A^+AA^+ &= A^+, \\
( AA^+ )^* &= AA^+, \\
(A^+ A)^* &= A^+ A.
\end{align}

### 2. SRD of rectangular quaternionic matrix pairs

**Lemma 2.1** ([3] Singular-Value Decomposition(SVD)). Let $A \in \mathbb{Q}^{m \times n}$ be of rank $r$. Then there exist $U \in U_m(\mathbb{Q})$ and $V \in U_n(\mathbb{Q})$ such that

$$U^*AV = \Sigma_A = \begin{bmatrix} D_A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $D_A = \text{diag}(d_1, \ldots, d_r)$ and the $d$'s are the positive singular values of $A$ such that $d_1 \geq \cdots \geq d_r > 0$.

We also give out a well-known theorem as a lemma without proof:

**Lemma 2.2** (Spectral Theorem for Hermitian Quaternionic Matrices). $A \in SC_n(\mathbb{Q})$ if and only if there exist $U \in U_n(\mathbb{Q})$ and $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$A = U\Lambda U^*,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Further, $A \in SC_n^+(\mathbb{Q})$ if and only if $\lambda_i > 0$ ($1 \leq i \leq n$).

Now, we give out the main result of this section:

**Theorem 2.3.** Let $A$, $B \in \mathbb{Q}^{m \times n}$. Then the following statements are equivalent:

(I) there exist $U \in U_m(\mathbb{Q})$ and $V \in U_n(\mathbb{Q})$ such that

$$\Sigma_A = U^*AV, \quad \Sigma_B = U^*BV$$

are both real diagonal matrices, and $\Sigma_A$ has all elements nonnegative;

(II) $AB^*$ and $B^*A$ are both hermitian.

In order to prove the theorem, we first introduce a lemma:

**Lemma 2.4.** Let $S \in \mathbb{Q}^{r \times r}$, $D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. If we rewrite $D$ as follows by grouping all the same valued $\sigma_i$ terms together,

$$D = \begin{bmatrix} \mu_1I_{r_1} \\ & \ddots \\ & & \mu_lI_{r_l} \end{bmatrix},$$

where $\mu_1 > \mu_2 > \cdots > \mu_l > 0$, $\mu_i \in \{ \sigma_1, \ldots, \sigma_r \}$ and $r_i$ is the multiplicity of $\mu_i$ ($1 \leq i \leq l$). Obviously, $\sum_{i=1}^l r_i = r$. Then the following statements are equivalent:

(I) $SD$ and $DS$ are both hermitian;

(II) $S$ is a partitioned block diagonal matrix with all hermitian blocks as

$$S = \begin{bmatrix} S_1 \\ \vdots \\ S_l \end{bmatrix},$$

where $S_j \in \mathbb{Q}^{r_i \times r_j}$ ($1 \leq i, j \leq l$).

**Proof.** (I) $\Rightarrow$ (II) Since $D$ has form of Eq. (2.4), we can write

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1l} \\ \vdots & \ddots & \vdots \\ S_{l1} & \cdots & S_{ll} \end{bmatrix}.$$


where \( S_{ij} \in Q \) \((1 \leq i, j \leq l)\). Then since \( SD \) is hermitian, we have
\[
\mu_i S_{ij}^* - \mu_j S_{ji} = 0.
\]
(2.7)
Similarly, since \( DS^* \) is hermitian, we have
\[
\mu_j S_{ij}^* - \mu_i S_{ji} = 0.
\]
(2.8)
According to Eqs. (2.7) and (2.8), we have
\[
(\mu_i - \mu_j)(S_{ij} + S_{ji}^*) = 0
\]
and
\[
(\mu_i + \mu_j)(S_{ij} - S_{ji}^*) = 0.
\]
When \( i \neq j \), under the assumption, \( \mu_i \neq \mu_j \), then
\[
S_{ij} + S_{ji}^* = 0 \quad S_{ij} - S_{ji}^* = 0
\]
which means \( S_{ij} = S_{ji}^* = 0 \). When \( i = j \), it’s easy to get
\[
S_{ii} = S_{ii}^*.
\]
For convenience, write \( S_i \equiv S_{ii} \) \((1 \leq i \leq l)\). Hence
\[
S = \begin{pmatrix}
S_1 \\
\vdots \\
S_l
\end{pmatrix}
\]
with hermitian \( S_i \) \((1 \leq i \leq l)\).

