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Positive Solutions of Two-Point Right Focal Boundary Value Problems on Time Scales

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Abstract—We consider the following boundary value problem,

$$\begin{aligned} (-1)^{n-1} y^{\Delta^n}(t) &= (-1)^{p+1} F\left(t, y\left(\sigma^{n-1}(t)\right)\right), && t \in [a,b] \cap \mathbf{T}, \\ y^{\Delta^i}(a) &= 0, && 0 \le i \le p-1, \\ y^{\Delta^i}(\sigma(b)) &= 0, && p \le i \le n-1, \end{aligned}$$

where $n \ge 2$, $1 \le p \le n-1$ is fixed and $\mathbf{T}$ is a time scale. Criteria for the existence of single, double, and multiple positive solutions of the boundary value problem are developed. Upper and lower bounds for these positive solutions are established for two special cases that arise from some physical phenomena. We also include several examples to illustrate the usefulness of the results obtained.

Keywords—Positive solutions, Boundary value problems, Two-point right focal boundary conditions, Time scales.

1. INTRODUCTION

In this paper, we present results governing the existence of positive solutions to the differential equation on time scales of the form,

$$(-1)^{n-1} y^{\Delta^n}(t) = (-1)^{p+1} F\left(t, y\left(\sigma^{n-1}(t)\right)\right), \qquad t \in [a,b], \tag{1.1}$$

subject to the two-point right focal boundary conditions,

$$\begin{aligned} y^{\Delta^i}(a) &= 0, \qquad 0 \le i \le p-1, \\ y^{\Delta^i}(\sigma(b)) &= 0, \qquad p \le i \le n-1, \end{aligned} \tag{1.2}$$

where p, n are fixed integers satisfying $n \ge 2$, $1 \le p \le n-1$, $a, b \in \mathbf{T}$ with $a < \sigma(b)$, and $\rho(\sigma(b)) = b$.

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To understand the notations used in (1.1), we recall some standard definitions as follows. The reader may refer to [1] for an introduction to the subject.

(a) Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a closed subset of $\mathbb{R}$. We assume that $\mathbf{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. Throughout, for any c, d $(> c)$, the interval $[c,d]$ is defined as $[c,d] = \{t \in \mathbf{T} \mid c \leq t \leq d\}$. We also use the notation $\mathbb{R}[c,d]$ to denote the real interval $\{t \in \mathbb{R} \mid c \leq t \leq d\}$ and $\mathbb{Z}[c,d]$ to represent the set $\{t \in \mathbb{Z} \mid c \leq t \leq d\}$. Analogous notations for open and half-open intervals will also be used.

(b) For $t < \sup \mathbf{T}$ and $s > \inf \mathbf{T}$, the *forward jump operator* σ and the *backward jump operator* ρ respectively are defined by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T} \qquad \text{and} \qquad \rho(s) = \sup\{\tau \in \mathbf{T} \mid \tau < s\} \in \mathbf{T}.$$

We define $\sigma^n(t) = \sigma(\sigma^{n-1}(t))$ with $\sigma^0(t) = t$. Similar definition is used for $\rho^n(s)$.

(c) Fix $t \in \mathbf{T}$. Let $y : \mathbf{T} \to \mathbb{R}$. We define $y^{\Delta}(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$, there is a neighbourhood U of t such that for all $s \in U$,

$$\left|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]\right| < \epsilon\,|\sigma(t) - s|.$$

We call $y^{\Delta}(t)$ the *delta derivative* of $y(t)$. Define $y^{\Delta^n}(t)$ to be the delta derivative of $y^{\Delta^{n-1}}(t)$, i.e., $y^{\Delta^n}(t) = (y^{\Delta^{n-1}}(t))^{\Delta}$.

(d) If $F^{\Delta}(t) = f(t)$, then we define the integral,

$$\int_a^t f(\tau)\,\Delta\tau = F(t) - F(a).$$

A solution of (1.1), (1.2) will be sought in $C[a, \sigma^n(b)]$, the space of continuous functions $\{y : [a, \sigma^n(b)] \to \mathbb{R}\}$. We say that y is a *positive solution* if $y(t) \geq 0$, for $t \in [a, \sigma^n(b)]$. By utilizing some fixed point theorems, we shall develop criteria for the existence of single, double, and multiple positive solutions of (1.1), (1.2). In addition, we shall consider the following special cases of (1.1),(1.2) when $n = 2$ and $p = 1$:

$$\begin{gathered} y^{\Delta^2}(t) + h(t)\left([y(\sigma(t))]^{\alpha} + [y(\sigma(t))]^{\beta}\right) = 0, \qquad t \in [a,b], \\ y(a) = y^{\Delta}(\sigma(b)) = 0, \end{gathered} \tag{Q1}$$

and

$$\begin{gathered} y^{\Delta^2}(t) + h(t)\, e^{\zeta[y(\sigma(t))]} = 0, \qquad t \in [a,b], \\ y(a) = y^{\Delta}(\sigma(b)) = 0. \end{gathered} \tag{Q2}$$

It is assumed that $0 \leq \alpha < 1 < \beta$, $\zeta > 0$ and h is nonnegative. We shall provide conditions under which (Q1) and (Q2) have double positive solutions, and also establish upper and lower bounds for these solutions. The importance of (Q1) is illustrated in [2,3] where particular cases in the real and discrete domains are discussed. Boundary value problem (Q2) arises in applications involving the diffusion of heat generated by positive temperature-dependent sources [4]. For instance, when $\zeta = 1$ the boundary value problem (Q2) occurs in the analysis of Joule losses in electrically conducting solids as well as in frictional heating.

Boundary value problems have attracted a lot of attention in the recent literature, due mainly to the fact that they model many physical phenomena which, besides (Q1) and (Q2), include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, just to name a few. In all these problems, only *positive* solutions are meaningful.

Many papers have discussed the existence of positive solutions of boundary value problems on the real and discrete domains, we refer to [5–9] and the monographs [10,11] which give a good documentary of the literature. A recent trend is to consider boundary value problems on time scales, which include the real and the discrete as special cases, see [12–17]. Our approach in the present work not only unifies the analysis for the real and the discrete cases in [18,19], but also leads to *new* results which, when reduced to $\mathbb{R}$ and $\mathbb{Z}$, are also *new* in the literature.

The outline of the paper is as follows. In Section 2, we shall state two fixed-point theorems, namely a nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorem, and also present some properties of certain Green's function which are needed later. Existence criteria for single, double, and multiple positive solutions of (1.1),(1.2) are developed in Section 3. As an application of the results obtained, in Section 4, we establish the existence of double positive solutions of (Q1) and (Q2) as well as upper and lower bounds for these solutions. Throughout, examples are included to illustrate the importance of the results obtained.

2. PRELIMINARIES

We shall first state two fixed-point theorems. The first theorem is the Leray-Schauder alternative [20] while the second is due to Krasnosel'skii [21].

THEOREM 2.1. *(See [20].) Let B be a Banach space with $E \subseteq B$ closed and convex. Assume U is a relatively open subset of E with $0 \in U$ and $S : \bar{U} \to E$ is a continuous and compact map. Then, either*

(a) *S has a fixed point in $\bar{U}$, or*
(b) *there exists $y \in \partial U$ and $\lambda \in \mathbb{R}$ $(0,1)$, such that $y = \lambda S y$.* ∎

THEOREM 2.2. *(See [21].) Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C(\subset B)$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of B with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\bar{\Omega}_2 \setminus \Omega_1) \to C$$

be a continuous and completely continuous operator such that, either

(a) *$\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_1$, and $\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_2$, or*
(b) *$\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_1$, and $\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_2$.* ∎

Then, S has a fixed point in $C \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

The Green's function related to (1.1),(1.2) plays a central role in the development of our results. We shall first state some definitions and notations, followed by the explicit expression of the Green's function, as well as some related inequalities.

DEFINITION 2.1.

(a) *Define the functions $h_k : \mathbf{T} \times \mathbf{T} \to \mathbb{R}$, $k \in \{0, 1, \dots\}$, recursively as*

$$h_0(t,s) = 1, \qquad \text{for all } s, t \in \mathbf{T},$$

and

$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\,\Delta\tau, \qquad \text{for all } s,t \in \mathbf{T}, \quad k = 0, 1, \dots.$$

(b) *Let t_i, $1 \le i \le n$ be such that*

$$a = t_1 = \cdots = t_p < t_{p+1} = \cdots = t_n = \sigma(b).$$

(c) *Define $T_i : [a,b] \to \mathbb{R}$, $0 \le i \le n-1$ as*

$$T_0(t) \equiv 1$$

and

$$T_i(t) = T_i(t : t_1, \dots, t_i) = \int_{t_1}^{t} \int_{t_2}^{\tau_1} \cdots \int_{t_i}^{\tau_{i-1}} \Delta\tau_i \dots \Delta\tau_2 \Delta\tau_1, \qquad 1 \le i \le n-1.$$

To obtain a solution for (1.1),(1.2), we require a mapping whose kernel $G(t,s)$ is the Green's function of boundary value problem (1.2),

$$(-1)^{n-1} y^{\Delta^n}(t) = 0, \qquad t \in [a,b]. \tag{2.1}$$

THEOREM 2.3. *(See [22].) The Green's function $G(t,s)$ of boundary value problem (2.1),(1.2) can be expressed as*

$$G(t,s) = \begin{cases} (-1)^{n-1} \sum_{i=0}^{p-1} T_i(t)\, h_{n-1-i}(a,\sigma(s)) + (-1)^n h_{n-1}(t,\sigma(s)), & t \le \sigma(s), \\ (-1)^{n-1} \sum_{i=0}^{p-1} T_i(t)\, h_{n-1-i}(a,\sigma(s)), & t \ge \sigma(s), \end{cases}$$

where $t \in [a,\sigma^n(b)]$ and $s \in [a,b]$. ∎

LEMMA 2.4. *(See [24].) For $(t,s) \in [a,\sigma^n(b)] \times [a,b]$,*

$$0 \le (-1)^{p+1} G(t,s) \le (-1)^{p+1} G(\sigma^n(b), s).$$ ∎

REMARK 2.1. In [22], it is noted that $(-1)^{p+1}G(t,s)$ is a nondecreasing function in $t \in [a,\sigma^n(b)]$.

