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# Logic of agreement: foundations, semantic system and proof theory

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**Abstract**

In this paper a multi-valued propositional logic – logic of agreement – in terms of its model theory and inference system is presented. This formal system is the natural consequence of a new way to approach concepts as commonsense knowledge, uncertainty and approximate reasoning – the point of view of agreement. Particularly, it is discussed a possible extension of the Classical Theory of Sets based on the idea that, instead of trying to conceptualize sets as "fuzzy" or "vague" entities, it is more adequate to define membership as the result of a partial agreement among a group of individual agents. Furthermore, it is shown that the concept of agreement provides a framework for the development of a formal and sound explanation for concepts (e.g. fuzzy sets) which lack formal semantics. According to the definition of agreement, an individual agent agrees or not with the fact that an object possesses a certain property. A clear distinction is then established, between an individual agent – to whom deciding whether an element belongs to a set is just a yes or no matter – and a commonsensical agent – the one who interprets the knowledge shared by a certain group of people. Finally, the logic of agreement is presented and discussed. As it is assumed the existence of several individual agents, the semantic system is based on the perspective that each individual agent defines her/his own conceptualization of reality. So the semantics of the logic of agreement can be seen as being similar to a semantics of possible worlds, one for each individual agent. The proof theory is an extension of a natural deduction system, using supported formulas and incorporating only inference rules. Moreover, the soundness and completeness of the logic of agreement are also presented. © 1999 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

Commonsense reasoning and representation are particularly relevant topics in Artificial Intelligence [23]. The present approach to these issues is based upon the idea that commonsense derives from the knowledge that is held, used, and shared by a group of individuals. Recall that the etymology of the word "common-sense" is the "sharing of feelings" (among people about themselves and the world). In fact, it can be hypothesized that what people call *reality* derives from the agreement among a set of individuals. As a consequence, a commonsensical concept like "tall", when characterizing a certain person, for instance, should be framed with respect to a certain group of individuals. Of course, being considered as tall differs strongly if commonsense emerges from a group of bushmen or swedish individuals.

The fact that agreement among individuals is not always perfect suggests a kind of uncertainty exhibiting some similarities with the concept of fuzziness introduced by Lofti Zadeh [27]. However, the fuzzy approach has been criticized due to its subjectivity: this is the reason logicists argue against "fuzzy logics" (see a reply from Sheridan to Dubois and Prade [8]).

The controversy that has emerged since the introduction of Fuzzy Set Theory by Zadeh [27] involving their supporters and detractors is not surprising. On the one hand, the former assert that it captures the intuition regarding a special kind of uncertainty – fuzziness – and, on the other hand, the later claim against the lack of a precise characterization of the theory and, particularly, the arbitrariness of the choice of operators and membership functions [4,16,20]. In fact, the lack of a strict semantic characterization of fuzzy sets makes impossible the task of establishing a well-founded fuzzy logic – a formal system which allows inference regarding some conceptualization of reality. For those who favour rigorous and precise foundations, it is clear that fuzzy logic should be improved.

Although there are (several) different interpretations of the partial membership concept, most of them are based on intuitive grounds, being presented a posteriori in order to justify the utilization of fuzzy sets. Moreover, the fuzzy operators commonly used are not semantically supported. As a consequence they could provide unacceptable results.

In this paper a research exploring the idea of agreement and its use to model human knowledge and reasoning in what concerns commonsense is presented. Therefore, the concept of agreement is used for establishing a strict semantic characterization of fuzzy sets and operators, revisiting its foundations, redefining the basic operators, providing a rigorous meaning to them and shedding light on certain difficulties which were already pointed out by others [3,5,10,16,20]. Also, the new definition of partial membership provides a framework for the development of a formal system to approximate reasoning – the logic of agreement, a multi-valued extension of the classical logic [2]. In Section 2 the concept of agreement is discussed in terms of commonsense knowledge, as a measure of a certain kind of uncertainty. It is also compared with other concepts that were presented to explain different types of uncertainty, namely fuzziness. In Section 3 the main definitions and results of the agreement notion – in terms of a set theory – are introduced. In Section 4, an example exploring the differences between agreement and classic fuzzy based operators is presented. In Section 5 some consequences of the agreement idea are presented. In Section 6 the semantic system of the logic of agreement is introduced and presented. Section 7 describes the corresponding proof theory. Section 8 discusses the operationalization issues of the logic of agreement, and finally in Section 9 conclusions and some considerations about current and future work are presented.

## 2. Motivation and philosophical issues

The point of view of the present work is that fuzziness results from an incomplete agreement among agents when faced with the characterization of a certain object. Moreover, it is hypothesized that each agent when considered separately is able of deciding on a yes or no basis, whether a certain object belongs or not to a given set. That "Paul is tall", "Mary is middle-aged" and, "seven is much greater than two", and so on, raises no difficulties of membership characterization to a single agent. [1] This fact has already been stressed by Hayes [14]. A membership grade different from zero or one only comes up when someone tries to represent not a personal but a commonsensical interpretation of the reality. Reality – in fact, a certain conceptualization of it – derives from agreement among agents and, if no agreement at all is reached about an object, a fact, or an event, it is not possible to decide them as being "real". In classical terms, for instance, when defining the semantics of First Order Logic it is usual considering not the "reality" (something that exists outside the agent) but a conceptualization of it, accepted by the agent [13]. This is a way to circumvent the philosophical problems which derive from discussing the existence of reality. In fact, the philosophical foundations of the present approach are based on constructing the reality through agreement. This perspective is similar to the one suggested by Edgar Morin when

[1] Notice that that is not a problem of evaluating the logical value of a proposition, but merely the acceptance or not acceptance of the meaning of the proposition.

discussing objectivity in science: objectivity is nothing but accepted inter-subjectivity.

Therefore, if someone intends to characterize fuzziness, a clear distinction should be made: when an agent expresses a personal point of view ("an individual agent") and when an agent interprets the points of view of a group ("a commonsensical agent"). Frequently, when confronting points of view with others individuals discover that some of their beliefs have less epistemic entrenchment than others.

The proposed semantic characterization of fuzzy sets is based on the concept of agreement: the ratio of agents agreeing with the membership of an object to a set w.r.t. the total number of questioned agents. This seems to be a neat and simple frequencist approach which has been around for centuries, successfully applied to other areas of research.

Consider, for instance, the task of evaluating a paper submitted to a conference. Assume that each reviewer is only asked whether the paper should be accepted or not. The easiest way to sort the papers is to grade each paper in terms of the number of acceptances w.r.t. the total number of reviewers. The program chair of the conference can establish a semantics in terms of grades and labels: grade 0.0, definitely not accepted, grade 1.0, definitely accepted, grade 0.5 corresponding to maximum ignorance (or minimal discrimination between the two possible choices).

Now imagine that the program chair asks reviewers to evaluate soundness and originality of papers in digital terms (accepted as sound, rejected as original, and so on). After processing the data, that is to say after calculating the agreement, the program chair has two grades (between zero and one) assigned to each paper – one for soundness and one for originality. How should the program chair sort the papers? Probably, she/he will be tempted to sort the papers on the basis of the conjunction (sort the papers on the basis of being sound *and* original). For instance, if the program chair is an expert in fuzzy logic it is likely that she/he will use the min operator. Another approach is to find the ratio between the number of reviewers who gave positive answers to both aspects (the individual conjunctions) w.r.t. the total number of reviewers. The former procedure should not be used because it is generally unsound – usually it provides different results when compared with the later. Another possibility could be the utilization of the product (instead of min); however, this procedure is only sound when the two aspects of the problem are independent (in fact, soundness and originality are not independent [2]).

Therefore, an essential result to achieve (with a semantic characterization) is demonstrating that if from a set of $N$ agents result two membership grades for objects $A$ and $B$ in a set $S$, and if, from the very same agents result a

[2] This fact was already verified by the authors when organizing a conference.

membership grade for $A \cup B$ (or $A \cap B$) in the same set $S$, the chosen operators for union (and for intersection, respectively) should respect the results previously determined. In a nut shell, the algebraic manipulation (syntax) should be consistent with the meaning (semantics) assigned to fuzzy sets and operators. It should be pointed out that the known operators (min, max, product, drastic sum, etc.) do not exhibit the intended characteristic of soundness.

On the other hand, human beings usually prefer to produce simple evaluations and fast decisions instead of complex and time-consuming analyses. As a consequence of this quest for fast judgements, human reasoning loses frequently in terms of precision, raising therefore the issue of uncertainty. Moreover, the complexity of real situations is often too high to allow a complete and detailed description [28]. Also, since commonsense knowledge could usually be linked to symbols in some language, it raises the problem of symbol ambiguity, as it has been recognized by several authors [17,29].

These two aspects – lack of precision and symbol ambiguity – lead frequently to human reasoning and communication involving different types of uncertainty. For instance, fuzzy theory researchers consider a kind of uncertainty called fuzziness, which is usually associated with vagueness concerning the description of the semantic meaning of events or concepts [29]. On the other hand, Dubois and Prade [9] consider that "modelling vagueness is a problem of representing what is sometimes called lexical imprecision of linguistic terms".

In order to understand concepts as grade of membership it is important to study the way human beings define and establish categories or classes. The definition of a prototype constitutes one possibility to characterize the class and to determine the grade of membership for each element, being defined by the grade of *similarity* between the element and the prototype. However, this method is not applicable to problems where there are no way to find a prototype (e.g. the class of the numbers much greater than 2) or when it is possible to choose several prototypes (e.g. the class of good tennis players).

Nevertheless, studies carried out on the area of psychology suggest that membership is not a "primitive" concept, meaning that a grade of membership is not generally defined absolutely by an individual [7]. Indeed, it varies from individual to individual according to certain internal and external factors, which in fact influence and characterize each individual.

Albeit the initial idea of phenomena being inherently vague, it is now widely accepted that vagueness emerges from the observation and description of the world, i.e., it depends on the human observer. So, different observers could have different interpretations of the phenomena under consideration [9,19].

