J. Differential Equations 248 (2010) 2736–2745

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On globally nilpotent differential equations

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ARTICLE INFO

Article history:
Received 22 February 2008
Revised 29 July 2009
Available online 3 March 2010

ABSTRACT

In a previous work of the authors, a middle convolution operation on the category of Fuchsian differential systems was introduced. In this note we show that the middle convolution of Fuchsian systems preserves the property of global nilpotence.
This leads to a globally nilpotent Fuchsian system of rank two which does not belong to the known classes of globally nilpotent rank two systems.

0. Introduction

A unifying description of all irreducible and physically rigid local systems on the punctured affine line was given by Katz [9]. The main tool therefore is a middle convolution functor on the category of perverse sheaves [9, Chap. 5]. In [6], the authors give a purely algebraic analogon of this convolution functor. This functor is a functor of the category of finite dimensional K-modules of the free group F_r on r generators to itself (K denoting a field). It depends on a scalar $\lambda \in K^\times$ and is denoted by MC_λ.

By the Riemann–Hilbert correspondence (see [3]), a construction parallel to MC_λ should exist in the category of Fuchsian systems of differential equations. In [7], such a construction is given, leading to a description of rigid Fuchsian systems which is parallel to Katz description of rigid local systems. The convolution depends on a parameter $\mu \in \mathbb{C}$ and carries a Fuchsian system F to another Fuchsian system, denoted by $\mathrm{mc}_\mu(F)$, see Section 1.2.

In this note we study how the $\mathfrak{p}$-curvature (for $\mathfrak{p}$ a prime of a number field K) of a Fuchsian system F having coefficients in the function field $K(t)$ changes under the convolution process. The $\mathfrak{p}$-curvature is a matrix $\bar{C}_\mathfrak{p}(F)$ with coefficients in the function field over a finite field which is obtained from a p-fold iteration of F (where p is the prime number below $\mathfrak{p}$) and reduction modulo $\mathfrak{p}$, see Section 2. The $\mathfrak{p}$-curvature matrices encode many arithmetic and geometric properties of a Fuchsian system. For example, the Bombieri–Dwork conjecture predicts that if the $\mathfrak{p}$-curvature $\bar{C}_\mathfrak{p}(F)$ is nilpo-

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.02.009

tent for almost all primes $\mathfrak{p}$ of K (i.e., F is *globally nilpotent*), then F is arising from geometry, see [1]. In this note we prove the following result (see Theorem 2.6):

Theorem 1. *Let F be a Fuchsian system, let $\mu \in \mathbb{Q}$ and let $\mathrm{mc}_\mu(F)$ be the middle convolution of F with respect to μ. Then the following holds*:

(i) *If $\mathrm{ord}_p(\mu) \geqslant 0$ and if the p-curvature of F is nilpotent of rank k, then the p-curvature of the middle convolution $\mathrm{mc}_\mu(F)$ is nilpotent of rank $r \in \{k-1, k, k+1\}$.*
(ii) *If F is globally nilpotent, then $\mathrm{mc}_\mu(F)$ is globally nilpotent.*

Of course, the second statement of the theorem follows immediately from the first. The second statement can be deduced alternatively from the stability of global nilpotence under pullback, tensor product, higher direct image, see Katz [8, Sections 5.7 through 5.10] (it follows from [7, Thm. 4.7], and [5, Rem. 3.3.7], that mc_μ corresponds under the Riemann–Hilbert correspondence to the middle convolution MC_χ of local systems $\mathcal{V}$ with Kummer sheaves — which is, by construction, a higher direct image sheaf).

The proof of the first statement relies on the closed formula of the p-curvature of Okubo systems (compare to Remark 2.4) and on the fact that the middle convolution of a Fuchsian system is a factor system of an Okubo system.

In Section 3, we apply the middle convolution to the globally nilpotent Fuchsian system of rank two which appears in the work of Krammer [10]. This leads again to a globally nilpotent Fuchsian system of rank two. It is a new type of a globally nilpotent rank two system because it is neither a pullback of a hypergeometric system nor arithmetic, see Theorem 3.1.