Throughout this paper, for a fixed number $\delta \in \mathbb{R}\ (0,1/2)$, we let

$$\begin{aligned} &c = \min\{t \in \mathbf{T} \mid t \ge a + \delta(\sigma^n(b) - a)\}, \\ &d = \max\{t \in \mathbf{T} \mid t \le \sigma^n(b) - \delta(\sigma^n(b) - a)\}, \\ &\text{and assume the existence of } c \text{ and } d, \text{ such that } a < c < \rho^{n-1}(d) < \sigma(b). \end{aligned} \tag{2.2}$$

LEMMA 2.5. *(See [22].) For $(t,s) \in [c,d] \times [a,b]$,*

$$(-1)^{p+1} G(t,s) \ge k(-1)^{p+1} G(\sigma^n(b), s),$$

where $0 < k < 1$ is a constant given by

$$k = \inf_{s\in[a,b]} \frac{G(c,s)}{G(\sigma^n(b), s)}.$$ ∎

3. EXISTENCE RESULTS FOR (1.1),(1.2)

In this section, we let the Banach space $B = C[a,\sigma^n(b)]$ be equipped with the norm,

$$\|y\| = \sup_{t\in[a,\sigma^n(b)]} |y(t)|.$$

Let the operator $S : B \to B$ be defined by

$$Sy(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) F\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s, \qquad t \in [a,\sigma^n(b)]. \tag{3.1}$$

It is clear that a fixed point of the operator S is a solution of (1.1),(1.2).

For clarity, we shall now list some of the conditions used later. In these conditions, we let

$$\tilde{K} = \{y \in B \mid y(t) \ge 0,\ t \in [a,\sigma^n(b)]\}$$

and

$$K = \left\{y \in \tilde{K} \mid y(t) > 0,\ \text{for some } t \in [a,\sigma^n(b)]\right\} = \tilde{K} \setminus \{0\}.$$

(B1) The function F is continuous on $[a, \sigma(b)] \times \tilde{K}$ with

$$F(t,y) \geq 0, \qquad (t,y) \in [a, \sigma(b)] \times \tilde{K},$$

and

$$F(t,y) > 0, \qquad (t,y) \in [a, \sigma(b)] \times K.$$

(B2) There exist continuous functions w, q with $w : \mathbb{R}\,[0,\infty) \to \mathbb{R}\,[0,\infty)$ nondecreasing and $q : [a, \sigma(b)] \to \mathbb{R}\,[0,\infty)$, such that for $(t,y) \in [a, \sigma(b)] \times \tilde{K}$,

$$F(t,y) \leq q(t)\, w(y).$$

(B3) There exists $\tau : [c, \rho^{n-1}(d)] \to \mathbb{R}^+$ such that for $(t,y) \in [c, \rho^{n-1}(d)] \times K$,

$$F(t,y) \geq \tau(t)\, w(y).$$

(B4) There exist continuous functions f, u, v with $f : \mathbb{R}\,[0,\infty) \to \mathbb{R}\,[0,\infty)$ and $u, v : [a, \sigma(b)] \to \mathbb{R}\,[0,\infty)$ such that for $(t,y) \in [a, \sigma(b)] \times \tilde{K}$,

$$u(t)\, f\left(y\left(\sigma^{n-1}(t)\right)\right) \leq F\left(t, y\left(\sigma^{n-1}(t)\right)\right) \leq v(t)\, f\left(y\left(\sigma^{n-1}(t)\right)\right).$$

(B5) $u(t)$ is nonzero for some $t \in [c, \rho^{n-1}(d))$, and there exists a number $\eta \in \mathbb{R}\,(0,1]$ such that $u(t) \geq \eta v(t)$, for $t \in [a, \sigma(b)]$.

(B6) $\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s < \infty$.

LEMMA 3.1. *(See [22].) Let $F : [a, \sigma(b)] \times \mathbb{R} \to \mathbb{R}$ be continuous. Then, the operator $S : B \to B$ is continuous and completely continuous.*

We shall now provide an existence criteria for a *general* (not necessarily positive) solution of (1.1),(1.2).

THEOREM 3.2. *Let $F : [a, \sigma(b)] \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose there exists a constant r, independent of λ, such that*

$$\|y\| \neq r,$$

for any solution $y \in B$ of the equation,

$$y(t) = \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t,s)\, F\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s, \qquad t \in [a, \sigma^n(b)], \tag{3.2}$$

where $\lambda \in \mathbb{R}\,(0,1)$. Then (1.1),(1.2) has at least one solution $y \in B$ such that $\|y\| \leq r$.

PROOF. Solving (3.2) is equivalent to obtaining a fixed point of the equation,

$$y = \lambda S y,$$

where S is defined in (3.1). By Lemma 3.1, S is continuous and completely continuous. Next, in the context of Theorem 2.1, we define

$$U = \{ y \in B \mid \|y\| < r \}.$$

The condition $\|y\| \neq r$ ensures that we cannot have Conclusion (b) of Theorem 2.1, hence, Conclusion (a) must hold, i.e., (1.1),(1.2) has a solution $y \in \bar{U}$ with $\|y\| \leq r$. ∎

The next result employs Theorem 3.2 to provide the existence of a *positive* solution.

THEOREM 3.3. *Let (B1) and (B2) hold. Suppose*

$$\text{there exists } \alpha > 0, \text{ such that } \alpha > \gamma \cdot w(\alpha), \quad \text{where } \gamma = \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) q(s) \Delta s. \tag{3.3}$$

Then, (1.1),(1.2) has a positive solution $y \in B$, *such that* $\|y\| < \alpha$, *i.e.,* $0 \le y(t) < \alpha$, *for* $t \in [a, \sigma^n(b)]$.

PROOF. Consider the equation,

$$y(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) \hat{F}\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s, \ t \in [a, \sigma^n(b)], \tag{3.4}$$

where $\hat{F} : [a, \sigma(b)] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\hat{F}(t,y) = F(t, |y|). \tag{3.5}$$

Since $|y| \in \tilde{K}$, by (B1) the function $\hat{F}$ is well-defined and is continuous.

We shall prove that (3.4) has a solution. For this, we consider the equation,

$$y(t) = \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) \hat{F}\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s, \ t \in [a, \sigma^n(b)], \tag{3.6}$$

where $\lambda \in \mathbb{R}$ $(0,1)$. Let $y \in B$ be any solution of (3.6). We shall show that $\|y\| \neq \alpha$, then it will follow from Theorem 3.2 that (3.4) has a solution.

By using (3.5), Lemma 2.4, and (B1), we get for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} y(t) &= \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) \hat{F}\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s \\ &= \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) F(s, \left|y\left(\sigma^{n-1}(s)\right)\right|) \Delta s \ge 0. \end{aligned}$$

This means that

$$|y(t)| = y(t), \qquad t \in [a, \sigma^n(b)]. \tag{3.7}$$

Applying (3.7), (B2), and Lemma 2.4, for $t \in [a, \sigma^n(b)]$, we see that

$$\begin{aligned} |y(t)| &= y(t) \\ &\le \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) F\left(s, \left|y\left(\sigma^{n-1}(s)\right)\right|\right) \Delta s \\ &\le \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) q(s) w\left(\left|y\left(\sigma^{n-1}(s)\right)\right|\right) \Delta s \\ &\le \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) q(s) w(\|y\|) \Delta s \\ &= \gamma \cdot w(\|y\|), \end{aligned}$$

which immediately leads to

$$\|y\| \le \gamma \cdot w(\|y\|). \tag{3.8}$$

Comparing (3.8) and (3.3), we conclude that $\|y\| \neq \alpha$.