\((\text{II}) \Rightarrow (\text{I})\) Since \( S \) has form of Eq. (2.5), then \( S \) is hermitian. Since \( D \) is a real diagonal matrix,
\[\begin{align*}
(\text{SD})^* &= D^*S^* = DS \\
&= \begin{pmatrix}
\mu_1^* & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \mu_l^*
\end{pmatrix} \begin{pmatrix}
S_1 \\
\vdots \\
S_l
\end{pmatrix} \\
&= \begin{pmatrix}
\mu_1 S_1 \\
\vdots \\
\mu_l S_l
\end{pmatrix} = \begin{pmatrix}
S_1 \mu_1 \\
\vdots \\
S_l \mu_l
\end{pmatrix} = SD,
\end{align*}\]
that is, \( SD \) is hermitian. Similarly, \( DS \) is also hermitian.

The proof is completed. ■

Now we prove Theorem 2.3:

**Proof.** \((\text{I}) \Rightarrow (\text{II})\) Since \( A \) and \( B \) have factorizations as Theorem 2.3, then
\[
\Sigma_A \Sigma_B = \Sigma_B \Sigma_A.
\]
Hence
\[
AB^* = U \Sigma_A V^* \Sigma_B U^* = U \Sigma_A \Sigma_B U^* = U \Sigma_B \Sigma_A U^* = U \Sigma_B V^* \Sigma_A U^* = BA^* = (AB^*)^*.
\]
Similarly, \( B^*A = A^*B = (B^*A)^* \).

\((\text{II}) \Rightarrow (\text{I})\) According to Lemma 2.1, there exist \( U_A \in U_m(Q) \) and \( V_A \in U_n(Q) \) such that
\[
\Sigma_A = U_A^* A V_A = \begin{pmatrix}
D_A & 0 \\
0 & 0
\end{pmatrix},
\]
where \( D_A = \text{diag}(\sigma_1, \ldots, \sigma_r), \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \).

Let \( C = U_A^* B V_A \). Since \( AB^* \) and \( B^*A \) are both hermitian, then
\[
\Sigma_A C^* = U_A^* A V_A U_B^* V_B A = U_A^* B A^* U_A = (\Sigma_A C^*)^*
\]

Let Lemma 2.4 provides a necessary and sufficient condition to the existence of the SRD of rectangular, for each hermitian (2.4)
Lemma 2.1, we give out the definition of the simultaneous real diagonalization (SRD) of two quaternionic matrices:

Theorem 2.3, we can have the following corollary:

Corollary 2.5. Let \( \mathcal{G} = \{A_i : 1 \leq i \leq s < \infty\} \subset \mathbb{Q}^{n \times n} \). Then there is \( U \in U_m(Q) \) and \( V \in U_n(Q) \) such that every \( A_i = U \Sigma_i V^* \) where \( \Sigma_i \in \mathbb{Q}^{n \times n} \) is diagonal if and only if \( A_i^T A_j^* \) and \( A_j^T A_i^* \) are both hermitian for \( 1 \leq i, j \leq s \).

Since Feng [9] studied the column left rank of matrix over quaternion which is useful in studying quaternionic matrix theory, we can have the following corollary:
Corollary 2.6. Let $A, B \in SC_n(Q)$. Then the following statements are equivalent:

(i) $(A, B)$ has SRD;

(ii) $AB = BA$;

(iii) There exist $C \in SC_n(Q)$ and real coefficient polynomials $P_1(x)$ and $P_2(x)$ such that $A = P_1(C)$ and $B = P_2(C)$.

Proof. (i) $\Rightarrow$ (ii) Since $A, B \in SC_n(Q)$,

$$AB = AB^* = (AB^*)^* = BA^* = BA.$$  

(ii) $\Rightarrow$ (i) Since $A, B \in SC_n(Q)$ and $AB = BA$, there is

$$(AB^*)^* = BA^* = BA = AB = AB^*,$$

which means $AB^*$ is hermitian. Similarly, $B^*A$ is hermitian. Then by Theorem 2.3, (i) is true.

(ii) $\Leftrightarrow$ (iii) see from [9].

Corollary 2.7. Let $\mathcal{Q} = \{A_i : 1 \leq i \leq s\} \subset SC_n(Q)$. Then there is $U \in U_n(Q)$ such that every $A_i = U \Sigma_i U^*$ where $\Sigma_i \in \mathbb{R}^{n \times n}$ is diagonal if and only if $A_i A_j = A_j A_i$ $(1 \leq i, j \leq s)$.

Remarks: We can also consider the generalized case that $\Sigma_A, \Sigma_B$ are complex instead of real, respectively. There are also some useful results for this case.

Since $A$ has SVD as Eq. (2.1), we can rewrite

$$A = U \Sigma_A V^* = U \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix} V^*,$$  

and the conjugate transpose and the Moore–Penrose inverse of $A$ can be presented as follows respectively:

$$A^* = V \Sigma_A^T U^* = V \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix} U^*,$$  

and

$$A^+ = V \Sigma_A^+ U^* = V \begin{bmatrix} D_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$  

It’s easy to verify that many classical conclusions of the Moore–Penrose inverses over the complex field still hold for the quaternion skew-field, such as $(A^+)^* = (A^*)^+$. 