The authors thank J. Aidan and J.A. Weil for pointing out misprints in the differential equation resp. in the monodromy group generators which appear in Theorem 3.1 (in an earlier version of this paper) and N. Katz for valuable comments on the nilpotence of higher direct image connections.

1. The Riemann–Hilbert correspondence of the middle convolution

1.1. *The tuple transformation* MC_λ

Let K be a field, let V be a finite dimensional vector space over K and let $M = (M_1, \dots, M_r)$ be an element of $\mathrm{GL}(V)^r$. For any $\lambda \in K^\times$ one can construct another tuple of matrices $(N_1, \dots, N_r) \in \mathrm{GL}(V^r)^r$, as follows: For $k = 1, \dots, r$, N_k maps a vector $(v_1, \dots, v_r)^{\mathrm{tr}} \in V^r$ to

$$\begin{pmatrix} 1 & 0 & & \dots & & 0 & \\ & \ddots & & & & & \\ & & 1 & & & & \\ (M_1-1) & \dots & (M_{k-1}-1) & \lambda M_k & \lambda(M_{k+1}-1) & \dots & \lambda(M_r-1) \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & \dots & & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ v_r \end{pmatrix}.$$

There are the following $\langle N_1, \dots, N_r \rangle$-invariant subspaces of V^r:

$$\mathcal{K}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \ker(M_k - 1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (k\text{-th entry}),\ k = 1, \ldots, r,$$

and

$$\mathcal{L} = \bigcap_{k=1}^{r} \ker(N_k - 1) = \ker(N_r \cdots N_1 - 1).$$

Let $\mathcal{K} := \bigoplus_{i=1}^{r} \mathcal{K}_i$.

Definition 1.1. Let $\mathrm{MC}_\lambda(M) := (\tilde{N}_1, \ldots, \tilde{N}_r) \in \mathrm{GL}(V^r/(\mathcal{K}+\mathcal{L}))^r$, where $\tilde{N}_k$ is induced by the action of N_k on $V^r/(\mathcal{K}+\mathcal{L})$. We call $\mathrm{MC}_\lambda(M)$ the *middle convolution* of M with λ.

1.2. The middle convolution of Fuchsian systems

Let $A = (A_1, \ldots, A_r)$, $A_k \in \mathbb{C}^{n\times n}$. For $\mu \in \mathbb{C}$ one can define block matrices B_k, $k = 1, \ldots, r$, as follows:

$$B_k := \begin{pmatrix} 0 & & & & \ldots & & 0 \\ & & \ddots & & & & \\ A_1 & \ldots & A_{k-1} & A_k + \mu & A_{k+1} & \ldots & A_r \\ & & & & \ddots & & \\ 0 & & & \ldots & & & 0 \end{pmatrix} \in \mathbb{C}^{nr\times nr},$$

where B_k is zero outside the k-th block row.

The tuple

$$c_\mu(A) := (B_1, \ldots, B_r) \tag{1}$$

is called the *naive convolution* of A with μ. There are the following left-$\langle B_1, \ldots, B_r\rangle$-invariant subspaces of the column vector space $\mathbb{C}^{nr}$ (with the tautological action of $\langle B_1, \ldots, B_r\rangle$):

$$\mathfrak{k}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \ker(A_k) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (k\text{-th entry}),\ k = 1, \ldots, r,$$

and

$$\mathfrak{l} = \bigcap_{k=1}^{r} \ker(B_k) = \ker(B_1 + \cdots + B_r).$$

Let $\mathfrak{k} := \bigoplus_{k=1}^{r} \mathfrak{k}_k$ and fix an isomorphism between $\mathbb{C}^{nr}/(\mathfrak{k}+\mathfrak{l})$ and $\mathbb{C}^m$.