It now follows from Theorem 3.2 that (3.4) has a solution $y_0 \in B$ with $\|y_0\| \le \alpha$, and

$$y_0(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) \hat{F}\left(s, y_0\left(\sigma^{n-1}(s)\right)\right) \Delta s, \qquad t \in [a, \sigma^n(b)].$$

Using a similar argument as above, it can be seen that

$$|y_0(t)| = y_0(t), \qquad t \in [a, \sigma^n(b)], \qquad \text{and} \qquad \|y_0\| \neq \alpha. \tag{3.9}$$

Thus, y_0 is positive and $\|y_0\| < \alpha$. Further, noting (3.5) and (3.9), we have for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} y_0(t) &= \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) F\left(s, \left|y_0\left(\sigma^{n-1}(s)\right)\right|\right) \Delta s \\ &= \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) F\left(s, y_0\left(\sigma^{n-1}(s)\right)\right) \Delta s. \end{aligned}$$

Hence, y_0 is in fact a solution of (1.1),(1.2). The proof is complete. ∎

REMARK 3.1. We note that the last inequality in (B1), *viz.*,

$$F(t,y) > 0, \qquad (t,y) \in [a, \sigma(b)] \times K,$$

is *not* needed in Theorem 3.3.

Theorem 3.3 provides the existence of a positive solution, which may be trivial. Our next result guarantees the existence of a *nontrivial* positive solution in the cone C_k, defined, for a fixed $\delta \in \mathbb{R}(0, 1/2)$, as

$$C_k = \left\{ y \in B \mid y(t) \geq 0,\ t \in [a, \sigma^n(b)];\ \min_{t \in [c,d]} y(t) \geq k\|y\| \right\}, \tag{3.10}$$

where k is given in Lemma 2.5.

THEOREM 3.4. *Let $\delta \in \mathbb{R}(0, 1/2)$ be fixed and assume (B1)–(B3) and (3.3) hold. Suppose*

$$\begin{gathered} \text{there exists } \beta > 0 \text{ such that for } x \in \mathbb{R}[k\beta, \beta], \text{ we have} \\ x \leq w(x) \int_c^{\rho^{n-1}(d)} k(-1)^{p+1} G(\sigma^n(b), s)\, \tau(s)\, \Delta s. \end{gathered} \tag{3.11}$$

Then, (1.1),(1.2) has a positive solution $y \in B$ such that

(a) $\alpha < \|y\| \leq \beta$ *and* $\min_{t\in[c,d]} y(t) > k\alpha$ *if* $\alpha < \beta$;
(b) $\beta \leq \|y\| < \alpha$ *and* $\min_{t\in[c,d]} y(t) \geq k\beta$ *if* $\alpha > \beta$.

PROOF. We shall employ Theorem 2.2. To begin, the operator $S : B \to B$ is continuous and completely continuous by Lemma 3.1.

Next, we shall show that S maps the cone C_k into C_k. For this, we let $y \in C_k$. Since $C_k \subseteq \tilde{K}$, it follows from (B1) that

$$F(t,y) \geq 0, \qquad (t,y) \in [a, \sigma(b)] \times C_k. \tag{3.12}$$

Noting (3.12), we obtain for $t \in [a, \sigma^n(b)]$,

$$Sy(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t,s) F\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s \geq 0. \tag{3.13}$$

In view of (3.13) and Lemma 2.4, we find

$$|Sy(t)| = Sy(t) \leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) F\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s, \qquad t \in [a, \sigma^n(b)],$$

which implies

$$\|Sy\| \leq \int_a^{\sigma(b)} (-1)^{p+1} G\left(\sigma^n(b), s\right) F\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s. \tag{3.14}$$

Now, using Lemma 2.5 and (3.14), we get for $t \in [c, d]$,

$$Sy(t) \geq \int_a^{\sigma(b)} k(-1)^{p+1} G\left(\sigma^n(b), s\right) F\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s \geq k\|Sy\|.$$

Hence,

$$\min_{t\in[c,d]} Sy(t) \geq k\|Sy\|. \tag{3.15}$$

Finally, we combine (3.13) and (3.15) to obtain $S(C_k) \subseteq C_k$.

To proceed to the next part of the proof, let

$$\Omega_\alpha = \{y \in B \mid \|y\| < \alpha\} \qquad \text{and} \qquad \Omega_\beta = \{y \in B \mid \|y\| < \beta\}.$$

We shall verify that

(i) $\|Sy\| \leq \|y\|$ for $y \in C_k \cap \partial\Omega_\alpha$, and
(ii) $\|Sy\| \geq \|y\|$ for $y \in C_k \cap \partial\Omega_\beta$.

To show (i), let $y \in C_k \cap \partial\Omega_\alpha$. Then, $\|y\| = \alpha$. Using (3.13), (B2), Lemma 2.4, and (3.3), we get for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned}|Sy(t)| &= Sy(t)\\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(t, s)\, q(s)\, w\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s\\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G\left(\sigma^n(b), s\right) q(s)\, w(\alpha)\, \Delta s = \gamma \cdot w(\alpha) < \alpha = \|y\|,\end{aligned}$$

and so

$$\|Sy\| \leq \|y\|.$$

Next, we shall prove (ii). Let $y \in C_k \cap \partial\Omega_\beta$. Then, $\|y\| = \beta$. Moreover, $k\beta \leq y(t) \leq \beta$, for $t \in [c, d]$, and this implies

$$k\beta \leq y\left(\sigma^{n-1}(t)\right) \leq \beta, \qquad t \in \left[c, \rho^{n-1}(d)\right].$$

Applying (B3) and (3.11), we find

$$\begin{aligned}|Sy(\sigma^n(b))| &= Sy(\sigma^n(b))\\ &= \int_a^{\sigma(b)} (-1)^{p+1} G\left(\sigma^n(b), s\right) F\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s\\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G\left(\sigma^n(b), s\right) F\left(s, y\left(\sigma^{n-1}(s)\right)\right) \Delta s\\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G\left(\sigma^n(b), s\right) \tau(s)\, w\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s\\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G\left(\sigma^n(b), s\right) \tau(s) \left(\frac{y\left(\sigma^{n-1}(s)\right)}{k\int_c^{\rho^{n-1}(d)} (-1)^{p+1} G\left(\sigma^n(b), x\right) \tau(x)\, \Delta x}\right) \Delta s\\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G\left(\sigma^n(b), s\right) \tau(s) \left(\frac{k\beta}{k\int_c^{\rho^{n-1}(d)} (-1)^{p+1} G\left(\sigma^n(b), x\right) \tau(x)\, \Delta x}\right) \Delta s\\ &= \beta = \|y\|,\end{aligned}$$

which implies

$$\|Sy\| \geq \|y\|.$$

Having established (i) and (ii), it follows from Theorem 2.2 that S has a fixed point $y \in C \cap (\overline{\Omega}_{\max\{\alpha,\beta\}} \backslash \Omega_{\min\{\alpha,\beta\}})$. Thus, $\min\{\alpha,\beta\} \leq \|y\| \leq \max\{\alpha,\beta\}$. Using a similar argument as in the first part of the proof of Theorem 3.3, we see that $\|y\| \neq \alpha$. Hence, we obtain the first part of Conclusions (a) and (b). Further, since $y \in C$, we have

$$\min_{t\in[c,d]} y(t) \geq k\|y\| \geq \min\{\alpha,\beta\},$$

which, together with $\|y\| \neq \alpha$, gives the second part of Conclusions (a) and (b). ∎

Our next result gives the existence of *double* positive solutions.

THEOREM 3.5. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed and assume (B1)–(B3), (3.3), and (3.11) hold with $\alpha < \beta$. Then, (1.1),(1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that*

$$0 \leq \|y_1\| < \alpha < \|y_2\| \leq \beta, \qquad \text{with } \min_{t\in[c,d]} y_2(t) > k\alpha.$$

PROOF. The existence of y_1 and y_2 is guaranteed by Theorems 3.3 and 3.4, respectively. ∎

In Theorem 3.5, it is possible to have $\|y_1\| = 0$. Our next result guarantees the existence of *two nontrivial* positive solutions.

THEOREM 3.6. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed and assume (B1)–(B3), (3.3), (3.11), and $(3.11)|_{\beta=\tilde{\beta}}$ hold, where $0 < \tilde{\beta} < \alpha < \beta$. Then (1.1),(1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that*

$$0 < \tilde{\beta} \leq \|y_1\| < \alpha < \|y_2\| \leq \beta, \qquad \min_{t\in[c,d]} y_1(t) \geq k\tilde{\beta}, \qquad \min_{t\in[c,d]} y_2(t) > k\alpha.$$

PROOF. The existence of y_1 and y_2 is guaranteed from Theorem 3.4(b) and 3.4(a), respectively. ∎

The next result generalizes Theorems 3.5 and 3.6 and gives the existence of *multiple* positive solutions of (1.1),(1.2).