Theorem 2.8. $A, B \in Q^{n \times n}$. The following statements are equivalent:

(i) there exist unitary matrices $U \in U_n(Q)$ and $V \in U_n(Q)$ such that $\Sigma_A = UAV^*$ and $\Sigma_B = UBV^*$ are both real diagonal matrices, and $\Sigma_A$ has all elements nonnegative;

(ii) $A^+ B$ and $B A^+$ are both hermitian.

Proof. (i) $\Rightarrow$ (ii) Since $\Sigma_A = UAV^*$ and $\Sigma_B = UBV^*$ are both real diagonal matrices, and $\Sigma_A$ has all elements nonnegative, then without loss of generality, we can rewrite $A$ as Eq. (2.10). Hence $A^+$ has form of Eq. (2.12)

$$A^+ = (U^* \Sigma_A V^*)^+ = V^* \Sigma_A^+ U.$$  

Then

$$(BA^+)^* = (A^+)^* B^* = (V^* \Sigma_A^+ U)^*(U^* \Sigma_B V)^*$$  

$$= U^* \Sigma_B^T VV^* \Sigma_B U = U^* \Sigma_A^+ \Sigma_B U$$  

$$= U^* \Sigma_A^+ U = U^* \Sigma_B VV^* \Sigma_A^+ U$$  

$$= BA^+.$$  

Similarly, $A^+ B$ is also hermitian.

(ii) $\Rightarrow$ (i) By Lemma 2.1, $A$ has form of Eq. (2.10)

$$A = U \Sigma_A V^* = \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix}$$

with $r = \text{rank} A$ and $D_A \in \mathbb{R}^{r \times r}$ has the form of Eq. (2.4). Assume

$$B = U^* BV = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$  

where $B_{11} \in Q^{r \times r}$. 

Since $A^+B$ is hermitian, 
$$A^+B = (A^+B)^* = B^* (A^+)^*,$$
we have
$$V^*A^+BV = V^*A^+UU^*BV = \Sigma^*_A \tilde{B} = \begin{bmatrix} D_{A}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} D_{A}^{-1}B_{11} & 0 \\ 0 & 0 \end{bmatrix},$$
and
$$V^*B^*(A^+)^*V = V^*B^*UU^*(A^+)^*V = \tilde{B}^*(\Sigma^*_A)^* = \begin{bmatrix} B_{11}^* & B_{12}^* \\ B_{21}^* & B_{22}^* \end{bmatrix} \begin{bmatrix} D_{A}^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_{11}^*D_{A}^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$
which means that $B_{12} = 0$ and $B_{11}^*D_{A}^{-1} = D_{A}^{-1}B_{11}$. Similarly, since $BA^+$ is hermitian, we can get $B_{21} = 0$ and $B_{11}D_{A}^{-1} = D_{A}^{-1}B_{11}$. By Lemma 2.4, $B_{11}$ is hermitian. Then $B$ is a partitioned diagonal matrix
$$B = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}$$
with $B_{11}$ hermitian. The next proof is similar to the proof of Theorem 2.3. The proof is completed. ■

Similarly, we have the following Corollary as generalization:

**Corollary 2.9.** Let $\mathfrak{N} = \{A_i: 1 \leq i \leq s < \infty\} \subset Q^{m \times n}$. Then there is $U \in U_m(\mathbb{Q})$ and $V \in U_n(\mathbb{Q})$ such that every $A_i = U\Sigma_i V^*$ where $\Sigma_i \in \mathbb{R}^{m \times n}$ is diagonal if and only if $A_i A_j^*$ and $A_i^* A_j$ are both hermitian for $1 \leq i, j \leq s$.

### 3. Algorithm for SRD

We give here a complete algorithm for computing the SRD of a given quaternionic matrix pair $(A, B)$ based on the proof of Theorem 2.3.