At first sight, a good candidate for representing and reasoning about vagueness is the fuzzy logic approach. However, "fuzzy logic" cannot be yet considered as a logic in the sense that, (i) it does not have a formal model theory – a semantic system, (ii) it does not rely on a unique inference system – there are several inference mechanisms to be applied according to the situation, and (iii) fundamental concepts, as the membership function, for instance, are ill-defined. As recognized by Zimmermann [29]: "Fuzziness has so far not been defined uniquely semantically [...]".

So, in order to handle these problems, the following aspects of FST should be taken into account:

(i) the choice of the rules of inference is usually arbitrary or at most based on algebraic arguments,

(ii) whenever such a choice is not supported by a formal semantic justification, soundness cannot be ensured,

(iii) in order to develop a formal model theory it is necessary to give a (formal and informal) semantic definition of basic concepts and operators (e.g., fuzzy connectives).

Otherwise, when FST is applied to certain problems (see the pencil of Sheridan-Fine example in Section 4) it delivers puzzling answers and weird results.

Nevertheless, the main problems are not in the "logic" itself but in its very foundations: the Fuzzy Set Theory. The basic element of FST is the notion of fuzzy set presented by Zadeh as a generalization of the classical set. Although there is some consensus about the intuitive meaning of a fuzzy set, the lack of a unified formal semantic characterization of this concept is a drawback of FST. Furthermore, basic operations such as union and intersection of fuzzy sets can be performed using different mechanisms as the min, max, drastic product and sum, to name but a few.

As it was mentioned before, the representation of vague concepts depends on the observer (or agent). Considering this idea, fuzzy approaches usually assume that each agent is individually capable of defining the most adequate fuzzy set to represent a vague concept. So, the majority of fuzzy applications (including fuzzy expert systems) starts from this point, by defining fuzzy sets for the problem in hand, choosing the fuzzy operators and establishing inference rules, only based on a problem-dependent basis. This methodology can be understood because of pragmatic reasons, as efficiency, computation time, and development effort. However, when one wants to justify the options made, the question "how the fuzzy sets were chosen", i.e., "where did they come from", should be considered seriously.

A work that tries to answer these questions is the TEE model presented by E. Hisdal. This model is based on an assumption that the subjective meaning which an individual assigns to a fuzzy concept can be measured by performing three experiments: (i) a labelling experiment, in which an individual assigns a particular label (e.g. tall), from a set of labels (e.g. small, medium, tall), to every object at a given time, (ii) a yes–no experiment, in which the individual is required to answer with yes or no concerning the adequacy of a label to every object, and (iii) a membership experiment, in which the individual is asked to give a degree, between zero and one, concerning the fitness of a label to every object [16]. Using these experiments together with the probabilistic theory, Hisdal introduces a model for interpreting and explaining how an individual defines her/his membership function for a particular fuzzy concept. Based on the assumptions considered in the TEE model, the definitions of the fuzzy set operators are reformulated and justified, and some difficulties of the fuzzy set theory are circumvented.

The main differences between the TEE model and the agreement framework lie not only on their foundations but also on their objectives: (i) the TEE model considers only an individual whereas the agreement concept considers a set of individuals, (ii) the degree of membership obtained by the TEE model is based on a probabilistic analysis of the experiments performed whereas the degree of agreement establishes a measure of consensus among a group of individuals, (iii) the TEE model provides a numeric formalism based on probability for handling fuzzy concepts, whereas the agreement framework underlies a rigorous semantic system and a proof theory for a symbolic multi-valued logic that is capable to cope with partial agreement, and (iv) the TEE model assumes that the individual is capable of choosing membership degrees whereas the agreement framework is based on two kinds of agents: the individual agent, which can only give yes or no answers and the commonsensical agent, which interprets the points of view of a group of individual agents. So, the individual considered in the TEE model can be seen as a commonsensical agent. In fact, when Hisdal discusses the third experiment – the membership degree assignment – she questions where does a particular degree, between zero and one, which an individual assigns to an object, come from. Hisdal assumes that an individual must have some internally stored procedure for choosing this degree. The agreement framework is a possible way to explain how such procedure can be establish. For instance, Dubois and Prade [9] say that "Classification itself is not objective. Our way of classifying objects may differ from our neighbors'. At best, it may be a matter of consensus".

As mentioned before, agents can have two differing roles: as *individual* agents and as *commonsensical* agents. An individual agent is an abstract entity which uses the classical set theory (and classical logic) to make decisions. So, when asked if a women with, say, 1.70 height is tall, a particular individual agent answers on a yes or no basis. In this case, it is clearly a matter of true/false decision on the logic value of a proposition [12,14]. However, when raising the same question to a group of individuals it is conceivable to finding a degree of acceptance of that particular proposition ranging from zero (complete rejection) to one (complete acceptance). When an agent associates non zero/one degrees of acceptance to propositions she/he is behaving as a commonsensical agent, who interprets the feelings of a group with a sufficient number of individuals.

Therefore, to characterize a fuzzy set, several individual agents are questioned concerning the membership of an element on a particular class or category. The proportion of the number of positive answers w.r.t. the total number of answers is called degree of agreement. If this degree is interpreted as the membership grade of the element in the class, it is possible to define a membership function representing the sethood of the class. For instance, if the number of individual agents that agree with the fact that a 1.7 m height individual is tall, is 80 out of 100, then the degree of agreement within the group is 0.8. So the membership grade of a 1.7 m height element in the set of tall individuals is 0.8. A commonsensical agent would assign the logical value 0.8 to the proposition "An individual 1.7 m height is tall".

The idea of defining (or interpreting) the membership degree as the proportion of positive answers among a population was already suggested by others. For instance, Baldwin [1] introduced an interpretation of fuzzy set based on a voting model with constant thresholds. Each voter (person) accepts or rejects that a specific object satisfies a particular fuzzy concept. The proportion of voters who accept is associated with the membership degree. The constant threshold assumption means that anyone who accepts an object with a certain membership level will accept all objects with a higher membership level. That is to say, a voter should either know a priori the membership level for a particular object or define a metric suitable for classifying the object, in order to be consistent with the constant threshold assumption. However, both hypotheses can be questioned. In the former case, if a voter knows the membership level, how does it come from and what is the purpose of the voting process? In the later case, the problem lies on the fact that most of the fuzzy concepts do not have an unique and objective metric. On the contrary, within the agreement framework, an individual agent needs not neither an a priori membership level nor a specific metric. Also, instead of considering the results of the voting process as a way to establish the probability that a member of population, drawn at random, answers in a particular manner, these results can be seen as a measure of the degree of agreement among the voters.

## 3. Foundations of the agreement logic

As it is assumed that individual agents use the classic set theory in order to answer questions relatively to the membership of an element in a set, the following axioms express the traditional definitions of set complement, union and intersection.

Let $A$ be a subset of $X$ and $x$ an element of $X$, the universe of discourse.

**Axiom 1** (*Omniscience*). When an individual agent is questioned about the belongingness of an element $x$ in a set $A$, she/he is always capable of deciding in an yes/no basis.

This axiom excludes the possibility of getting answers like "I do not know", "maybe", among others. There are two kinds of situations which can be found: one that presupposes a set with clear cut boundaries, and another where sets have blurred frontiers. As examples, consider respectively the questions "Is ten greater than one?", and "Is ten much greater than one?". The Classical Set Theory (CST) was defined mainly to handle the former. Some authors consider that the later should also be handled by CST as agents are always capable of establishing a clear cut frontier and act accordingly. In what concerns a single agent, this hypothesis can be accepted provided that this agent performs as an abstract entity. However, it is interesting to notice that children tend to use a kind of digital process for categorizing objects and, therefore, they show some difficulties to understand that usually things are not only "black and white". In fact, when asking children about the belongingness of elements to sets answers are quick and direct. On the contrary, adults tend to feel uncomfortable when asked by children to answer in an yes/no basis, trying in certain cases to pathetically answer in a fuzzy fashion. That is to say, adults act normally as commonsensical agents (possibly because they were exposed to the opinions of others during their long life time).

Therefore, it can be hypothesized that when a real agent (an adult human being) expresses doubts about the belongingness of an element in a set she/he is acting as a "commonsensical agent" who takes into consideration a lack of agreement about a particular concept within a group of individuals. A difficult issue deserving discussion is the consistency of answers along time and in different situations. An agent could consider Mary as a tall person and Bea as a very tall, although they have the same height, just because Mary is fat and Bea is not. Or considering Mary as tall now and very tall when wearing a vertical stripped dress. In order to avoid this kind of difficulty, the proposed approach does not impose constraints or assumptions on how an individual agent should perform, beyond the acceptance of the present axioms.

To represent the individual agent's opinion a ternary function is defined as follows:

**Definition 1** (*Concordant function*). The belief of individual agent $i$ with respect to the membership of $x$ in a set $A$ is represented by a concordant function $cc(i,x,A)$ defined as:

$$cc(i,x,A) = \begin{cases} 1 & \text{if agent } i \text{ agrees with membership of } x \text{ in } A, \\ 0 & \text{otherwise.} \end{cases}$$

**Axiom 2** (*Negation*). When an individual agent believes that an element $x$ belongs to a set $A$ then she/he also accepts that $x$ does not belong to the complement of $A$, $\bar{A}$, and vice-versa.

Certain human beings, in particular examples, do not respect this axiom. However, these cases correspond to elements *presumably* located near an imaginary borderline establishing the boundary between the set $A$ and its complement. As real agents should perform like ideal (rational) individual agents, it is always possible to clear things up and restore consistency. In fact, the experiments of Hersh and Caramazza [15] also confirm the hypothesis stated in the Axiom 2.

Based on Axiom 2 the concordance of an agent relatively to the membership in a complement set is given through the function $cc$.