THEOREM 3.7. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed and assume (B1)–(B3) hold. Let (3.3) be satisfied with $\alpha = \alpha_l$, $l = 1, 2, \ldots, r$, and (3.11) be satisfied with $\beta = \beta_l$, $l = 1, 2, \ldots, m$.*

(a) *If $m = r + 1$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_r < \alpha_r < \beta_{r+1}$, then (1.1),(1.2) has (at least) $2r$ positive solutions $y_1, \ldots, y_{2r} \in B$ such that*

$$0 < \beta_1 \leq \|y_1\| < \alpha_1 < \cdots < \beta_r \leq \|y_{2r-1}\| < \alpha_r < \|y_{2r}\| \leq \beta_{k+1}.$$

(b) *If $m = r$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_r < \alpha_r$, then (1.1),(1.2) has (at least) $2r - 1$ positive solutions $y_1, \ldots, y_{2r-1} \in B$ such that*

$$0 < \beta_1 \leq \|y_1\| < \alpha_1 < \cdots \leq \beta_r \leq \|y_{2r-1}\| < \alpha_r.$$

(c) *If $r = m + 1$ and $0 < \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m < \alpha_{m+1}$, then (1.1),(1.2) has (at least) $2m + 1$ positive solutions $y_0, \ldots, y_{2m} \in B$ such that*

$$0 \leq \|y_0\| < \alpha_1 < \|y_1\| \leq \beta_1 \leq \cdots \leq \beta_m \leq \|y_{2m}\| < \alpha_{m+1}.$$

(d) *If $r = m$ and $0 < \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m$, then (1.1),(1.2) has (at least) $2m$ positive solutions $y_0, \ldots, y_{2m-1} \in B$ such that*

$$0 \leq \|y_0\| < \alpha_1 < \|y_1\| \leq \beta_1 \leq \cdots < \alpha_m < \|y_{2m-1}\| \leq \beta_m.$$

PROOF. Repetitive applications of Theorem 3.4 yield (a) and (b). In (c) and (d), Theorem 3.3 is used to obtain the existence of $y_0 \in B$ with $0 \leq \|y_0\| < \alpha_1$. The results then follow by repeated use of Theorem 3.4. ∎

EXAMPLE 3.1. Let $\mathbf{T} = \mathbb{R}[0,1) \cup \mathbb{Z}^+$. Consider the boundary value problem,

$$\begin{gathered} y^{\Delta^2}(t) + \frac{2e^{y(\sigma(t))}}{e^{33\sigma(t)-[\sigma(t)]^2}} = 0, \qquad t \in [0,15], \\ y(0) = y^{\Delta}(\sigma(15)) = 0. \end{gathered} \tag{3.16}$$

Here, $n = 2$, $p = 1$, $a = 0$, and $b = 15$. We shall verify that the hypotheses (B1)–(B3), (3.3), and (3.11) are satisfied. Note that (B1) obviously holds with $F(t,y) = 2e^y/e^{33\sigma(t)-[\sigma(t)]^2}$. For Conditions (B2) and (B3), we may choose

$$q(t) = \tau(t) = \frac{2}{e^{33\sigma(t)-[\sigma(t)]^2}} \qquad \text{and} \qquad w(y) = e^y.$$

Next, we compute directly to get

$$\begin{aligned} \gamma &= \int_0^{\sigma(15)} G\left(\sigma^2(15), s\right) q(s)\,\Delta s \\ &= \int_0^1 \frac{2s}{e^{33s-s^2}}\,ds + \sum_{s=1}^{15} \frac{2(s+1)}{e^{33(s+1)-(s+1)^2}} \\ &= 1.847 \times 10^{-3}. \end{aligned}$$

It follows that the inequality $\alpha > \gamma \cdot e^{\alpha}$ is true provided $1.85 \times 10^{-3} < \alpha < 8.43$. We may choose $\alpha = 8.4$ so that (3.3) is satisfied.

Using Theorem 2.3 with $n = 2$ and $p = 1$, we get

$$G(t,s) = \begin{cases} t - a, & t \leq \sigma(s), \\ \sigma(s) - a, & t \geq \sigma(s). \end{cases} \tag{3.17}$$

Let $\delta = 0.2$. From (2.2), we get $c = 4$ and $d = 13$. So, noting (3.17) and Lemma 2.5, we find

$$\begin{aligned} k &= \inf_{s\in[0,15]} \frac{G(4,s)}{G\left(\sigma^2(15), s\right)} \\ &= \min\left\{ \inf_{\sigma(s)\in[0,4)} \frac{\sigma(s)}{\sigma(s)}, \inf_{\sigma(s)\in[4,\sigma(15)]} \frac{4}{\sigma(s)} \right\} \\ &= \frac{4}{\sigma(15)} = \frac{1}{4}. \end{aligned}$$

Now, the inequality in (3.11) reduces to

$$x \leq w(x) \int_c^{\rho(d)} kG\left(\sigma^2(15), s\right) \tau(s)\,\Delta s = \frac{1}{2}e^x \int_4^{12} \frac{\sigma(s)}{e^{33\sigma(s)-[\sigma(s)]^2}}\Delta s = \left(3.95 \times 10^{-61}\right) e^x,$$

and is true if $x \leq 3.95 \times 10^{-61}$ or $x \geq 144.05$. Hence, (3.11) is fulfilled if we choose $\beta = 580$ or $\beta = 3.9 \times 10^{-61}$. It now follows from Theorem 3.6 that boundary value problem (3.16) has (at least) two positive solutions y_1 and y_2 such that

$$\begin{gathered} 3.9 \times 10^{-61} \leq \|y_1\| < 8.4 < \|y_2\| \leq 580, \\ \min_{t\in[4,13]} y_1(t) \geq 9.75 \times 10^{-62}, \qquad \min_{t\in[4,13]} y_2(t) > 2.1. \end{gathered} \tag{3.18}$$

In fact, a known positive solution is $y(t) = t(33 - t)$ with $\|y\| = 272$ and $\min_{t\in[4,13]} y(t) = 116$, both are within the ranges given in (3.18).

We have so far established results based on assumptions (B1)–(B3) on the nonlinear term F. We shall now consider Conditions (B4)–(B6) on F and develop further existence criteria.

Note that if (B4) holds, then for $(t, y) \in [a, \sigma^n(b)] \times \tilde{K}$,

$$\begin{aligned} &\int_a^{\sigma(b)} (-1)^{p+1} G(t,s)\, u(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \\ &\quad \le Sy(t) \\ &\quad \le \int_a^{\sigma(b)} (-1)^{p+1} G(t,s)\, v(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s. \end{aligned} \tag{3.19}$$

To obtain a positive solution of (1.1),(1.2), we shall seek a fixed point of the operator S in the cone C, defined, for a fixed $\delta \in \mathbb{R}(0, 1/2)$, as

$$C = \left\{ y \in B \mid y(t) \ge 0,\ t \in [a, \sigma^n(b)];\ \min_{t\in[c,d]} y(t) \ge \theta \|y\| \right\} \tag{3.20}$$

where $\theta = k\eta$, k and η are defined in Lemma 2.5 and (B5), respectively. It is clear that $0 < \theta < 1$.

LEMMA 3.8. *Let (B4) and (B5) hold. Then, the operator S maps C into C.*

PROOF. Let $y \in C$. Then, for all $t \in [a, \sigma^n(b)]$, we see from (3.19) that

$$Sy(t) \ge \int_a^{\sigma(b)} (-1)^{p+1} G(t,s)\, u(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \ge 0. \tag{3.21}$$

It also follows from (3.19) and Lemma 2.4 that for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\le \int_a^{\sigma(b)} (-1)^{p+1} G(t,s)\, v(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \\ &\le \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s. \end{aligned}$$

Hence,

$$\|Sy\| \le \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s. \tag{3.22}$$

For $t \in [c, d]$, using (3.19), Lemma 2.5, (B5), and (3.22) provides

$$\begin{aligned} Sy(t) &\ge \int_a^{\sigma(b)} k(-1)^{p+1} G(\sigma^n(b), s)\, u(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \\ &\ge \int_a^{\sigma(b)} k(-1)^{p+1} G(\sigma^n(b), s)\, \eta v(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \ge \theta \|Sy\|. \end{aligned}$$

Thus,

$$\min_{t\in[c,d]} Sy(t) \ge \theta \|Sy\|. \tag{3.23}$$

Having established (3.21) and (3.23), we have shown that $Sy \in C$. ∎

Next, we introduce the following notations which will be used in subsequent results,

$$\begin{aligned} \underline{f}_0 &= \liminf_{x\to 0} \frac{f(x)}{x}, & \overline{f}_0 &= \limsup_{x\to 0} \frac{f(x)}{x}, \\ \underline{f}_\infty &= \liminf_{x\to \infty} \frac{f(x)}{x}, & \overline{f}_\infty &= \limsup_{x\to \infty} \frac{f(x)}{x}. \end{aligned}$$

THEOREM 3.9. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed and assume (B4)–(B6) hold. Let $w > 0$ be given and suppose that f satisfies*

$$0 < f(x) \le w \left[\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, \Delta s \right]^{-1}, \qquad 0 < x \le w. \tag{3.24}$$

(a) *If $\underline{f}_0 = \infty$, then (1.1),(1.2) has a positive solution $y_1 \in B$ such that $0 < \|y_1\| \le w$.*
(b) *If $\underline{f}_\infty = \infty$, then (1.1),(1.2) has a positive solution $y_2 \in B$ such that*

$$\|y_2\| \ge w, \qquad \text{with } \min_{t \in [c,d]} y_2(t) \ge \theta w.$$

(c) *If $\underline{f}_0 = \underline{f}_\infty = \infty$, then (1.1),(1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that*

$$0 < \|y_1\| \le w \le \|y_2\|, \qquad \text{with } \min_{t \in [c,d]} y_2(t) \ge \theta w.$$

PROOF.