Definition 1.2. The tuple of matrices $\mathrm{mc}_\mu(A) := (\tilde{B}_1, \ldots, \tilde{B}_r) \in \mathbb{C}^{m\times m}$, where $\tilde{B}_i$ is induced by the action of B_i on $\mathbb{C}^m (\simeq \mathbb{C}^{nr}/(\mathfrak{k}+\mathfrak{l}))$, is called the *middle convolution* of A with μ.

Let $S := \mathbb{C} \setminus \{t_1, \ldots, t_r\}$ and $A := (A_1, \ldots, A_r)$, $A_i \in \mathbb{C}^{n\times n}$. The Fuchsian system

$$F : Y' = \sum_{i=1}^{r} \frac{A_i}{t - t_i} Y$$

is called *the Fuchsian system associated to the tuple A*.

Definition 1.3. Let $A := (A_1, \ldots, A_r)$, $A_i \in \mathbb{C}^{n\times n}$, and $\mu \in \mathbb{C}$. Let F be the Fuchsian system associated to A. Then the Fuchsian system which is associated to the middle convolution tuple $\mathrm{mc}_\mu(A)$ is denoted by $\mathrm{mc}_\mu(F)$ and is called the *middle convolution* of F with the parameter μ. The Fuchsian system which is associated to the naive convolution tuple $\mathrm{c}_\mu(A)$ is denoted by $\mathrm{c}_\mu(F)$ and is called the *naive convolution* of F with the parameter μ.

Fix a set of homotopy generators δ_i, $i = 1, \ldots, r$, of $\pi_1(S, t_0)$ by traveling from the base point t_0 to t_i, then encircling t_i counterclockwise, and then going back to t_0. Then, the analytic continuation of solutions along δ_i defines a linear isomorphism M_i, $i = 1, \ldots, r$, of the vector space $V \simeq \mathbb{C}^n$ of holomorphic solutions of F which are defined in a small neighborhood of t_0. We call the tuple $(M_1, \ldots, M_r)$ the *monodromy tuple* of F. The Riemann–Hilbert correspondence says that F is determined up to isomorphism by M, see [3].

The following result is an explicit realization of the Riemann–Hilbert correspondence for a convoluted Fuchsian system, see [7, Thm. 4.7]:

Theorem 1.4. *Let F be an irreducible Fuchsian system associated to $A = (A_1, \ldots, A_r) \in (\mathbb{C}^{n\times n})^r$. Fix a set of homotopy generators*

$$\delta_1, \ldots, \delta_r \in \pi_1\bigl(\mathbb{A}^1 \setminus \{t_1, \ldots, t_r\}\bigr)$$

of the fundamental group as above. Let $M = (M_1, \ldots, M_r) \in \mathrm{GL}_n(\mathbb{C})^r$ be the monodromy tuple of F (with respect to $\delta_1, \ldots, \delta_r$). Assume that there exist at least two different elements of M that are $\neq 1$ and that

$$\mathrm{rk}(A_i) = \mathrm{rk}(M_i - 1), \qquad \mathrm{rk}(A_1 + \cdots + A_r + \mu) = \mathrm{rk}(\lambda \cdot M_1 \cdots M_r - 1).$$

Then the monodromy tuple of $\mathrm{mc}_\mu(F)$ is given by $\mathrm{MC}_\lambda(M)$, where $\lambda = e^{2\pi i\mu}$.

2. Transformation of the p-curvature under mc_μ

Let K be a number field and let $F : Y' = CY$ be a system of linear differential equations, where $C \in K(t)^{n\times n}$. Successive application of differentiation yields differential systems

$$F^{(n)} : Y^{(n)} = C_n Y \quad \text{for each } n \in \mathbb{N}_{>0}. \tag{2}$$

In the following, $\mathfrak{p}$ always denotes a prime of K which lies over a prime number p. For almost all primes p, one can reduce the entries of the matrices C_p modulo $\mathfrak{p}$ in order to obtain the $\mathfrak{p}$-*curvature matrices of F*

$$\bar{C}_\mathfrak{p} = \bar{C}_\mathfrak{p}(F) := C_p \mod \mathfrak{p}.$$

Definition 2.1. A system of linear differential equations F *can be written in Okubo normal form*, if F can be written as

$$F : Y' = (t - T)^{-1} BY,$$

where $B \in \mathbb{C}^{n \times n}$ and T is a diagonal matrix $T = \mathrm{diag}(t_1, \ldots, t_n)$ with $t_i \in \mathbb{C}$ (here possibly $t_i = t_j$ for $i \neq j$).