**Algorithm 1 (SRD of a Quaternionic Matrix Pair).**
Step 1: Input $m \times n$ quaternionic matrices $A$ and $B$;

Step 2: Calculate $A^+, B^+, AB^+, BA^+$. Compare $AB^+$ with $BA^+$, and $BA^+$ with $A+B$. If $AB^+ = BA^+$ and $BA^+ = A^+B$ both come into existence, then goto step 3; else, stop and output: {A, B DO NOT HAVE SRD;}

Step 3: Compute the SVD of $A$ using the algorithm of quaternion singular value decomposition [6], to obtain $U_A \in U_m(\mathbb{Q})$, $\Sigma_A \in \mathbb{R}^{m \times n}$ and $V_A \in U_n(\mathbb{Q})$, such that $U_A \Sigma_A V_A^* = A$ where $\Sigma_A = D_A \oplus 0$, $D_A =$ diag$\{\sigma_1, \ldots, \sigma_r\} \in \mathbb{R}^{r \times r}$, $r \equiv \text{rank } A$;

Step 4: Since the singular values of $A$, $\sigma_1, \ldots, \sigma_r$, which are got in step 3 may not satisfy the condition that $\sigma_1 \geq \cdots \geq \sigma_r > 0$, find the permutation matrix $P \in \mathbb{R}^{r \times r}$ such that $P^T D_A P = \text{diag}\{\sigma_1^{(1)}, \ldots, \sigma_i^{(1)}, \ldots, \sigma_i^{(1)}, \ldots, \sigma_r^{(1)}\}$ where the set

$$(\sigma_1^{(1)}, \ldots, \sigma_i^{(1)}) = (\sigma_1, \ldots, \sigma_r), \sigma_i^{(1)} \geq \cdots \geq \sigma_i^{(1)} > 0, \text{ and } \sum_{i=1}^r r_i = r.$$

Then $\Sigma_A$ can be partitioned to a block diagonal matrix as $\Sigma_A = \sigma_1 l_{i1} \oplus \cdots \oplus \sigma_i l_{ii} \oplus 0$, where $l_{ii}$ is the $r_i \times r_i$ unit matrix and $0$ is the $(m-r) \times (n-r)$ zero matrix;

Step 5: Compute $C = (P^T \oplus I_{m-r}) U_A^* V_A$, $V = V_A (P \oplus I_{n-r}) (U_1 \oplus \cdots \oplus U_{l-1})$, and $\Sigma = A_1 \oplus \cdots \oplus A_i \oplus A_i^{*}$ where for $1 \leq i \leq l_i$, $C_i \in \mathbb{C}_{r_i} (\mathbb{Q})$, and $C_{l+1} \in \mathbb{Q}^{(m-r)(n-r)}$;

Step 6: For $1 \leq i \leq l_i$, compute the Schur decomposition of $C_i$ using the quaternion QR algorithm [7], to obtain $U_i \in U_{r_i}(\mathbb{Q})$ and $A_i \in \mathbb{R}^{r \times n}$, such that $C_i = U_i A_i U_i^*$;

Step 7: Compute the SVD of $C_{l+1}$ using the algorithm of quaternion singular value decomposition [6], to obtain $U_{l+1}, \Sigma_{l+1}$ and $V_{l+1}$, such that $C_{l+1} = U_{l+1} \Sigma_{l+1} V_{l+1}^*$;

Step 8: Let $U = U_A (P \oplus I_{m-r}) (U_1 \oplus \cdots \oplus U_{l-1}), V = V_A (P \oplus I_{n-r}) (U_1 \oplus \cdots \oplus U_l \oplus V_{l+1})$, and $\Sigma = A_1 \oplus \cdots \oplus A_l \oplus \Sigma_{l+1}$;

Step 9: Output $U, V, \Sigma_A$ and $\Sigma_B$.

### 4. Conclusion

Because of the noncommutativity of quaternions, the study of quaternionic matrices is more difficult than the study of matrices over the real or complex field. A lot of classical conclusions in real or complex matrix theory may not hold for matrices over the quaternion skew-field any longer, such as $(AB)^T \neq B^TA^T$ in general and $AB \neq BA$ in general for two quaternionic matrices $A$ and $B$. But there are still many authors who want to do the possible extensions, such as Janovská et al. [10], who gave an extension of Givens' transformation from real matrices to quaternionic matrices, and Bunse-Gerstner et al., who gave a quaternionic QR algorithm. The SVD over the real or complex fields is considered to be a very useful and versatile tool in a lot of areas [11,12]. That is why we decided to extend the conclusions about the SVD of real or complex matrices to the quaternionic case.
There are few articles about the generalized singular value decomposition (GSVD) of two quaternionic matrices. In this paper, we are interested in the problem of the simultaneous real diagonalization of two quaternionic matrices of the form of SVD, which can be considered as a special case of GSVD. Some necessary and sufficient conditions have been derived which can be used to judge whether two given quaternionic matrices have SRD. Finally, a complete algorithm for computing the SRD of a rectangular quaternionic matrix pair has been provided. The conclusions obtained in this paper are expected to have more applications in the near future. In fact, we have made certain progress in obtaining the solutions and least squares solutions of some special quaternionic matrix equations by using SRD of quaternionic matrix pairs.

References