(a) We let

$$Q = \left[\theta \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s)\, u(s)\, \Delta s \right]^{-1}. \tag{3.25}$$

Since $\underline{f}_0 = \infty$, there exists $0 < r < w$, such that

$$f(x) \ge Qx, \qquad 0 < x \le r. \tag{3.26}$$

Let $y \in C$ be such that $\|y\| = r$. Then, applying (3.19), (3.26), and (3.25), we find

$$\begin{aligned} Sy(\sigma^n(b)) &\ge \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, u(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \\ &\ge \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s)\, u(s)\, Qy\left(\sigma^{n-1}(s)\right) \Delta s \\ &\ge Q \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s)\, u(s)\, \theta \|y\|\, \Delta s = \|y\|. \end{aligned}$$

This implies that

$$\|Sy\| \ge \|y\|. \tag{3.27}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < r\}$, then $\|Sy\| \ge \|y\|$ for $y \in C \cap \partial\Omega_1$.

Next, we let $y \in C$ be such that $\|y\| = w$. Applying (3.19), Lemma 2.4, and (3.24), we get for $t \in [a, \sigma^n(b)]$,

$$Sy(t) \le \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \le w = \|y\|.$$

Hence, we have

$$\|Sy\| \le \|y\|. \tag{3.28}$$

If we set $\Omega_2 = \{y \in B \mid \|y\| < w\}$, then $\|Sy\| \le \|y\|$ for $y \in C \cap \partial\Omega_2$.

Having obtained (3.27) and (3.28), it now follows from Theorem 2.2 that S has a fixed point $y_1 \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $r \le \|y_1\| \le w$. It is clear that y_1 is a positive solution of (1.1),(1.2).

(b) From the proof of Theorem 3.9(a), we note that the condition (3.24) gives rise to (3.28). Hence, by letting $\Omega_1 = \{y \in B \mid \|y\| < w\}$, we have $\|Sy\| \le \|y\|$, for $y \in C \cap \partial\Omega_1$.

Next, let Q be defined as in (3.25). Since $\underline{f}_\infty = \infty$, we may choose $T > w$ such that

$$f(x) \ge Qx, \qquad x \ge T. \tag{3.29}$$

Let $y \in C$ be such that $\|y\| = T/\theta$. Then, for $t \in [c,d]$, we have

$$y(t) \ge \theta \|y\| = \theta \times \frac{T}{\theta} = T.$$

So, it follows from (3.29) that

$$f(y(t)) \ge Qy(t), \qquad t \in [c,d],$$

which implies

$$f\left(y\left(\sigma^{n-1}(t)\right)\right) \ge Qy\left(\sigma^{n-1}(t)\right), \qquad t \in \left[c, \rho^{n-1}(d)\right]. \tag{3.30}$$

Now, using (3.19), (3.30), and (3.25), we have

$$\begin{aligned} Sy(\sigma^n(b)) &\ge \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) u(s) f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \\ &\ge \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) Qy\left(\sigma^{n-1}(s)\right) \Delta s \\ &\ge Q \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \theta \|y\| \Delta s = \|y\|, \end{aligned}$$

which leads to (3.27). If we set $\Omega_2 = \{y \in B \mid \|y\| < T/\theta\}$, then $\|Sy\| \ge \|y\|$ for $y \in C \cap \partial\Omega_2$.

It follows from Theorem 2.2 that S has a fixed point $y_2 \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $w \le \|y_2\| \le T/\theta$. It is clear that y_2 is a positive solution of (1.1),(1.2).

(c) This follows from (a) and (b). ∎

THEOREM 3.10. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed and assume (B4)–(B6) hold. Suppose that $\underline{f}_0 = \underline{f}_\infty = \infty$ and f satisfies (3.24) with $w = w_l$, $l = 1, \dots, m$, where $w_1 < w_2 < \cdots < w_m$. Then, (1.1),(1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that*

$$0 < \|y_1\| \le w_1, \qquad \|y_2\| \ge w_m, \qquad \min_{t \in [c,d]} y_2(t) \ge \theta w_m.$$

PROOF. The result follows by repeated use of Theorem 3.9. ∎

THEOREM 3.11. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed and assume (B4)–(B6) hold. Let $w > 0$ be given and suppose that f satisfies*

$$f(x) \ge w \left[\int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s\right]^{-1}, \qquad \theta w \le x \le w. \tag{3.31}$$

(a) *If $\overline{f}_0 = 0$, then (1.1),(1.2) has a positive solution $y_1 \in B$ such that $0 < \|y_1\| \le w$.*

(b) *If $\overline{f}_\infty = 0$ and f is nondecreasing, then (1.1),(1.2) has a positive solution $y_2 \in B$ such that*

$$\|y_2\| \ge w, \qquad \text{with } \min_{t \in [c,d]} y_2(t) \ge \theta w.$$

(c) *If $\overline{f}_0 = \overline{f}_\infty = 0$ and f is nondecreasing, then (1.1),(1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that*

$$0 < \|y_1\| \le w \le \|y_2\|, \qquad \text{with } \min_{t \in [c,d]} y_2(t) \ge \theta w.$$

PROOF.

(a) We define

$$\epsilon = \left[\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, \Delta s\right]^{-1}. \tag{3.32}$$

Since $\overline{f}_0 = 0$, there exists $0 < \gamma < w$ such that

$$f(x) \le \epsilon x, \qquad 0 < x \le \gamma. \tag{3.33}$$

Let $y \in C$ be such that $\|y\| = \gamma$. Then, applying (3.19), Lemma 2.4, (3.33), and (3.32), we find for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\le \int_a^{\sigma(b)} (-1)^{p+1} G(t,s)\, v(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \\ &\le \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, \epsilon y\left(\sigma^{n-1}(s)\right) \Delta s \\ &\le \epsilon \|y\| \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, \Delta s = \|y\|. \end{aligned}$$

This immediately implies (3.28). If we set $\Omega_1 = \{y \in B \mid \|y\| < \gamma\}$, then $\|Sy\| \le \|y\|$ for $y \in C \cap \partial\Omega_1$.

Next, let $y \in C$ be such that $\|y\| = w$. Clearly,

$$\theta w \le y\left(\sigma^{n-1}(t)\right) \le w, \qquad t \in \left[c, \rho^{n-1}(d)\right],$$

which, together with (3.19) and (3.31), gives

$$Sy(\sigma^n(b)) \ge \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s)\, u(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \ge w = \|y\|.$$

Hence, we have (3.27). Set $\Omega_2 = \{y \in B \mid \|y\| < w\}$, then we get $\|Sy\| \ge \|y\|$ for $y \in C \cap \partial\Omega_2$.

It now follows from Theorem 2.2 that S has a fixed point $y_1 \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $\gamma \le \|y_1\| \le w$. It is clear that y_1 is a positive solution of (1.1),(1.2).

(b) From the proof of Theorem 3.11(a), we note that the condition (3.31) gives rise to (3.27). So, by letting $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then $\|Sy\| \ge \|y\|$ for $y \in C \cap \partial\Omega_1$.

Next, let ϵ be defined as in (3.32). Since $\overline{f}_\infty = 0$, we may choose $T > w$, such that

$$f(x) \le \epsilon x, \qquad x \ge T. \tag{3.34}$$

Let $y \in C$ be such that $\|y\| = T$. Then, using (3.19), Lemma 2.4, the fact that f is nondecreasing, (3.34) and (3.32), we find for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\le \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, f\left(y\left(\sigma^{n-1}(s)\right)\right) \Delta s \\ &\le \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, f(\|y\|)\, \Delta s \\ &\le \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s)\, v(s)\, \epsilon \|y\|\, \Delta s = \|y\|. \end{aligned}$$

Thus, we have (3.28). If we set $\Omega_2 = \{y \in B \mid \|y\| < T\}$, then $\|Sy\| \le \|y\|$ for $y \in C \cap \partial\Omega_2$.

We have now obtained (3.27) and (3.28). Once again, it follows from Theorem 2.2 that S has a fixed point $y_2 \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $w \le \|y_2\| \le T$. It is clear that y_2 is a positive solution of (1.1),(1.2).