The following proposition is obvious from the definitions:

Proposition 2.2. *If F is a Fuchsian system, then the naive convolution $\mathrm{c}_\mu(F)$ of F can be written in Okubo normal form.*

An induction yields the following formula for the $\mathfrak{p}$-curvature matrix of a system in Okubo normal form:

Lemma 2.3. *Let*

$$F : Y' = CY = (t - T)^{-1} BY$$

be a system of linear differential equations which can be written in Okubo normal form. Then

$$C_n = (t - T)^{-1}(B - n + 1) \cdot (t - T)^{-1}(B - n + 2) \cdots (t - T)^{-1}(B - 1) \cdot (t - T)^{-1} B. \tag{3}$$

Especially, if the matrix B has coefficients in a number field K, then the $\mathfrak{p}$-curvature $\bar{C}_\mathfrak{p}$ of F has the form

$$(t - T)^{-1}(B - p + 1) \cdot (t - T)^{-1}(B - p + 2) \cdots (t - T)^{-1}(B - 1) \cdot (t - T)^{-1} B \mod \mathfrak{p}.$$

Remark 2.4. The above proposition is interesting because there is no closed formula known for the computation of the $\mathfrak{p}$-curvature if the rank is > 2 (compare to [12] and [11]). The above lemma yields such a closed formula for Okubo systems. On the other hand, every irreducible Fuchsian system is a subfactor of an Okubo system (this is known by the work of Okubo and follows also from Proposition 2.2 and the fact that, by Eq. (4) below, the middle convolution mc_{-1} induces the identity transformation of Fuchsian systems).

The following technical proposition will be used below:

Proposition 2.5. *Let K be a number field, let $F : Y' = CY$, $C \in K(t)^{n \times n}$, be a Fuchsian system and let $\mathrm{c}_0(F) : Y' = DY$ be the naive convolution of F with the parameter 0. Let p be a prime number and let $\mathfrak{p}$ denote a prime of K which lies over p such that the entries of C can be reduced modulo $\mathfrak{p}$. If $C_p^k \equiv 0 \mod \mathfrak{p}$, then $D_{kp+1} \equiv 0 \mod \mathfrak{p}$.*

Proof. The Fuchsian system F is the Fuchsian system associated to some tuple $(A_1, \ldots, A_r) \in (K^{n \times n})^r$. By Proposition 2.2, the naive convolution of F with the parameter -1 can be written in Okubo normal form

$$\mathrm{c}_{-1}(F) : Y' = (t - T)^{-1} \tilde{B} Y = \tilde{D} Y.$$

Here, T is the diagonal matrix $T = \mathrm{diag}(t_1, \ldots, t_1, \ldots, t_r, \ldots, t_r)$ (every t_k occurs n times) and

$$\tilde{B} = (\tilde{B}_{i,j}), \quad \tilde{B}_{i,j} = A_j - \delta_{i,j} E_n$$

($E_n \in \mathrm{GL}_n(K)$ denoting the identity matrix). Using the base change which is induced by the matrix $H(t-T)$, where

$$H := \begin{pmatrix} E_n & -E_n & 0 & \dots \\ 0 & \ddots & \ddots & \\ \vdots & & & -E_n \\ 0 & \dots & & E_n \end{pmatrix},$$

one can verify that the naive convolution $\mathrm{c}_{-1}(F)$ is equivalent to the following system:

$$Y' = GY = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ \frac{A_1}{t-t_1} & (\frac{A_1}{t-t_1} + \frac{A_2}{t-t_2}) & \dots & (\frac{A_1}{t-t_1} + \dots + \frac{A_r}{t-t_r}) \end{pmatrix} Y. \tag{4}$$