If $\bar{A}$ is the complement of a set $A$ then

$$cc(i,x,\bar{A}) = \begin{cases} 1 & \text{if} cc(i,x,A) = 0 \\ 0 & \text{if} cc(i,x,A) = 1 \end{cases}$$
$$= 1 - cc(i,x,A).$$

**Axiom 3** (*Conjunction*). When an individual agent believes that $x$ belongs to a set $A$ and also believes that $x$ belongs to a set $B$, then she/he believes that $x$ belongs to the intersection of sets $A$ and $B$, and vice-versa.

Human agents exhibit two kinds of behaviour when categorizing objects: a *best-fit* approach (when the agent assigns one and only one label to the object, corresponding to the best describing characteristic), and a *subsumption* perspective (when the agent assigns several labels which are "acceptable" to describe the object). For instance, consider an individual who is asked whether John Smith (2.35 m) is *tall*, and whether he is *very tall*. In the former case (best fit), the agent will only assign the label very tall, and in the later the agent will assign both labels, as *all very tall persons are also tall*. Notice that this difficulty only arises when the labels are associated with the same characteristic (attribute) under evaluation (in this case, height).

A more subtle example is the assignment of colors to objects. Suppose, say, a reddish orange pencil. Some individuals will consider its color as red and some as orange. On the one hand, the very same individual seldom considers the pencil as red *and* orange. However, on the other hand, almost all individuals accept the fact that deep orange is a kind of light red. So there is an overlapping between these two sets (red and orange).

Of course, when applying the best fit approach, individuals are considering sets as disjoint (e.g., tall/very tall, red/orange, and so on). Also, the *order* by which the questions are raised is relevant to the answers got (if the question "Is John Smith tall?" is asked before the question "Is John Smith very tall?", it is likely to get two positive answers even when a best fit approach is considered). As the present axiom does not consider the order of conjuncts as a relevant aspect, a special care must be taken to avoid erroneous results.

Another interesting case involving colors in a conjunction is the Portuguese flag example: the flag is divided vertically with colors green and red (the area of red being slightly bigger than the area of green). In the vertical line there is an heraldic sphere (yellow) with a complex drawing including small castles, shields in blue and white. When asked about the color of the flag, *very few* people accept green as the color of the flag. On the other hand, *few* people accept red as the color of the flag. However, almost all consider the Portuguese flag as being *green and red*. Notice that there is nothing fuzzy in the flag (colors are primitive, by definition). This is a case where people, although accepting the conjunction, do not accept each one of the conjuncts. (Can this be explained because the *green and red* is now a single predicate?). This example illustrates the difficulties of formalizing natural language concepts.

It is important to notice that in any case (best fit or subsumption approaches) the Logic of Agreement – which is based upon the present axioms – performs correctly although providing different results. It is however crucial that all agents (performing as individual agents) should adopt the very same methodology when categorizing objects, in order to maintain consistency. In fact, some interesting differences will arise depending on the approach taken: for instance, and not surprisingly, when the best fit approach is chosen, it is not possible to finding a "strong" implication between, say, being very tall and being tall as it happens if the other approach is considered.

Again the function $cc$ is used to represent the assumption introduced by Axiom 3.

If $A$ and $B$ are two sets then

$$cc(i,x,A\cap B) = cc(i,x,A)cc(i,x,B).$$

**Axiom 4** (*Disjunction*). When an individual agent believes that $x$ belongs to a set $A$ or that $x$ belongs to a set $B$ (or both) then she/he should also accept that $x$ belongs to the union of sets $A$ and $B$, and vice-versa.

This axiom establishes the semantics of the connective "or". In natural language, the "or" is, in most cases, utilized in the exclusive sense, *presumably* because it allows direct inferences. For instance, knowing that "Mary is either tall or very tall," as soon as it is found out that "Mary is tall" it can be inferred that "Mary is not very tall" (best fit approach). However, this axiom is based on an inclusive sense for the connective "or" because it is more general than the previous one. (Notice that it is always possible to associate conjunctions and disjunctions for expressing propositions involving exclusive "or"). Therefore, agents performing as individual agents should understand the "or" in the inclusive sense.

Once again the function $cc$ is utilized to represent the assumption introduced by Axiom 4.

If $A$ and $B$ are two sets then

$$cc(i,x,A\cup B) = cc(i,x,A) + cc(i,x,B) - cc(i,x,A)cc(i,x,B).$$

Based on the former axioms it is possible to define the degree of agreement for operations on sets. Suppose that $N$ individual agents are asked concerning the membership of $x$ in a set $A$.

**Definition 2** (*Degree of agreement*). The degree of agreement $ac(x\in A)$ among $N$ individual agents relatively to the membership of $x$ in a set $A$ is defined by the proportion of the number of agents agreeing that $x$ belongs to $A$, i.e.

$$ac(x\in A) = \frac{\sum_{i=1}^{N} cc(i,x,A)}{N}.$$

This definition of the concept of agreement among a group of individual agents assumes basically that the process is *democratic*, i.e. (i) the group includes a relevant sample of individual agents, (ii) the agents are not influenced by others, (iii) all agents have the same relative importance in terms of the agreement calculus, and (iv) the agreement value depends only of the agents' opinion.

Using this definition, the extension of the concept of agreement for set operations can be easily performed.

**Proposition 1** (Degree of agreement for the complement of a set). *The degree of agreement $ac(x\in\bar{A})$ among N individual agents w.r.t. the membership of x in a set $\bar{A}$ is*

$$\begin{aligned} ac(x\in\bar{A}) &= \frac{\sum_{i=1}^{N} cc(i,x,\bar{A})}{N} \\ &= \frac{\sum_{i=1}^{N} 1 - cc(i,x,A)}{N} \\ &= 1 - ac(x\in A). \end{aligned}$$

**Proposition 2** (Degree of agreement for the intersection of two sets). *The degree of agreement $ac(x\in A\cap B)$ among N individual agents w.r.t. the membership of x in a set $A\cap B$ is*

$$ac(x\in A\cap B) = \frac{\sum_{i=1}^{N} cc(i,x,A)cc(i,x,B)}{N}.$$

**Proposition 3** (Degree of agreement for the union of two sets). *The degree of agreement $ac(x\in A\cup B)$ among N individual agents w.r.t. the membership of x in a set $A\cup B$ is*

$$ac(x\in A\cup B) = \frac{\sum_{i=1}^{N} cc(i,x,A) + cc(i,x,B) - cc(i,x,A)cc(i,x,B)}{N}.$$

The following definition establishes a function that assesses the covariance between the two collection of answers obtained concerning the membership of an element $x$ in a set $A$ and in a set $B$. It measures the quantity of individual agents giving the same answer for both questions and the quantity of agents giving different answers.

**Definition 3** (*Covariance*). A covariance function in terms of agreement is defined as

$$\mathrm{cov}(x \in A, x \in B) = \frac{\sum_{i=1}^{N}[cc(i,x,A) - ac(x \in A)][cc(i,x,B) - ac(x \in B)]}{N}.$$

Now the following propositions can be proved: [3]

**Proposition 4.**

$$ac(x \in A \cap B) = ac(x \in A)ac(x \in B) + \mathrm{cov}(x \in A, x \in B). \tag{1}$$

**Proposition 5.**

$$ac(x \in A \cup B) = ac(x \in A) + ac(x \in B) - ac(x \in A \cap B). \tag{2}$$

**Proposition 6.**

$$ac(x \in A \cap B) \leqslant ac(x \in A) \text{ (or } ac(x \in B)) \leqslant ac(x \in A \cup B). \tag{3}$$

**Proposition 7.**

$$\mathrm{cov}(x \in A, x \in A) = ac(x \in A)[1 - ac(x \in A)]. \tag{4}$$

**Proposition 8.**

$$\mathrm{cov}(x \in A, x \in \bar{B}) = -\mathrm{cov}(x \in A, x \in B). \tag{5}$$

**Proposition 9.**

$$\mathrm{cov}(x \in \bar{A}, x \in \bar{B}) = \mathrm{cov}(x \in A, x \in B). \tag{6}$$

**Proposition 10.**

$$\begin{aligned} &ac(x \in A)ac(x \in B) + ac(x \in A)ac(x \in \bar{B}) \\ &\quad + ac(x \in \bar{A})ac(x \in B) + ac(x \in \bar{A})ac(x \in \bar{B}) = 1. \end{aligned} \tag{7}$$

[3] For sake of paper length the proofs of the following propositions are not included. However, all of them are available in Ref. [6].

**Proposition 11.**

$$ac(x \in A \cap C) = \operatorname{cov}(x \in A, x \in C) + \frac{\operatorname{cov}(x \in A, x \in B) - ac(x \in A \cap B)}{ac(x \in B)} \times \frac{\operatorname{cov}(x \in B, x \in C) - ac(x \in B \cap C)}{ac(x \in B)} \quad \text{if } ac(x \in B) > 0. \tag{8}$$

**Proposition 12.**

$$|\operatorname{cov}(x \in A, x \in B)| \leqslant 0.25. \tag{9}$$

## 4. The pencil of Sheridan-fine: a simple example

In this section, an example suggested by Sheridan in a reply to Dubois and Prade [8] is presented. The example is particularly interesting because it raises some of the difficulties found when Fuzzy Set Theory is used.

Sheridan writes:

"Say we have a pencil that is fairly red, and fairly orange. I claim that it is at least *conceivable* that it is very red or orange. (In fact there is such a pencil, but that does not matter.) With the "most popular" choice of operatos (Dubois and Prade's equations (14) and (15)) this is impossible: the pencil is fairly red or orange ($t(P \vee Q) = \max(t(P), t(Q))$. $t(P)$ is the degree of truth of the proposition $P$.)".

(Sheridan's emphasis; Eq. (14) and (15) correspond to max and min operators, respectively).

The purpose of presenting this example is twofold: on the one hand, it will be used to demonstrate that the agreement-based operators do not exhibit the type of problems pointed out by Sheridan and, on the other hand, to illustrate the two possible ways individuals use for categorizing objects – best fit or subsumption (see Section 3). Suppose that $N$ agents were questioned concerning the acceptance of the membership of $p$ in the set of orange pencils $O$ and in the set of red pencils $R$. To ease the understanding of this example, assume that the following labels are associated with degrees of agreement: 0.6 "slightly", 0.7 "fairly", 0.8 "strongly", 0.9 "very", and 1.0 "completely".