(c) This follows from (a) and (b). ∎

THEOREM 3.12. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed and assume (B4)–(B6) hold. Suppose that $\overline{f}_0 = \overline{f}_\infty = 0$, f is nondecreasing and f satisfies (3.31) with $w = w_l$, $l = 1, \dots, m$, where $w_1 < w_2 < \cdots < w_m$. Then, (1.1),(1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that*

$$0 < \|y_1\| \le w_1, \qquad \|y_2\| \ge w_m, \qquad \min_{t\in[c,d]} y_2(t) \ge \theta w_m.$$

PROOF. Repeated applications of Theorem 3.11 yield the result. ∎

EXAMPLE 3.2. Let $m \in \mathbb{R}^+$ and $\mathbf{T} = m\mathbb{Z}$. Consider the boundary value problem,

$$y^{\Delta^2}(t) + \frac{24\left[y(\sigma(t))^2 + 1872m^4\right]}{\left[300m\sigma(t) - 12(\sigma(t))^2\right]^2 + 1872m^4} = 0, \qquad t \in [0, 11m],$$
$$y(0) = y^{\Delta}(12m) = 0. \tag{3.35}$$

Here, $n = 2$, $p = 1$, $a = 0$, and $b = 11m$. Taking $f(y) = y^2 + 1872m^4$, we may choose

$$u(t) = v(t) = \frac{24}{\left[300m\sigma(t) - 12(\sigma(t))^2\right]^2 + 1872m^4}.$$

It is easy to check that (B4)–(B6) are satisfied with $\eta = 1$. Clearly, $\underline{f}_0 = \underline{f}_\infty = \infty$. We shall find some $w > 0$ such that condition (3.24) is fulfilled. First, using (3.17) we obtain

$$\begin{aligned}
\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s)\,\Delta s &= \int_0^{12m} (s+m) \frac{24}{\left[300m\sigma(s) - 12(\sigma(s))^2\right]^2 + 1872m^4} \Delta s \\
&\ge 24 \int_0^{12m} \frac{s+m}{(1872m^2)^2 + 1872m^4} \Delta s \\
&= \frac{1872m^2}{(1872m^2)^2 + 1872m^4} = \frac{1}{1873m^2}
\end{aligned}$$

where we have substituted $s = 11m$ to get the inequality. Next, to ensure that (3.24) is true, we set

$$\begin{aligned}
0 < f(y) &\le w^2 + 1872m^4 \\
&\le w \left[\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s)\,\Delta s\right]^{-1} \\
&\le 1873m^2 w, \qquad 0 < y \le w,
\end{aligned}$$

which gives the inequality $w^2 + 1872m^4 \le 1873m^2 w$. This holds if and only if

$$m^2 \le w \le 1872m^2. \tag{3.36}$$

Hence, (3.24) holds for any $w \in \mathbb{R}[m^2, 1872m^2]$. By Theorem 3.10, there exist two positive solutions y_1 and y_2 of (3.35) such that for a fixed $\delta \in \mathbb{R}(0,1/2)$,

$$0 < \|y_1\| \le m^2, \qquad \|y_2\| \ge 1872m^2, \qquad \min_{t\in[c,d]} y_2(t) \ge \theta\left(1872m^2\right). \tag{3.37}$$

Let $\delta = 0.3$. Then, $c = 4m$, $d = 9m$ and $\theta = k = 1/3$. We note that one positive solution of (3.35) is given by $y(t) = 12t(25m - t)$ with $\|y\| = 1872m^2$ and $\min_{t\in[4m,9m]} y(t) = 1008m^2$, both are within the ranges given in (3.37).

4. APPLICATIONS

In this section, we shall apply the results obtained in Section 3 to two special cases of (1.1),(1.2), namely, (Q1) and (Q2). In addition to providing easily verifiable criteria for the existence of double positive solutions, we also establish upper and lower bounds for these solutions.

Before proceeding further, it is noted that (Q1) and (Q2) are particular cases of (1.1),(1.2) when $n = 2$ and $p = 1$. Hence, by Theorem 2.3 the related Green's function is explicitly given in (3.17). Moreover, for a fixed $\delta \in \mathbb{R}(0, 1/2)$, using Lemma 2.5 and (3.17), we get

$$\begin{aligned} k &= \inf_{s\in[a,b]} \frac{G(c,s)}{G(\sigma^2(b),s)} = \inf_{s\in[a,b]} \frac{G(c,s)}{\sigma(s)-a} \\ &= \min\left\{ \inf_{\sigma(s)\in[\sigma(a),c]} \frac{\sigma(s)-a}{\sigma(s)-a}, \inf_{\sigma(s)\in[c,\sigma(b)]} \frac{c-a}{\sigma(s)-a} \right\} = \frac{c-a}{\sigma(b)-a}. \end{aligned} \tag{4.1}$$

We shall begin with the boundary value problem,

$$\begin{gathered} y^{\Delta^2}(t) + h(t)\left([y(\sigma(t))]^\alpha + [y(\sigma(t))]^\beta\right) = 0, \qquad t \in [a,b] \\ y(a) = y^\Delta(\sigma(b)) = 0 \end{gathered} \tag{Q1}$$

where $0 \le \alpha < 1 < \beta$. It is assumed that

(C1) $h(t)$ is continuous and nonnegative on $[a, \sigma(b)]$;
(C2) $h(t)$ is nonzero for some $t \in [c, \rho(d))$;
(C3) $\int_a^{\sigma(b)} G(\sigma^2(b), s)h(s)\Delta s = \int_a^{\sigma(b)} [\sigma(s) - a]h(s)\Delta s < \infty$.

THEOREM 4.1. *Let $\delta \in \mathbb{R}(0, 1/2)$ be fixed and let $w > 0$ be given. Suppose that*

$$\int_a^{\sigma(b)} [\sigma(s)-a]\, h(s)\, \Delta s \le \frac{w}{w^\alpha + w^\beta}. \tag{4.2}$$

Then, boundary value problem (Q1) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that

$$0 < \|y_1\| \le w \le \|y_2\|, \qquad \text{with } \min_{t\in[c,d]} y_2(t) \ge \frac{(c-a)\,w}{\sigma(b)-a}.$$

PROOF. The boundary value problem (Q1) is a particular case of (1.1),(1.2) when $F(t,y) = h(t)(y^\alpha + y^\beta)$. Pick $f(x) = x^\alpha + x^\beta$ and $u(t) = v(t) = h(t)$. Then, in view of (C1)–(C3), Conditions (B4)–(B6) are now satisfied with $\eta = 1$. Moreover, $\underline{f}_0 = \underline{f}_\infty = \infty$. We shall apply Theorem 3.9. Since we have

$$0 < f(x) \le w^\alpha + w^\beta, \qquad 0 < x \le w,$$

to ensure that (3.24) is satisfied, we shall impose (noting (3.17))

$$w^\alpha + w^\beta \le w\left[\int_a^{\sigma(b)} G(\sigma^2(b), s)\, v(s)\, \Delta s\right]^{-1} = w\left\{\int_a^{\sigma(b)} [\sigma(s)-a]\, h(s)\, \Delta s\right\}^{-1},$$

which leads to (4.2). The conclusion now follows immediately from Theorem 3.9(c) and (4.1). ∎

THEOREM 4.2. *Let $\delta \in \mathbb{R}(0, 1/2)$ be fixed. Suppose (4.2) is satisfied with $w = w_l$, $l = 1, \dots, m$, where $w_1 < w_2 < \cdots < w_m$. Then, (Q1) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that*

$$0 < \|y_1\| \le w_1, \qquad \|y_2\| \ge w_m, \qquad \min_{t\in[c,d]} y_2(t) \ge \frac{(c-a)\,w_m}{\sigma(b)-a}.$$

PROOF. The result follows by repeated use of Theorem 4.1. ∎

We shall now establish upper and lower bounds for the two positive solutions of (Q1).

THEOREM 4.3. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed. We define*

$$\phi(x) = \left(\frac{c-a}{\sigma(b)-a}\right)^x \int_c^{\rho(d)} [\sigma(s)-a]\, h(s)\,\Delta s.$$

Let

$$w_1 = [\phi(\alpha)]^{1/1-\alpha} \qquad \text{and} \qquad w_2 = [\phi(\beta)]^{1/1-\beta}.$$

Suppose that $w > 0$ is given and (4.2) holds. Then, boundary value problem (Q1) has (at least) two positive solutions $y_1, y_2 \in C[a,\sigma^2(b)]$ such that

(a) *if $w < \min\{w_1, w_2\}$, then*

$$0 < \|y_1\| \le w \le \|y_2\| \le \min\{w_1, w_2\}, \qquad \text{with } \min_{t\in[c,d]} y_2(t) \ge \frac{(c-a)\,w}{\sigma(b)-a};$$

(b) *if $\min\{w_1,w_2\} < w < \max\{w_1,w_2\}$, then*

$$\min\{w_1,w_2\} \le \|y_1\| \le w \le \|y_2\| \le \max\{w_1,w_2\},$$

$$\min_{t\in[c,d]} y_1(t) \ge \frac{(c-a)\min\{w_1,w_2\}}{\sigma(b)-a}, \qquad \min_{t\in[c,d]} y_2(t) \ge \frac{(c-a)\,w}{\sigma(b)-a};$$

(c) *if $w > \max\{w_1,w_2\}$, then*

$$\max\{w_1,w_2\} \le \|y_1\| \le w \le \|y_2\|,$$

$$\min_{t\in[c,d]} y_1(t) \ge \frac{(c-a)\max\{w_1,w_2\}}{\sigma(b)-a}, \qquad \min_{t\in[c,d]} y_2(t) \ge \frac{(c-a)\,w}{\sigma(b)-a}.$$