By a straightforward computation, one sees that

$$G = H(t-T) \cdot D \cdot \big(H(t-T)\big)^{-1}, \tag{5}$$

where D is the matrix appearing in the naive convolution $\mathrm{c}_0(F) : Y' = DY$ with the parameter 0. Using the Leibnitz rule, one can further verify that

$$G_p = \big(H(t-T)\big) \cdot \tilde{D}_p \cdot \big(H(t-T)\big)^{-1} \tag{6}$$

(note that $\tilde{D}_p$ belongs to the naive convolution $\mathrm{c}_{-1}(F)$ with parameter -1). It follows from Eq. (4) that

$$G_p = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ * & * & * & C_p \end{pmatrix}.$$

Using this and the assumption $C_p^k \equiv 0 \bmod \mathfrak{p}$ one sees that

$$G_p^k G = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ * & * & * & 0 \end{pmatrix} \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ * & * & * & * \end{pmatrix} \equiv 0 \mod \mathfrak{p}.$$

Thus, by Eqs. (5) and (6),

$$H(t-T) \cdot \tilde{D}_p^k \cdot D \cdot \big(H(t-T)\big)^{-1} \equiv 0 \mod \mathfrak{p}$$

and thus

$$\tilde{D}_p^k \cdot D \equiv 0 \mod \mathfrak{p}.$$

The claim follows now from this and the equality

$$\tilde{D}_p^k \cdot D \equiv D_{pk+1} \mod \mathfrak{p},$$

which is immediate from Eq. (3) appearing in Lemma 2.3. □

Theorem 2.6. *Let K be a number field, let*

$$F : Y' = CY, \quad C \in K(t)^{n\times n},$$

be a Fuchsian system and let $\mu \in \mathbb{Q}$. Let p be a prime number and let $\mathfrak{p}$ denote a prime of K which lies over p such that the number μ and the entries of C can be reduced modulo $\mathfrak{p}$. If the $\mathfrak{p}$-curvature of F is nilpotent of rank k (i.e., $\bar{C}_p(F)^k \equiv 0$), then the $\mathfrak{p}$-curvature of the middle convolution $\mathrm{mc}_\mu(F)$ is nilpotent of rank $r \in \{k-1, k, k+1\}$.

Proof. We first prove that the $\mathfrak{p}$-curvature of the naive convolution $\mathrm{c}_\mu(F)$ is nilpotent of rank at most $k+1$: Let $\mathrm{c}_0(F) : Y' = DY$ be the naive convolution of F with the parameter 0 and let $\mathrm{c}_\mu(F) : Y' = D^\mu Y$ be the naive convolution of F with the parameter μ. By assumption, one has $C_p^k \equiv 0 \mod \mathfrak{p}$. Thus, by the preceding proposition,

$$D_{kp+1} \equiv 0 \mod \mathfrak{p}. \tag{7}$$

One can easily deduce from Eq. (3) of Lemma 2.3, that the matrix $D_{kp+1} \mod \mathfrak{p}$ appears as a factor in the product formula (3) for $D^\mu_{p(k+1)}$ taken modulo $\mathfrak{p}$. Thus, by Eq. (7) and again by Lemma 2.3,

$$0 \equiv D^\mu_{p(k+1)} \equiv \left(D_p^\mu\right)^{k+1} = \bar{C}_p\left(\mathrm{mc}_\mu(F)\right)^{k+1} \mod \mathfrak{p}.$$

Since the middle convolution $\mathrm{mc}_\mu(F)$ is a factor of the naive convolution $\mathrm{c}_\mu(F)$, it follows that the $\mathfrak{p}$-curvature of the middle convolution $\mathrm{mc}_\mu(F)$ is nilpotent of rank at most $k+1$. Now the claim follows from the fact that $\mathrm{mc}_{-\mu} \circ \mathrm{mc}_\mu = \mathrm{id}$. □