Consider first the subsumption approach. In this case,

$$ac(p \in O) = 0.7, \qquad ac(p \in R) = 0.7.$$

That is to say, 70% of the individual agents consider that the pencil belongs to the set of orange pencils (the commonsensical agent considers it fairly orange), and the same proportion of the individuals agents consider that the pencil belongs to the set of red pencils (the commonsensical agent considers it fairly red). This means that there are individual agents considering that the pencil belongs to both sets – red and orange. In fact, at least 40% of them, or at most 70% of them. [4] This is not surprising as the pencil is reddish orange (the only explanation for the answers got) and because of the approach taken – subsumption. On the other hand, as a direct consequence of proposition 2, it is not possible to find the degree of agreement for a disjunction only on the basis of the degrees of agreement of each one of the disjuncts: it is also necessary to know the degree of agreement for the conjunction. (Union and intersection are the two side of the very same coin). For those who are used to evaluate membership degrees utilizing the max and min operators this conclusion could seem a bit complex; however, this is the price to pay on being precise. Particularly, in this example, the utilization of max leads to the (strange) conclusion that the pencil is also fairly red or orange. Now suppose that the degree of agreement for the membership of $p$ in the set $O \cap R$ is 0.5, i.e. $ac(p \in O \cap R) = 0.5$. In other words, 50% of the individual agents consider the pencil simultaneously red and orange. (This means that there is a maximum disagreement concerning that $p \in O \cap R$). Based on expression 2, it is possible to determine $ac(p \in O \cup R)$.

$$\begin{aligned} ac(p \in O \cup R) &= ac(p \in O) + ac(p \in R) - ac(p \in O \cap R) \\ &= 0.7 + 0.7 - 0.5 = 0.9. \end{aligned}$$

Then it can be concluded that the pencil is very red or orange (the conclusion suggested by Sheridan). As the union (disjunction) operation depends on the intersection (conjunction) one, it is possible to obtain different grades of agreement for $p \in O \cup R$. For instance, if the conjunction degree of agreement is 0.4, the disjunction degree is 1.0, meaning that the pencil is completely red or orange. On the other hand, if the conjunction degree of agreement is 0.7, the disjunction degree of agreement is 0.7. Only in this case the result given by the fuzzy operator *max* equals the one obtained by agreement-based operators.

The mentioned conclusions deserve some comments. First of all, how can be explained that the degree of agreement of the disjunction is less than 1.0? The only explanation is that there are individual agents considering simultaneously that the pencil does not belong to the set $O$ and does not belong to the set $R$. Perhaps they consider the pencil as yellow. Possibly there are individual agents considering that the color of the pencil is reddish-orange and decided not to accept the only two labels provided: red, orange. This situation is strange as agents were advised to use the subsumption approach. Recall that in this approach, individual agents assume that there could be an overlapping between sets and they should answer yes to both membership questions when the

[4] From Proposition 2, $ac(x \in A \cup B) = ac(x \in A) + ac(x \in B) - ac(x \in A \cap B) \leqslant 1$, it can be derived that $ac(x \in A \cap B) \geqslant ac(x \in A) + ac(x \in B) - 1$, and so $ac(x \in A \cap B) \geqslant 0.4$.

element belongs to the intersection. Notice that, due to the fact that the perception mechanisms of individuals are different, allows to getting, for the very same pencil, the answers "red", "orange", and "red and orange".

Consider now the best fit approach. In this case individual agents will assign one and only one color label to the pencil. So, the sum of the degrees of agreement for both colors cannot be higher than 1.0.

Suppose that the degrees of agreement for both membership relations, $p \in O$ and $p \in R$, is 0.5. This means that there is a maximum disagreement among individual agents and, therefore, they do not provide any useful information for discriminating between red and orange as the color of the pencil. As the degree of agreement for the conjunction is zero (best fit approach), using proposition 2 it is obtained,

$$\begin{aligned} ac(p \in O \cup R) &= ac(p \in O) + ac(p \in R) - ac(p \in O \cap R) \\ &= 0.5 + 0.5 - 0 = 1. \end{aligned}$$

That is to say, the pencil is definitely red or orange. This is an interesting case of maximum ignorance on the disjuncts (impossibility of discriminating between red and orange) and minimum ignorance on the disjunction (it is certain that the pencil is either red or orange).

Of course, depending on the color of the pencil, it is possible to obtain, say, $ac(p \in O) = 0.3$ and $ac(p \in R) = 0.7$. In this case, the commonsensical agent considers the pencil as fairly red, and fairly not orange. But, it is also possible that $ac(p \in O) = 0.4$ and $ac(p \in R) = 0.4$. In this case there are, possibly, agents who think that the label reddish-orange is a better classification for $p$. As individual agents are answering under a best fit approach, the commonsensical agent is not completely sure that the pencil is either orange or red ($ac(p \in O \cup R) = 0.8$).

It should be stressed that the agreement-based operators perform correctly in both cases – best fit and subsumption – provided that the individual agents be advised to maintain consistency in their judgements. It is claimed that the formal semantics which results from the agreement approach delivers conclusions that are acceptable in a commonsense framework. In other words, the formal semantics fits the informal one – in a nut shell, it makes sense.

## 5. Consequences of the agreement-based semantics

Interpreting the grade of membership $\mu_A(x)$ of an element $x$ in a fuzzy set $A$ as the degree of agreement among $N$ individual agents w.r.t. the membership of $x$ in $A$, demonstrates that the operators usually used in fuzzy set theory should be re-evaluated.

The most popular choice of operators for intersection and union is min and max, respectively [7,29,26]. Based on the new interpretation of grade of membership it is clear that both operators are not adequate considering Propositions 1 and 2. Also the pair product/probabilistic sum has been frequently used as operators for those operations. However, based on the agreement definitions, this pair is also inadequate to represent the intersection and union of fuzzy sets. Notice that only when the covariance is zero, the intersection is given by the product.

With the agreement interpretation of the meaning of vague concepts, it is possible to define a rigorous but clear semantics of a logic involving classic and non-classic propositions. A classic one is represented by a degree of agreement that it is one (or zero), i.e., all the agents agree (or disagree) on the membership of an element in a class.

Since it is supposed that each agent is a rational individual using first order logic to reason about the world [24], the logic of agreement is an extension of the classical logic. When propositions involved are classic ones (0,1), the logic provides the same results as the classical logic, giving only two logical values: true and false.

Moreover, each proposition is logically evaluated by a set of $N$ individual agents generating a degree of agreement in the interval $[0, 1]$. Based on the proof theory, presented in Section 7, this logic is capable of inferring new propositions, with the corresponding degrees of agreement.

The key issue in what regards reasoning under agreement is to determine whether the mechanisms of derivation used by people are sound with respect to a given semantics. In other words, for instance, if a group of individuals agrees, in some extent, with propositions $A$, $A \rightarrow B$, and $B$, (given a degree of agreement for each proposition), then, there should be an inference system capable of deriving, say $B$ (and the corresponding degree of agreement) given $A$, and $A \rightarrow B$. If such an inference system would not exist, then it would be impossible to reason under agreement (at least with some degree of accuracy). This should be a very puzzling conclusion as people usually derive conclusions from premises stated under agreement. For instance, most of the people agrees that Spielberg movies are good. When a new Spielberg movie comes up, people should agree, at least a priori, that it should be good. Of course, the degree of agreement of the former proposition should positively influence the degree of the latter.

In terms of inference, the logic of agreement is based on a natural deduction system using therefore only rules of inference. It takes propositions and their respective degrees of agreement and it generates other propositions and the corresponding degrees of agreement. It works as multi-valued logic through the manipulation of degrees of agreement. By comparison with fuzzy logic, the logic of agreement is less expressive, since the former uses membership functions in its inference process and it allows the inference of new membership functions. However, this process is justified only in mathematical terms using the idea of fuzzy relation as an extension of classic (crisp) relation [7]. As there is no semantic system, there are no semantic justifications for this kind of inference methodology. Besides that there are several multi-valued logics [7] each one based on its own set theory, i.e., each one satisfying a particular set of axioms, but none having a semantic framework to justify its utilization.

## 6. Semantic system

In order to create the semantics of a logic, it is necessary to establish a conceptualization of reality, to define the relevant objects of the world for a specific problem, as well as to establish the relations among the objects. The set of objects of the conceptualization is called universe of discourse.

Since the existence of several individual agents is assumed, the semantic system incorporates the perspective that each agent defines her/his own conceptualization of reality. However, when agents are asked about a property of an object, each one conceptualizes the property differently, but the object involved should be the same so the aggregation of answers would make sense. For instance, if agents are asked whether John is tall, all the agents must be considering the same John. The same hypothesis is assumed for functions. For instance, when the question involves the son of Peter (for example, John), all agents should consider the same John. Finally for the relations of the conceptualizations of reality, it is assumed that the agents have total freedom of conceptualizing each relevant property. Therefore, supposing that there are $N$ agents, the semantic system considers $N$ conceptualizations of reality, with the same objects and functions, but (possibly) different relations. Prior to the full definition of the semantic system, it is essential to have a formal language allowing the representation of propositions about the conceptualized world. These propositions are represented by formulas of the language, according to a specific alphabet and certain rules of formation.

The evaluation of a formula depends on the interpretation given to each element in the formula, i.e., it depends on the relation between the elements of the language and the elements of the conceptualization – objects, functions and relations. As usual, the concept of interpretation $I$ is defined as the mapping from the elements of the language to the elements of the conceptualization.

Since there are different conceptualizations, it is also assumed that there are different interpretations $I_i$, one for each agent. As the objects and functions are the same in all conceptualizations, all the interpretations map an object or function constant of the language into the same objects or functions of the conceptualization.