PROOF. Since (4.2) is satisfied, it follows from Theorem 4.1 that (Q1) has double positive solutions y_3 and y_4 such that

$$0 < \|y_3\| \le w \le \|y_4\|. \tag{4.3}$$

Next, the operator S defined in (3.1) becomes

$$Sy(t) = \int_a^{\sigma(b)} G(t,s)\,h(s)\left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta\right)\Delta s, \qquad t \in \left[a,\sigma^2(b)\right], \tag{4.4}$$

where $G(t,s)$ is given in (3.17). Moreover, the cone $C_k \subset B = C[a,\sigma^2(b)]$ defined in (3.10) becomes (noting (4.1))

$$C_k = \left\{ y \in B \mid y(t) \ge 0,\ t \in \left[a,\sigma^2(b)\right];\ \min_{t\in[c,d]} y(t) \ge \frac{c-a}{\sigma(b)-a}\|y\| \right\}. \tag{4.5}$$

We shall employ Theorem 2.2. Let $y \in C_k$ be such that $\|y\| = w$. Then, in view of Lemma 2.4, (3.17) and (4.2), we have

$$\begin{aligned} Sy(t) &\le \int_a^{\sigma(b)} G\left(\sigma^2(b),s\right) h(s)\left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta\right)\Delta s \\ &\le \int_a^{\sigma(b)} [\sigma(s)-a]\,h(s)\left(w^\alpha + w^\beta\right)\Delta s \le w = \|y\|, \qquad t \in \left[a,\sigma^2(b)\right]. \end{aligned}$$

Thus,

$$\|Sy\| \le \|y\|. \tag{4.6}$$

By setting $\Omega = \{y \in B \mid \|y\| < w\}$, we see that (4.6) holds for $y \in C_k \cap \partial\Omega$.

Next, it is clear that for $y \in C_k$,

$$\begin{aligned}
\|Sy\| &= \sup_{t\in[a,\sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)\,h(s)\left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta\right)\Delta s \\
&= \int_a^{\sigma(b)} G\left(\sigma^2(b), s\right) h(s)\left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta\right)\Delta s \\
&\geq \int_c^{\rho(d)} [\sigma(s) - a]\,h(s)\left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta\right)\Delta s \\
&\geq \int_c^{\rho(d)} [\sigma(s) - a]\,h(s)\left[\left(\frac{c-a}{\sigma(b)-a}\right)^\alpha \|y\|^\alpha + \left(\frac{c-a}{\sigma(b)-a}\right)^\beta \|y\|^\beta\right]\Delta s.
\end{aligned}$$

It follows that

$$\|Sy\| \geq \phi(\alpha)\,\|y\|^\alpha + \phi(\beta)\,\|y\|^\beta, \qquad y \in C_k. \tag{4.7}$$

Let $y \in C_k$ be such that $\|y\| = w_1$. Then, (4.7) provides

$$\|Sy\| \geq \phi(\alpha)\,\|y\|^\alpha \geq \phi(\alpha)\,\|y\|^{\alpha-1}\,\|y\| = \|y\|. \tag{4.8}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < w_1\}$, then (4.8) holds for $y \in C_k \cap \partial\Omega_1$. Having obtained (4.6) and (4.8), it follows from Theorem 2.2 that S has a fixed point $y_5 \in C_k$ such that

$$\min\{w_1, w\} \leq \|y_5\| \leq \max\{w_1, w\}. \tag{4.9}$$

Similarly, if we let $y \in C_k$ be such that $\|y\| = w_2$, then from (4.7), we get

$$\|Sy\| \geq \phi(\beta)\,\|y\|^\beta \geq \phi(\beta)\,\|y\|^{\beta-1}\,\|y\| = \|y\|. \tag{4.10}$$

By setting $\Omega_2 = \{y \in B \mid \|y\| < w_2\}$, we see that (4.10) holds for $y \in C_k \cap \partial\Omega_2$. With (4.6) and (4.10), once again by Theorem 2.2, we conclude that S has a fixed point $y_6 \in C_k$, such that

$$\min\{w_2, w\} \leq \|y_6\| \leq \max\{w_2, w\}. \tag{4.11}$$

Our result follows by combining (4.3), (4.9), and (4.11). For Case (a), we may pick

$$y_1 = y_3 \qquad \text{and} \qquad y_2 = \begin{cases} y_5, & w_1 \leq w_2, \\ y_6, & w_1 \geq w_2. \end{cases}$$

In Case (b), we choose

$$(y_1, y_2) = \begin{cases} (y_5, y_6), & w_1 \leq w_2, \\ (y_6, y_5), & w_1 \geq w_2. \end{cases}$$

Finally, in Case (c), we take

$$y_1 = \begin{cases} y_6, & w_1 \leq w_2, \\ y_5, & w_1 \geq w_2, \end{cases} \qquad \text{and} \qquad y_2 = y_4. \qquad \blacksquare$$

EXAMPLE 4.1. Consider boundary value problem (Q1) with

$$\mathbf{T} = \left\{2^k \mid k \in \mathbb{Z}\right\} \cup \{0\}, \qquad a = 1, \quad b = 8.$$

Let $w = 1$. Then, condition (4.2) reduces to

$$\int_1^{\sigma(8)} (2s - 1)\,h(s)\,\Delta s \leq \frac{1}{2}. \tag{4.12}$$

By Theorem 4.1, for any $h(t)$ that satisfies (4.12), the boundary value problem has (at least) double positive solutions y_1 and y_2 such that for a fixed $\delta \in \mathbb{R}(0, 1/2)$,

$$0 < \|y_1\| \leq 1 \leq \|y_2\| \qquad \text{with } \min_{t\in[c,d]} y_2(t) \geq \frac{c-1}{15}.$$

Some examples of such $h(t)$ are $1/180$ and $1/160 \sin^2 t$.

EXAMPLE 4.2. *Let $\mathbf{T} = \{2^k \mid k \in \mathbb{Z}\} \cup \{0\}$. Consider boundary value problem,*

$$y^{\Delta^2}(t) + \frac{[y(\sigma(t))]^{0.1} + [y(\sigma(t))]^{1.1}}{((1/2)\sigma(t)+5)^2} = 0, \qquad t \in \left[\frac{1}{2}, 4\right], \qquad y\left(\frac{1}{2}\right) = y^{\Delta}(4) = 0. \tag{4.13}$$

Here, $\alpha = 0.1$, $\beta = 1.1$ and $h(t) = ((1/2)\sigma(t)+5)^{-2}$. Condition (4.2) yields

$$\frac{w}{w^{0.1}+w^{1.1}} \geq \int_{1/2}^{\sigma(4)} [\sigma(s) - a]\, h(s)\, \Delta s = \int_{0.5}^{8} \frac{2s - 0.5}{((1/2)\sigma(s)+5)^2} \Delta s = 0.563,$$

and this is satisfied for any $1.39 \leq w \leq 302$.

Let $\delta = 0.05$. Then, $c = 2$ and $d = 8$. By direct computation, we get

$$w_1 = [\phi(0.1)]^{10/9} = 0.0962 \qquad \text{and} \qquad w_2 = [\phi(1.1)]^{-10} = 1.38 \times 10^{16}.$$

Since $w_1 < w < w_2$, by Theorem 4.3(b), boundary value problem (4.13) has (at least) two positive solutions y_1 and y_2 such that

$$0.0962 \leq \|y_1\| \leq w \leq \|y_2\| \leq 1.38 \times 10^{16}, \qquad \min_{t\in[2,8]} y_1(t) \geq 0.01924, \qquad \min_{t\in[2,8]} y_2(t) \geq 0.2w.$$

Since $1.39 \leq w \leq 302$, we further conclude that

$$0.0962 \leq \|y_1\| \leq 1.39, \qquad 302 \leq \|y_2\| \leq 1.38 \times 10^{16},$$
$$\min_{t\in[2,8]} y_1(t) \geq 0.01924, \qquad \min_{t\in[2,8]} y_2(t) \geq 0.2(302) = 60.4.$$

For the rest of this section, we shall consider boundary value problem,

$$y^{\Delta^2}(t) + h(t)\, e^{\zeta[y(\sigma(t))]} = 0, \qquad t \in [a,b], \qquad y(a) = y^{\Delta}(\sigma(b)) = 0 \tag{Q2}$$

where $\zeta > 0$. It is assumed that $h(t)$ satisfies Conditions (C1)–(C3).

THEOREM 4.4. *Let $\delta \in \mathbb{R}(0, 1/2)$ be fixed and let $w > 0$ be given. Suppose that*

$$\int_a^{\sigma(b)} [\sigma(s) - a]\, h(s)\, \Delta s \leq w e^{-\zeta w}. \tag{4.14}$$

Then, boundary value problem (Q2) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that

$$0 < \|y_1\| \leq w \leq \|y_2\|, \qquad \text{with } \min_{t\in[c,d]} y_2(t) \geq \frac{(c-a)\,w}{\sigma(b) - a}.$$

PROOF. Boundary value problem (Q2) is a special case of (1.1),(1.2) when $F(t,y) = h(t)e^{\zeta y}$. Choose $f(x) = e^{\zeta x}$ and $u(t) = v(t) = h(t)$. Then, in view of (C1)–(C3), Conditions (B4)–(B6) are now satisfied with $\eta = 1$. Further, $\underline{f}_0 = \underline{f}_\infty = \infty$. We shall employ Theorem 3.9. Since

$$f(x) \leq e^{\zeta w}, \qquad 0 < x \leq w,$$

condition (3.24) will be satisfied if we set

$$e^{\zeta w} \leq w \left[\int_a^{\sigma(b)} G\left(\sigma^2(b), s\right) v(s)\, \Delta s\right]^{-1} = w \left\{\int_a^{\sigma(b)} [\sigma(s) - a]\, h(s)\, \Delta s\right\}^{-1},$$

which is inequality (4.14). The conclusion is now immediate from Theorem 3.9(c) and (4.1). ∎

THEOREM 4.5. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed. Suppose (4.14) is satisfied with $w = w_l$, $l = 1, \dots, m$, where $w_1 < w_2 < \cdots < w_m$. Then, (Q2) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that*

$$0 < \|y_1\| \le w_1, \qquad \|y_2\| \ge w_m, \qquad \min_{t\in[c,d]} y_2(t) \ge \frac{(c-a)\, w_m}{\sigma(b) - a}.$$

PROOF. The result is obtained by repeated use of Theorem 4.4. ∎

The next result offers upper and lower bounds for the two positive solutions of (Q2).

THEOREM 4.6. *Let $\delta \in \mathbb{R}(0,1/2)$ be fixed and $i, j \in \mathbb{Z}[2,\infty) \cup \{0\}$ be fixed distinct integers. We define*

$$\psi(x) = \frac{1}{x!}\left(\frac{\zeta(c-a)}{\sigma(b)-a}\right)^x \int_c^{\rho(d)} (\sigma(s) - a)\, h(s)\, \Delta s.$$

Let

$$w_1 = [\psi(j)]^{1/(1-j)} \qquad \text{and} \qquad w_2 = [\psi(i)]^{1/(1-i)}.$$

Suppose that $w > 0$ is given and (4.14) holds. Then, the boundary value problem (Q2) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that Conclusions (a)–(c) of Theorem 4.3 hold.

PROOF. Since (4.14) is satisfied, by Theorem 4.4 the boundary value problem (Q2) has double positive solutions y_3 and y_4 such that (4.3) holds.

Next, the operator S defined in (3.1) becomes

$$Sy(t) = \int_a^{\sigma(b)} G(t,s)\, h(s)\, e^{\zeta y(\sigma(s))} \Delta s, \qquad t \in \left[a, \sigma^2(b)\right], \tag{4.15}$$

where $G(t,s)$ is given in (3.17). Moreover, the cone C_k defined in (3.10) reduces to that in (4.5). We shall once again employ Theorem 2.2. Let $y \in C_k$ be such that $\|y\| = w$. Then, in view of Lemma 2.4, (3.17), and (4.14), for $t \in [a, \sigma^2(b)]$ we have

$$Sy(t) \le \int_a^{\sigma(b)} G\left(\sigma^2(b), s\right) h(s)\, e^{\zeta y(\sigma(s))} \Delta s \le \int_a^{\sigma(b)} [\sigma(s) - a]\, h(s)\, e^{\zeta w} \Delta s \le w = \|y\|.$$

Thus, (4.6) holds. By setting $\Omega = \{y \in B \mid \|y\| < w\}$, we see that $\|Sy\| \le \|y\|$ for $y \in C \cap \partial\Omega$.

Now, using the inequality,

$$e^x \ge \frac{x^i}{i!} + \frac{x^j}{j!}, \qquad x > 0, \tag{4.16}$$

we find for $y \in C_k$,

$$\begin{aligned}
\|Sy\| &= \sup_{t\in[a,\sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)\, h(s)\, e^{\zeta y(\sigma(s))} \Delta s \\
&= \int_a^{\sigma(b)} G\left(\sigma^2(b), s\right) h(s)\, e^{\zeta y(\sigma(s))} \Delta s \\
&\ge \int_c^{\rho(d)} [\sigma(s) - a]\, h(s)\, e^{\zeta y(\sigma(s))} \Delta s \\
&\ge \int_c^{\rho(d)} [\sigma(s) - a]\, h(s) \exp\left(\frac{\zeta(c-a)}{\sigma(b)-a}\|y\|\right) \Delta s \\
&\ge \int_c^{\rho(d)} [\sigma(s) - a]\, h(s) \left[\left(\frac{\zeta(c-a)}{\sigma(b)-a}\right)^i \frac{\|y\|^i}{i!} + \left(\frac{\zeta(c-a)}{\sigma(b)-a}\right)^j \frac{\|y\|^j}{j!}\right] \Delta s.
\end{aligned}$$

Hence, we have

$$\|Sy\| \ge \psi(i)\, \|y\|^i + \psi(j)\, \|y\|^j, \qquad y \in C_k. \tag{4.17}$$

Using a similar technique as in the proof of Theorem 4.3, from (4.17), we get

$$\|Sy\| \geq \|y\|, \tag{4.18}$$

for $y \in C \cap \partial\Omega_1$ and for $y \in C \cap \partial\Omega_2$, where $\Omega_1 = \{y \in B \mid \|y\| < w_1\}$ and $\Omega_2 = \{y \in B \mid \|y\| < w_2\}$. Having obtained (4.6) and (4.18), by Theorem 2.2 the operator S has fixed points y_5 and y_6 satisfying (4.9) and (4.11). Finally, just as in the proof of Theorem 4.3, Conclusions (a)–(c) follow from a combination of (4.3), (4.9), and (4.11). ∎

EXAMPLE 4.3. Let $\mathbf{T} = m\mathbb{Z}$ where $m \in \mathbb{R}^+$. Consider the boundary value problem

$$\begin{gathered} y^{\Delta^2}(t) + h(t)\, e^{(1/2)y(\sigma(t))} = 0, \qquad t \in [0, 11m] \\ y(0) = y^{\Delta}(12m) = 0. \end{gathered} \tag{4.19}$$

Here $\zeta = \frac{1}{2}$. Let $w = 1$ be given. Then, condition (4.14) reduces to

$$\int_0^{12m} (s+m)h(s)\Delta s \leq e^{-1/2}. \tag{4.20}$$

By Theorem 4.4, if $h(t)$ satisfies (4.20), then the boundary value problem (4.19) has double positive solutions y_1 and y_2 such that for a fixed $\delta \in \mathbb{R}(0, 1/2)$,

$$0 < \|y_1\| \leq 1 \leq \|y_2\| \qquad \text{with } \min_{t\in[c,d]} y_2(t) \geq \frac{c}{12m}.$$

Some examples of such $h(t)$ are $1/150m^2$ and $1/130\cos^2 t$.

EXAMPLE 4.4. Let $\mathbf{T} = \mathbb{R}[0, 1) \cup \mathbb{Z}^+$. Consider the boundary value problem,

$$\begin{gathered} y^{\Delta^2}(t) + \frac{2e^{y(\sigma(t))/2}}{e^{[19(\sigma(t))-(\sigma(t))^2]/2}} = 0, \qquad t \in [0, 8] \\ y(0) = y^{\Delta}(\sigma(8)) = 0. \end{gathered} \tag{4.21}$$

Here, $\zeta = 1/2$ and $h(t) = 2e^{-[19(\sigma(t))-(\sigma(t))^2]/2}$. We check that condition (4.14) is true provided

$$we^{-(1/2)w} \geq \int_0^{\sigma(8)} [\sigma(s)]\, h(s)\,\Delta s = \int_0^1 \frac{2s}{e^{[19s-s^2]/2}}\,ds + \sum_{s=1}^{8} \frac{2(s+1)}{e^{[19(s+1)-(s+1)^2]/2}} = 0.0229,$$

and this inequality is satisfied for any $0.0232 \leq w \leq 12.6$.

Fix $\delta = 0.1$, $i = 3$, and $j = 0$. Then, $c = 1$, $d = 9$, and by direct computation, we have

$$w_1 = \zeta(0) = 1.6582 \times 10^{-7} \qquad \text{and} \qquad w_2 = [\zeta(3)]^{-1/2} = 459370.$$

Since $w_1 < w < w_2$, by Theorem 4.6(b), boundary value problem (4.21) has two positive solutions y_1 and y_2 such that

$$1.6582\times 10^{-7} \leq \|y_1\| \leq w \leq \|y_2\| \leq 459370, \qquad \min_{t\in[1,9]} y_1(t) \geq 1.8424\times 10^{-8}, \qquad \min_{t\in[1,9]} y_2(t) \geq \frac{w}{9}.$$

Since $0.0232 \leq w \leq 12.6$, we can further conclude that

$$\begin{aligned} &1.6582 \times 10^{-7} \leq \|y_1\| \leq 0.0232, && 12.6 \leq \|y_2\| \leq 459370, \\ &\min_{t\in[1,9]} y_1(t) \geq 1.8424 \times 10^{-8}, && \min_{t\in[1,9]} y_2(t) \geq \frac{12.6}{9} = 1.4. \end{aligned} \tag{4.22}$$

In fact, a positive solution is given by $y(t) = t(19 - t)$ and we notice that $\|y\| = 90$ and $\min_{t\in[1,9]} y(t) = 18$ are well within the ranges obtained in (4.22).

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