Each agent should conceptualize a property by a classical relation through the definition of a classical set of objects that satisfy the property. Fig. 1 shows a graphical representation of the connection among conceptualizations, interpretations and language.

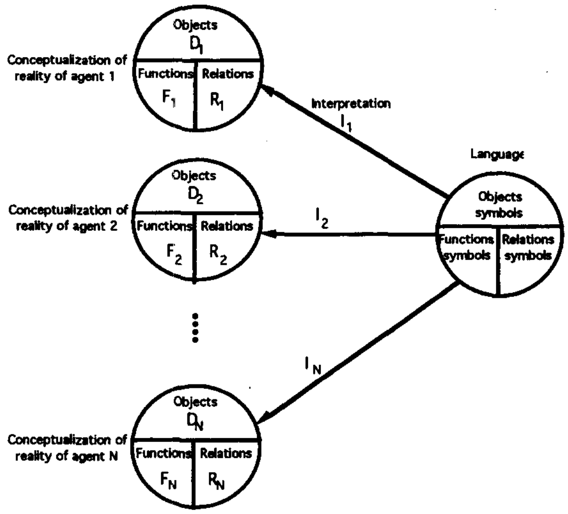


Fig. 1. Representation of the concept of interpretation. All sets of objects and functions are identical, i.e. $D_i = D_j$ and $F_i = F_j$.

**Definition 4** (*Interpretation*). An interpretation $I_i$ (used by the agent $\mathscr{A}_i$) is a mapping from the elements of the language to the elements of the conceptualization satisfying the following properties:

(i) if $c$ is a constant symbol of object, then $I_i(c) \in D$ and $\forall_j\, I_i(c) = I_j(c)$, where $D$ represents the universe of discourse common to all conceptualizations.

(ii) if $f$ is a constant symbol of function, with rank $n$, then $I_i(f) : D^n \to De$ $\forall_j\, I_i(f) = I_j(f)$.

(iii) if $p$ is a constant symbol of relation, with rank $n$, then $I_i(p) \subseteq D^n$.

Now it is possible to define the concept of *concordance*, denoted by $\models_{I_i} \phi$, representing the fact that the individual agent using the interpretation $I_i$ accepts the proposition represented by the formula $\phi$.

**Definition 5** (*Concordance*).

(i) an agent $\mathscr{A}_i$ concords with the proposition represented by the atomic formula $p(c_1, \ldots, c_n)$, according to her/his interpretation $I_i$, if and only if $\langle I_i(c_1), \ldots, I_i(c_n)\rangle \in I_i(p)$, i.e.,

$$\models_{I_i} p(c_1, \ldots, c_n) \;\text{ iff }\; \langle I_i(c_1), \ldots, I_i(c_n)\rangle \in I_i(p).$$

(ii) an agent $\mathscr{A}_i$ concords with the proposition represented by the formula $\neg\phi$, according to her/his interpretation $I_i$, if and only if the agent $A_i$ does not concord with the proposition represented by the formula $\phi$, i.e.,

$$\models_{I_i} \neg\phi \;\text{ iff }\; \not\models_{I_i}\phi.$$

(iii) an agent $\mathscr{A}_i$ concords with the proposition represented by the formula $\phi \wedge \psi$, according to his/her interpretation $I_i$, if and only if the agent $A_i$ concords with the proposition represented by the formula $\phi$ and concords with the proposition represented by the formula $\psi$, i.e.,

$$\models_{I_i} \phi \wedge \psi \quad \text{iff} \quad \models_{I_i} \phi \text{ and } \models_{I_i} \psi.$$

(iv) an agent $\mathscr{A}_i$ concords with the proposition represented by the formula $\phi \vee \psi$, according to her/his interpretation $I_i$, if and only if the agent $A_i$ concords with the proposition represented by the formula $\phi$ or concords with the proposition represented by the formula $\psi$ (or both), i.e.,

$$\models_{I_i} \phi \vee \psi \quad \text{iff} \quad \models_{I_i} \phi \text{ or } \models_{I_i} \psi.$$

(v) an agent $\mathscr{A}_i$ concords with the proposition represented by the formula $\phi \rightarrow \psi$, according to her/his interpretation $I_i$, if and only if the agent $A_i$ does not concord with the proposition represented by the formula $\phi$ or concords with the proposition represented by the formula $\psi$, i.e.,

$$\models_{I_i} \phi \rightarrow \psi \quad \text{iff} \quad \not\models_{I_i} \phi \text{ or } \models_{I_i} \psi.$$

(vi) an agent $\mathscr{A}_i$ concords with the proposition represented by the formula $\phi \leftrightarrow \psi$, according to his interpretation $I_i$, if and only if the agent $A_i$ concords with the proposition represented by the formula $\phi \wedge \psi$ or concords with the proposition represented by the formula $\neg\phi \wedge \neg\psi$, i.e.,

$$\models_{I_i} \phi \leftrightarrow \psi \quad \text{iff} \quad \models_{I_i} \phi \wedge \psi \text{ or } \models_{I_i} \neg\phi \wedge \neg\psi.$$

The concordance concept can be defined by a mathematical function $conc : I \times \mathscr{L} \rightarrow \{0, 1\}$, where $I$ is the set of interpretations and $\mathscr{L}$ is the set of formulas. The new form of the former definition is as below:

**Definition 6** (*Function conc*).

(i)

$$\mathrm{conc}(I_i, p(c_1, \ldots, c_n)) = \begin{cases} 1 & \text{if} \langle I_i(c_1), \ldots, I_i(c_n) \rangle \in I_i(p). \\ 0 & \text{otherwise,} \end{cases}$$

(ii)

$$\mathrm{conc}(I_i, \neg\phi) = 1 \ \text{ iff } \ \mathrm{conc}(I_i, \phi) = 0.$$

(iii)

$$\mathrm{conc}(I_i, \phi \wedge \psi) = 1 \ \text{ iff } \ \mathrm{conc}(I_i, \phi) = 1 \ \text{ and } \ \mathrm{conc}(I_i, \psi) = 1$$

i.e.,

$$\mathrm{conc}(I_i, \phi \wedge \psi) = \mathrm{conc}(I_i, \phi)\, \mathrm{conc}(I_i, \psi).$$

(iv)

$$\mathrm{conc}(I_i, \phi \vee \psi) = 1 \ \text{ iff } \ \mathrm{conc}(I_i, \phi) = 1 \ \text{ or } \ \mathrm{conc}(I_i, \psi) = 1$$

i.e.,

$$\mathrm{conc}(I_i, \phi \vee \psi) = \mathrm{conc}(I_i, \phi) + \mathrm{conc}(I_i, \psi) \\ - \mathrm{conc}(I_i, \phi)\, \mathrm{conc}(I_i, \psi).$$

(v)

$$\mathrm{conc}(I_i, \phi \rightarrow \psi) = 1 \text{ iff } \mathrm{conc}(I_i, \phi) = 0 \text{ or } \mathrm{conc}(I_i, \psi) = 1$$

i.e.,

$$\mathrm{conc}(I_i, \phi \rightarrow \psi) = \mathrm{conc}(I_i, \neg\phi \vee \psi).$$

(vi)

$$\mathrm{conc}(I_i, \phi \leftrightarrow \psi) = 1 \text{ iff } \mathrm{conc}(I_i, \phi \wedge \psi) = 1 \text{ or } \mathrm{conc}(I_i, \neg\phi \wedge \neg\psi) = 1$$

i.e.,

$$\begin{aligned} \mathrm{conc}(I_i, \phi \leftrightarrow \psi) &= \mathrm{conc}(I_i, (\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)) \\ &= \mathrm{conc}(I_i, \phi \wedge \psi) + \mathrm{conc}(I_i, \neg\phi \wedge \neg\psi). \end{aligned}$$

When an individual agent creates a conceptualization of reality, it can be said that the agent defines a possible world. Assuming $N$ agents, $N$ possible worlds are created, each one associated with a particular interpretation. Let $W = \{w_1, \ldots, w_N\}$ be the set of possible worlds created by $N$ agents.

**Definition 7.** Let $w$ be the possible world defined by an individual agent $\mathscr{A}$. A function $In : W \rightarrow I$ associating the possible world $w$ with the interpretation $In(w)$ used by the agent $\mathscr{A}$ is defined. Then, $In(W) = \{I_1, \ldots, I_N\} = I$ is the set of all interpretations.

In order to aggregate the information provided by $N$ agents, the concept of (agreed) satisfaction, in the logic of agreement (LA), is defined.

Let $A$ be a formula of the language and $\alpha$ a real number in the interval [0,1]. If the formula $A$ is satisfied (agreed) by a degree $\alpha$, in the set $W$ of possible worlds, this is denoted by $W \models_{\alpha}^{\mathrm{LA}} A$.

**Definition 8** (*Satisfaction*). The set $W$ of possible worlds satisfies with a degree $\alpha$ a formula in the following cases:

(1) $W \models_{\alpha}^{\mathrm{LA}} A$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\models_{In(w_i)} A$, where $A$ is an atomic formula, i.e.,

(1.1) $W \models_{\alpha}^{\mathrm{LA}} A$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\mathrm{conc}(In(w_i), A) = 1$, i.e.,

(1.2) $W \models_{\alpha}^{\mathrm{LA}} A$ iff

$$\frac{\sum_{i=1}^{N} \mathrm{conc}(In(w_i), A)}{N} = \alpha,$$

(2) $W \models_{\alpha}^{\mathrm{LA}} \neg A$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\models_{In(w_i)} \neg A$, i.e.,

(2.1) $W \models_{\alpha}^{\mathrm{LA}} \neg A$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\mathrm{conc}(In(w_i), A) = 0$, i.e., (2.2) $W \models_{\alpha}^{\mathrm{LA}} \neg A$ iff

$$\frac{\sum_{i=1}^{N} \mathrm{conc}(In(w_i), A)}{N} = 1 - \alpha,$$

(2.3) $W \models_{\alpha}^{\mathrm{LA}} \neg A$ iff $W \models_{1-\alpha}^{\mathrm{LA}} A$,
(3) $W \models_{\alpha}^{\mathrm{LA}} A \wedge B$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\models_{In(w_i)} A \wedge B$, i.e.,
(3.1) $W \models_{\alpha}^{\mathrm{LA}} A \wedge B$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\mathrm{conc}(In(w_i), A) = 1$ and $\mathrm{conc}(In(w_i), B) = 1$, i.e., $\mathrm{conc}(In(w_i), A)\ \mathrm{conc}(In(w_i), B) = 1$, i.e.,
(3.2) $W \models_{\alpha}^{\mathrm{LA}} A \wedge B$ iff

$$\frac{\sum_{i=1}^{N} \mathrm{conc}(In(w_i), A)\ \mathrm{conc}(In(w_i), B)}{N} = \alpha,$$

(4) $W \models_{\alpha}^{\mathrm{LA}} A \vee B$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\models_{In(w_i)} A \vee B$, i.e.,
(4.1) $W \models_{\alpha}^{\mathrm{LA}} A \vee B$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\mathrm{conc}(In(w_i), A) = 1$ or $\mathrm{conc}(In(w_i), B) = 1$, i.e., $\mathrm{conc}(In(w_i), A) + \mathrm{conc}(In(w_i), B) - \mathrm{conc}(In(w_i), A)\ \mathrm{conc}(In(w_i), B) = 1$, i.e.,
(4.2) $W \models_{\alpha}^{\mathrm{LA}} A \vee B$ iff

$$\frac{\sum_{i=1}^{N} \mathrm{conc}(In(w_i), A) + \mathrm{conc}(In(w_i), B) - \mathrm{conc}(In(w_i), A)\ \mathrm{conc}(In(w_i), B)}{N} = \alpha,$$

(5) $W \models_{\alpha}^{\mathrm{LA}} A \rightarrow B$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\models_{In(w_i)} A \rightarrow B$, i.e.,
(5.1) $W \models_{\alpha}^{\mathrm{LA}} A \rightarrow B$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\mathrm{conc}(In(w_i), A) = 0$ or $\mathrm{conc}(In(w_i), B) = 1$, i.e., $1 - \mathrm{conc}(In(w_i), A) + \mathrm{conc}(In(w_i), A)\ \mathrm{conc}(In(w_i), B) = 1$, i.e.,
(5.2) $W \models_{\alpha}^{\mathrm{LA}} A \rightarrow B$ iff

$$\frac{\sum_{i=1}^{N} 1 - \mathrm{conc}(In(w_i), A) + \mathrm{conc}(In(w_i), A)\ \mathrm{conc}(In(w_i), B)}{N} = \alpha,$$

(6) $W \models_{\alpha}^{\mathrm{LA}} A \leftrightarrow B$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\models_{In(w_i)} A \leftrightarrow B$, i.e.,
(6.1) $W \models_{\alpha}^{\mathrm{LA}} A \leftrightarrow B$ iff exist in $W$, $\alpha N$ worlds $w_i$ such that $\mathrm{conc}(In(w_i), A \wedge B) = 1$ or $\mathrm{conc}(In(w_i), \neg A \wedge \neg B) = 1$, i.e., $1 - \mathrm{conc}(In(w_i), A) - \mathrm{conc}(In(w_i), B) + 2\,\mathrm{conc}(In(w_i), A)\ \mathrm{conc}(In(w_i), B) = 1$, i.e.,
(6.2) $W \models_{\alpha}^{\mathrm{LA}} A \leftrightarrow B$ iff

$$\frac{\sum_{i=1}^{N} 1 - \mathrm{conc}(In(w_i), A) - \mathrm{conc}(In(w_i), B) + 2\,\mathrm{conc}(In(w_i), A)\ \mathrm{conc}(In(w_i), B)}{N} = \alpha.$$

**Definition 9** (*Logical value*). If a formula $A$ is satisfied in a value $\alpha$, by the set of possible worlds $W$, then it is said that $\alpha$ is the logical value of the formula $A$.

Next it will be defined the concept of collective or simultaneous satisfaction, that is to say the satisfaction of a set of formulas.

**Definition 10** (*Collective or simultaneous satisfaction*). A set of formulas $\{P_1, \ldots, P_m\}$ is satisfied by the set of possible worlds $W$, according to the set of logical values $\bar{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$, denoted by $W \models^{\mathrm{LA}}_{\bar{\alpha}} \{P_1, \ldots, P_m\}$, iff

$$W \models^{\mathrm{LA}}_{\alpha_i} P_i \quad \forall_i : i = 1, \ldots, m$$

i.e., the set of possible worlds satisfies the formula $P_i$ with the corresponding logical value $\alpha_i$.

**Definition 11** (*Logical consequence*). Let $\Gamma = \{P_1, \ldots, P_m\}$ be a set of formulas and $\bar{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$ a set of logical values, where $\alpha_i$ is the logical value of the formula $P_i$. Let $A$ also be a formula with the corresponding logical value $\alpha$. It is said that $A$ is a logical consequence of $\Gamma$, according to $\bar{\alpha}$ and $\alpha$, denoted by $\Gamma \models^{\mathrm{LA}}_{\bar{\alpha},\alpha} A$, if for the set of possible worlds $W$ $\Gamma$ has a collective satisfaction $\bar{\alpha}$, then $A$ is satisfied in the value $\alpha$.

## 7. Proof theory

The syntactic system of the logic of agreement is a *natural deduction* system: Natural deduction systems include, instead of logical axioms, as usual, a set of inference rules, normally two per connective (one for introduction and another for elimination). To start a proof, it is usually provided a "rule of hypothesis" which allows the introduction of assumptions that will be the premises of the derived conclusions.

The present natural deduction system has several predecessors, namely the systems of Lemmon [21], Fitch [11], Martins and Shapiro [22], and Pinto-Ferreira and Martins [25], among others. One of the major differences between the present proof theory and others is that some rules of inference need three (instead of two) formulas (premises) to infer a consequence. For instance, consider the Modus Ponens rule. In the logic of agreement, it is not enough $A$ and $A \rightarrow B$ to infer $B$. In fact, it is also needed $A \leftrightarrow B$, a "measure" of the proportion of agents having the same opinion (concordance or not) about $A$ and $B$. This could seem strange at first sight; however, to infer under agreement more information is needed.

The proof theory of the logic of agreement will be defined in terms of three aspects: alphabet, rules of formation and rules of inference.

### 7.1. Alphabet

The alphabet defines all symbols that could and will be used to establish the proof theory.

**Definition 12** (*Alphabet*). The alphabet of the syntactic system of the logic of agreement is the union of the following disjoint sets:

(i) $\mathscr{P}$: non-empty set of predicate symbols. Each element of $\mathscr{P}$ has associated a non-negative number called rank. $\mathscr{P}^0$ is the set of proposition symbols.

(ii) $\mathscr{F}$: set of function symbols. Each element of $\mathscr{F}$ has associated a non-negative number called rank. $\mathscr{F}^0$ is the set of constant symbols. This set includes all the symbols representing objects of the universe of discourse.

(iii) Set of connectives: $\{\neg, \rightarrow, \wedge, \vee, \leftrightarrow\}$.

(iv) Set of punctuation symbols: $\{(,),,\}$.

### 7.2. Rules of formation

This section presents rules that must be followed in order to define formulas accepted in the language of the logic of agreement. Since this logic is for now a zero-order logic, the rules of formation allow only the definition of zero-order formulas.

**Definition 13** (*Term*). A term $v$ is an element of the set $\mathscr{F}^0$ or $v = \phi(x_1, \ldots, x_n)$, where $\phi \in \mathscr{F}^n$ and $x_i$ is a term, $i = 1, \ldots, n$.

**Definition 14** (*Rules of formation*). The rules of formation of formulas of the logic of agreement are the following ones:

(i) if $\beta \in \mathscr{P}^n$ and $x_i$ is a term, $i = 1, \ldots, n$, then $\beta(x_1, \ldots, x_n)$ is a well-formed formula (wff) called atomic formula.

(ii) is $A$ and $B$ are well-formed formulas, then $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \leftrightarrow B$ are also well-formed formulas.

(iii) Nothing else is a well-formed formula.

Natural deduction systems allow the establishment of a distinction between hypothesis (assumptions) and derived formulas. As mentioned, assumptions are introduced by a rule of hypothesis and derived formulas are the ones inferred through the application of inference rules. Certain rules of inference demand the knowledge about which hypothesis underlie a given conclusion. This implies that the computational system which implements the proof theory ought to have a *dependency recording* mechanism, that is to say, to each derived formula is associated the set (or sets) of premises which were utilized to reach the conclusion. (In fact, this dependency record is one of the crucial aspects of the Assumption Based Truth Maintenance Systems – ATMS – see De Kleer [18], and Martins and Shapiro [22].)

On the other hand, as inference in the present proof theory should provide not only a derived formula but also its degree of credibility (a value in the interval [0,1] which is the counterpart of the degree of agreement in the proof theory), the inference system should process not only formulas but also *supported formulas*. Supported formulas are aggregated objects including a formula and the degree of credibility.

**Definition 15** (*Supported formula*). A supported formula is a pair $\langle A, cr\rangle$, where $A$ is a well-formed formula and $cr$ is the corresponding degree of credibility.

*7.3. Rules of inference*

This section introduces rules of inference allowing the derivation of a formula from a set of others. As any traditional natural deduction system, the logic of agreement includes typically two rules for each logical connective plus two rules for introducing and eliminating hypothesis and a *Reductio ad Absurdum* rule. Therefore, its rules of inference are as following (as usual, the symbol $\vdash$ means "infer"):

**Rule of Hypothesis Introduction (HIP-I).** At any point of a proof it is possible to introduce an hypothesis $\langle A, cr\rangle$.

**Rule of Hypothesis Elimination (HIP-E).** This rule allows the elimination of hypothesis, i.e.,

$$\begin{array}{ll} \text{If} & \{\langle A_1, cr_1\rangle, \ldots, \langle A_n, cr_n\rangle, \langle A, x\rangle\} \vdash \langle B, cr\rangle \\ \text{and} & cr \text{ does not depend on } x \\ \text{then} & \{\langle A_1, cr_1\rangle, \ldots, \langle A_n, cr_n\rangle\} \vdash \langle B, cr\rangle \end{array}$$

**Rule of Implication Introduction (→I).** Given a supported formula $\langle A, cr_A\rangle$, a supported formula $\langle A \leftrightarrow B, cr\rangle$ and a supported formula $\langle B, cr_B\rangle$, this rule allows to derive the formula $A \rightarrow B$ with the corresponding support.

$$\frac{\begin{array}{c}\langle A, cr_A\rangle \\ \langle A \leftrightarrow B, cr\rangle \\ \langle B, cr_B\rangle\end{array}}{\langle A \rightarrow B, \frac{1 - cr_A + cr + cr_B}{2}\rangle}$$

In order to guarantee consistency among formulas used as premises, the following condition should be satisfied:

$$\begin{aligned} &\max[1 - cr_A - cr_B, -(1 - cr_A - cr_B)] \\ &\quad \leqslant cr \leqslant \min[cr_A - cr_B + 1, cr_B - cr_A + 1]. \end{aligned}$$

This and the following conditions are obtained using a set of theorems presented in Ref. [6].

**Rule of Implication Elimination 1 (MP).** Given $\langle A, cr_A\rangle$, $\langle A \rightarrow B, cr_1\rangle$ and $\langle A \leftrightarrow B, cr_2\rangle$, this rule allows to derive the formula $B$ with the corresponding support.

$$\frac{\begin{array}{c}\langle A, cr_A\rangle \\ \langle A \rightarrow B, cr_1\rangle \\ \langle A \leftrightarrow B, cr_2\rangle\end{array}}{\langle B, 2cr_1 - cr_2 + cr_A - 1\rangle}$$

In order to guarantee consistency among formulas used as premises, the following conditions should be satisfied:

$$1 - cr_A \leqslant cr_1 \leqslant 1$$

and

$$cr_1 + cr_A - 1 \leqslant cr_2 \leqslant cr_1.$$

**Rule of Implication Elimination 2 (MT).** Given $\langle \neg B, cr_{\neg B}\rangle$, $\langle A \rightarrow B, cr_1\rangle$ and $\langle A \leftrightarrow B, cr_2\rangle$, this rule allows to derive the formula $\neg A$ with the corresponding support.

$$\frac{\begin{array}{c}\langle \neg B, cr_{\neg B}\rangle \\ \langle A \rightarrow B, cr_1\rangle \\ \langle A \leftrightarrow B, cr_2\rangle\end{array}}{\langle \neg A, 2cr_1 - cr_2 + cr_{\neg B} - 1\rangle}$$

In order to guarantee consistency among formulas used as premises, the following conditions should be satisfied:

$$1 - cr_{\neg B} \leqslant cr_1 \leqslant 1$$

and

$$cr_1 + cr_{\neg B} - 1 \leqslant cr_2 \leqslant cr_1.$$

**Rule of Negation Introduction (¬I).** Given any supported formula $\langle A, cr\rangle$, this rule allows to derive the negation of the formula $A$ and the corresponding support.

$$\frac{\langle A, cr\rangle}{\langle \neg A, 1 - cr\rangle}$$

**Rule of Negation Elimination (¬E).** Given any supported formula $\langle \neg A, cr\rangle$, this rule allows to derive the formula $A$ and the corresponding support.

$$\frac{\langle \neg A, cr\rangle}{\langle A, 1 - cr\rangle}$$

**Rule of Conjunction Introduction (∧I).** Given $\langle A, cr_A\rangle$, $\langle B, cr_B\rangle$ and $\langle A \leftrightarrow B, cr\rangle$, this rule allows to derive the formula $A \wedge B$ with the corresponding support.

$$\frac{\begin{array}{c}\langle A, cr_A\rangle\\ \langle B, cr_B\rangle\\ \langle A\leftrightarrow B, cr\rangle\end{array}}{\langle A\wedge B, \frac{cr+cr_A+cr_B-1}{2}\rangle}$$

In order to guarantee consistency among formulas used as premises, the following condition should be satisfied:

$$\max[1-cr_A-cr_B, -(1-cr_A-cr_B)] \\ \leqslant cr \leqslant \min[cr_A-cr_B+1, cr_B-cr_A+1].$$

**Rule of Conjunction Elimination ($\wedge$E).** Given $\langle A, cr_A\rangle$, $\langle A\wedge B, cr_1\rangle$ and $\langle A\leftrightarrow B, cr_2\rangle$, this rule allows to derive the formula $B$ with the corresponding support.

$$\frac{\begin{array}{c}\langle A, cr_A\rangle\\ \langle A\wedge B, cr_1\rangle\\ \langle A\leftrightarrow B, cr_2\rangle\end{array}}{\langle B, 2cr_1-cr_2-cr_A+1\rangle}$$

In order to guarantee consistency among formulas used as premises, the following conditions should be satisfied:

$$0\leqslant cr_1\leqslant cr_A$$

and

$$cr_1\leqslant cr_2\leqslant 1-cr_A+cr_1$$

**Rule of Disjunction Introduction ($\vee$I)].** Given $\langle A, cr_A\rangle$, $\langle B, cr_B\rangle$ and $\langle A\leftrightarrow B, cr\rangle$, this rule allows to derive the formula $A\vee B$ with the corresponding support.

$$\frac{\begin{array}{c}\langle A, cr_A\rangle\\ \langle B, cr_B\rangle\\ \langle A\leftrightarrow B, cr\rangle\end{array}}{\langle A\vee B, \frac{cr_A+cr_B-cr+1}{2}\rangle}$$

In order to guarantee consistency among formulas used as premises, the following condition should be satisfied:

$$\max[1-cr_A-cr_B, -(1-cr_A-cr_B)] \\ \leqslant cr \leqslant \min[cr_A-cr_B+1, cr_B-cr_A+1]$$

**Rule of Disjunction Elimination ($\vee$E).** Given $\langle A, cr_A\rangle$, $\langle A\vee B, cr_1\rangle$ and $\langle A\leftrightarrow B, cr_2\rangle$, this rule allows to derive the formula $B$ with the corresponding support.

$$\frac{\begin{array}{c}\langle A, cr_A\rangle\\ \langle A\vee B, cr_1\rangle\\ \langle A\leftrightarrow B, cr_2\rangle\end{array}}{\langle B, 2cr_1+cr_2-cr_A-1\rangle}$$

In order to guarantee consistency among formulas used as premises, the following conditions should be satisfied:

$$cr_A \leqslant cr_1 \leqslant 1$$

and

$$1-cr_1 \leqslant cr_2 \leqslant 1+cr_A-cr_1.$$

**Rule of Equivalence Introduction (↔I).** Given $\langle A, cr_A\rangle$, $\langle B, cr_B\rangle$ and one of the following supported formulas $\langle A\wedge B, cr\rangle$, $\langle A\vee B, cr\rangle$ or $\langle A\rightarrow B, cr\rangle$, this rule allows to derive the formula $A\leftrightarrow B$ with the corresponding support.

$$\frac{\begin{array}{c}\langle A, cr_A\rangle\\ \langle B, cr_B\rangle\\ \langle A\wedge B, cr\rangle\end{array}}{\langle A\leftrightarrow B, 1-cr_A-cr_B+2cr\rangle}$$

In order to guarantee consistency among formulas used as premises, the following condition should be satisfied:

$$\max[0, cr_A+cr_B-1] \leqslant cr \leqslant \min[cr_A, cr_B]$$

$$\frac{\begin{array}{c}\langle A, cr_A\rangle\\ \langle B, cr_B\rangle\\ \langle A\vee B, cr\rangle\end{array}}{\langle A\leftrightarrow B, 1+cr_A+cr_B-2cr\rangle}$$

In order to guarantee consistency among formulas used as premises, the following condition should be satisfied:

$$\max[cr_A, cr_B] \leqslant cr \leqslant \min[1, cr_A+cr_B]$$

$$\frac{\begin{array}{c}\langle A, cr_A\rangle\\ \langle B, cr_B\rangle\\ \langle A\rightarrow B, cr\rangle\end{array}}{\langle A\leftrightarrow B, cr_A-cr_B-1+2cr\rangle}$$

In order to guarantee consistency among formulas used as premises, the following condition should be satisfied:

$$\max[1-cr_A, cr_B] \leqslant cr \leqslant \min[1, 1-cr_A+cr_B]$$

**Rule of *Reductio ad Absurdum* (RAA).** Given a proof of $\langle B, y_1 \rangle$ derived from a set of supported formulas $\Delta = \{\langle A_1, cr_1 \rangle, \ldots, \langle A_n, cr_n \rangle\}$ and a proof of $\langle B, y_2 \rangle$, where $y_2 \neq y_1$ (a contradiction), derived from a set of formulas $\Delta \cup \{\langle A, x \rangle\}$, where $x \neq cr$. Then it is possible to infer $\langle A, cr \rangle$ only from the set $\Delta$.

**Rule of Classical Logic (LC).** Let $\Gamma = \{a_1, \ldots, A_n\}$ a set of formulas. From $\Delta = \{\langle A_1, 1 \rangle, \ldots, \langle A_n, 1 \rangle\}$, if $A$ is derivable from $\Gamma$ using classical logic, then it can be derived $\langle A, 1 \rangle$.

### *7.4. Derivability, soundness and completeness*

In this section, the concepts related to derivability in the logic of agreement are introduced. It is also shown that the logic is sound and complete, which means that a formula that was derived from a set of premises, by the application of inference rules, is a logical consequence of that set, and vice versa, respectively.

**Definition 16** (*Deductive sequence*). A deductive sequence is a finite sequence of supported formulas $P_1, P_2, \ldots, P_n$, where for all $i$, $1 \leqslant i \leqslant n$, $P_i$ is obtained through the application of an inference rule with the premises included in the set $\{P_1, \ldots, P_{i-1}\}$.

**Definition 17** (*Proof*). A proof of a supported formula $P$ from a set of supported formulas $\Delta$ is a deductive sequence $P_1, P_2, \ldots, P_n$, where $P_n = P$ and $\Delta \subseteq \{P_1, \ldots, P_n\}$.

**Definition 18** (*Derivability in the logic of agreement*). If there is a proof of a supported formula $P$ from a set of supported formulas $\Delta$, then it is said that $P$ is derived from $\Delta$, denoted by $\Delta \vdash P$.

The following two theorems show that the logic of agreement is both sound and complete.

**Theorem 1** (Soundness of the logic of agreement). *Let $\Delta = \{\langle A_1, cr_1 \rangle, \ldots, \langle A_n, cr_n \rangle\}$ be a set of supported formulas and $\langle A, cr \rangle$ a supported formula. If $\Delta \vdash \langle A, cr \rangle$ then $\Gamma \models^{\mathrm{LA}}_{\bar{\alpha},\alpha} A$, where $\Gamma = \{A_1, \ldots, A_n\}$, $\bar{\alpha} = \{cr_1, \ldots, cr_n\}$ and $\alpha = cr$.*

**Proof.** The proof is available in Ref. [6].

**Theorem 2** (Completeness of the logic of agreement). *Let $\Gamma = \{A_1, \ldots, A_n\}$ be a set of propositions and $A$ another proposition. If $\Gamma \models^{\mathrm{LA}}_{\bar{\alpha},\alpha} A$, where $\bar{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$, then $\Delta \vdash \langle A, \alpha \rangle$ where $\Delta = \{\langle A_1, \alpha_1 \rangle, \ldots, \langle A_n, \alpha_n \rangle\}$.*

**Proof.** The proof is available in Ref. [6].

## 8. Operationalization issues

A relevant aspect concerning a new theory is its operationalization. Does the agreement approach demand the elaboration of questionnaires to be distributed to a (high) number of individuals in order to get degrees of agreement? Is not it too heavy to have practical applicability? In fact, there are two perspectives. One concerns the field experiments to establish whether the system performs according to the human agreement approach. This phase demands experimentation with a high number of agents and is advisable to be performed (some work has already been done in this direction).

On the other hand, the utilization of the agreement-based approach can be done using a commonsensical agent who provides values for the degrees of agreement. However, she/he should be supported by an expert who should ensure the consistency of her/his assessments. (The commonsensical agent should understand, for instance, that a value near 0.5 means inability of discriminating (ignorance), a value near zero means not belongingness (negation), and so on.)

Of course, this is much more complicate than just using classical theory of sets. However, the semantics of the classical theory of sets is (at least at first sight) very simple and obvious, so it is possible to base inference only on algebraic arguments.

In order to illustrate how the logic of agreement can be utilized to perform inference consider the Sheridan-Fine pencil example discussed previously. Suppose that the knowledge base $\Delta$ is composed by the following formulas:

$$\Delta = \{\langle R, 0.7\rangle, \langle O, 0.7\rangle, \langle R \wedge O, 0.5\rangle\},$$

where $R$ and $O$ denote propositions "the pencil is red" and "the pencil is orange", respectively.

Based on $\Delta$ and the proof theory of the Logic of Agreement it is possible to perform the following inference steps:

1. using the rule of equivalence introduction ($\leftrightarrow I$ i)

$$\frac{\begin{array}{c}\langle R, 0.7\rangle \\ \langle O, 0.7\rangle \\ \langle R \wedge O, 0.5\rangle\end{array}}{\langle R \leftrightarrow O, 1 - 0.7 - 0.7 + 2 * 0.5\rangle}$$

2. using the rule of disjunction introduction ($\vee I$)

$$\frac{\begin{array}{c}\langle R, 0.7\rangle \\ \langle O, 0.7\rangle \\ \langle R \leftrightarrow O, 0.6\rangle\end{array}}{\langle R \vee O, \frac{0.7+0.7-0.6+1}{2}\rangle}.$$

So based on $\Delta$ it is inferred that $\langle R \vee O, 0.9 \rangle$, confirming therefore the result obtained in the Sheridan-Fine pencil example.

## 9. Conclusions

In this paper, a semantic approach to formalizing approximate reasoning, based on the concept of agreement, was introduced. The relation between agreement and fuzzy membership was used to emphasize some aspects of the fuzzy theory that have been criticize by other research communities. For instance, the notion of agreement was utilized to allow a formal semantic definition for the concept of degree of membership. Moreover, some fuzzy set operators were re-evaluated in terms of the new way to define fuzzy membership. Based on the agreement concept, a multi-valued propositional logic – logic of agreement – was developed, in terms of its semantic system and its proof theory which is supported in a natural deduction system. This logic is a extension of the classical logic and it was proved that is sound and complete.

One of the main conclusions of this work is the need for introducing either the covariance in the set operators or the equivalence connective in several inference rules of the proof theory. So, to infer under agreement more information is necessary. Albeit increasing complexity of the inference system, this is what one has to pay in order to guarantee a sound and complete logic.

Finally, although the traditional foundations of FST have been questioned in this paper, the authors consider that the fundamental ideas of FST continue to be relevant, interesting and worth pursuing. The objective was to give a formal semantic explanation for the definition of fuzzy sets, which should improve the confidence on the utilization of fuzzy set based approaches.

## References

[1] J. Baldwin, Evidential reasoning under probabilistic and fuzzy uncertainties, in: Yager and Zadeh (Eds.), An Introduction to Fuzzy Logic Applications in Intelligent Systems, Kluwer Academic Publishers, Dordrecht, 1992.

[2] L. Custódio, C. Pinto-Ferreira, Logic of Agreement, Technical Report ISR/IST 10-95, 1995.

[3] L. Custódio, C. Pinto-Ferreira, Agreement: a logical approach to approximate reasoning, 7th Portuguese Conference on Artificial Intelligence, Funchal, October 1995, also in Lectures Notes on Artificial Intelligence, vol. 990, Springer, Berlin, 1995.

[4] L. Custódio, C. Pinto-Ferreira, A critical review of the fuzzy theory progress, Advanced Manufacturing Forum, vol. 1, Scitec Publications, Trans. Tec Publishers, 1996.

[5] L. Custódio, C. Pinto-Ferreira, Agreement: a new perspective on fuzzy logic foundations, 5th IEEE International Conference on Fuzzy Systems, New Orleans, USA, September 1996.

[6] L. Custódio, A Lógica de Acordo e o Problema da Certeza Vaga, Ph.D. Thesis, Instituto Superior Técnico, 1997.

[7] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academy Press, New York, 1980.

[8] D. Dubois, H. Prade, An introduction to possibilistic and fuzzy logics, in: Smets et al. (Eds.), Non-Standard Logics for Automated Reasoning, Academic Press, New York, 1988.
[9] D. Dubois, H. Prade, Fuzzy Sets – A convenient fiction for modeling vagueness and possibility, IEEE Trans. on Fuzzy Systems 2 (1) (1994) 16–21.
[10] C. Elkan, The paradoxical success of fuzzy logic, Proceedings of the National Conference on Artificial Intelligence, The MIT and AAAI Press, 1993, pp. 698–703 .
[11] F. Fitch, Symbolic Logic: An Introduction, Ronald, New York, 1952.
[12] P. Gardenfors, Knowledge in Flux: Modeling the Dynamics of Epistemic States, MIT Press, Cambridge, MA, 1988.
[13] M. Genesereth, N. Nilsson, Logical Foundations of Artificial Intelligence, Morgan Kaufmann, Los Altos, CA, 1987.
[14] P. Hayes, Some problems and non-problems in representation theory, in: R. Brachman, H. Levesque (Eds.), Readings in Knowledge Representation, Morgan Kaufmann, Los Altos, CA, 1985, pp. 3–22.
[15] H. Hersh, A. Caramazza, A fuzzy set approach to modifiers and vagueness in natural language, Journal of Exp. Psychology: General 105 (1976) 254–276.
[16] E. Hisdal, A theory of logic based on probability: Part 1. The TEE model, Research Report No. 64, Institute of Informatics, University of Oslo, Norway, 1984.
[17] D. Hofstadter, Godel, Escher, Bach: An Eternal Golden Braid, Vintage Books, 1989.
[18] J. de Kleer, An assumption-based truth maintenance system, Artificial Intelligence 28 (2) (1986) 127–162.
[19] G. Klir, On the alleged superiority of probabilistic representation of uncertainty, IEEE Trans. on Fuzzy Systems 2 (1) (1994) 27–31.
[20] M. Laviolette, J. Seaman, The efficacy of fuzzy representations of uncertainty, IEEE Trans. on Fuzzy Systems 2 (1) (1994) 4–15.
[21] E. Lemmon, Beginning Logic, Hackett Publishing Company, 1978.
[22] J. Martins, S. Shapiro, A model for belief revision, Artificial Intelligence 35 (1) (1988) 25–79.
[23] J. McCarthy, Programs with common sense, Proceedings of the Teddington Conference on the Mechanization of Thought Processes, 1959.
[24] R. Omar, Artificial intelligence through logic, AICOM 7 (3/4) (1994).
[25] C. Pinto-Ferreira, J. Martins, The STRICT assumption: a propositional approach to change, Journal of Experimental and Theoretical Artificial Intelligence 5 (1993) 215–224.
[26] R. Yager, Expert systems using fuzzy logic, in: Yager et al. (Eds.), An Introduction to Fuzzy Logic Applications in Intelligent Systems, Kluwer Academic Publishers, Dordrecht, 1992.
[27] L. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338–353.
[28] L. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Trans. Syst. Man and Cyber. 3 (1973) 28–44.
[29] H. Zimmermann, Fuzzy Set Theory and its Applications, Kluwer-Nijhoff Publishing, 1985.