3. A new example of a globally nilpotent Fuchsian system

Dwork has conjectured that any globally nilpotent second order differential equation has either algebraic solutions or has a correspondence to a Gauss hypergeometric differential equation (see [10]). This conjecture was disproved by Krammer [10]. The counterexample which Krammer gives is the uniformizing differential equation K of an arithmetic Fuchsian lattice. It comes from the periods of a family of abelian surfaces over a Shimura curve $S \simeq \mathbb{P}^1 \setminus \{0, 1, 81, \infty\}$. The equation K is an irreducible ordinary second order differential equation which can easily be transformed into the following Fuchsian system of rank two:

$$K : Y' = \left(\frac{1}{t} \cdot \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + \frac{1}{t-1} \cdot \begin{pmatrix} 0 & 0 \\ \frac{4}{9} & -\frac{1}{2} \end{pmatrix} + \frac{1}{t-81} \begin{pmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}\right) \cdot Y.$$

The local monodromy of K at the finite singularities is given by three reflections, the local monodromy of K at ∞ is given by an element of order six.

Theorem 3.1. *The middle convolution $H := \mathrm{mc}_{\frac{1}{6}}(K)$ is equivalent to the following Fuchsian system:*

$$Y' = \left(\frac{1}{t} \begin{pmatrix} -\frac{19}{30} & \frac{19}{10} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix} + \frac{1}{t-1} \begin{pmatrix} -\frac{1}{3} & -\frac{7}{18} \\ 0 & 0 \end{pmatrix} + \frac{1}{t-81} \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{2}{3} \end{pmatrix}\right) Y.$$

The Fuchsian system H is globally nilpotent. Moreover, the system H has neither a correspondence to a hypergeometric differential system, nor a correspondence to a uniformizing differential equation of a Fuchsian lattice.

Proof. The first claim follows from Theorem 2.6. To prove the last statement, we use the Riemann–Hilbert correspondence: The monodromy tuple $A = (A_1, A_2, A_3)$ of K can be shown to be (cf. Ex. 4.7.5 in [2])

$$\left(i \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i \cdot \begin{pmatrix} 0 & -(\sqrt{5}+1)/2 \\ (\sqrt{5}-1)/2 & 0 \end{pmatrix}, i \cdot \begin{pmatrix} -\sqrt{3} & 1+\sqrt{5} \\ 1-\sqrt{5} & \sqrt{3} \end{pmatrix}\right),$$

where

$$(A_1 \cdot A_2 \cdot A_3)^{-1} = i \cdot \begin{pmatrix} (\sqrt{3}+\sqrt{15})/2 & -2 \\ 2 & (\sqrt{3}-\sqrt{15})/2 \end{pmatrix}.$$

In the following, the element ζ_n denotes the root of unity $e^{2\pi i/n}$. Using Theorem 1.4, it is a straightforward computation that the monodromy of H is given by

$$\begin{aligned} \mathrm{MC}_{\zeta_6}(A) &= (B_1, B_2, B_3) \\ &= \left(\frac{\zeta_3}{5}\begin{pmatrix} \zeta_{15}+2\zeta_{15}^2+\zeta_{15}^4-\zeta_{15}^7+2\zeta_{15}^8-2\zeta_{15}^{11}-\zeta_{15}^{13}-2\zeta_{15}^{14} & -6\zeta_{15}+6\zeta_{15}^2-6\zeta_{15}^4+6\zeta_{15}^7+6\zeta_{15}^8+4\zeta_{15}^{11}+6\zeta_{15}^{13}+4\zeta_{15}^{14} \\ 2\zeta_{15}+2\zeta_{15}^4+3\zeta_{15}^7+3\zeta_{15}^{13} & -6\zeta_{15}-7\zeta_{15}^2-6\zeta_{15}^4-4\zeta_{15}^7-7\zeta_{15}^8-3\zeta_{15}^{11}-4\zeta_{15}^{13}-3\zeta_{15}^{14} \end{pmatrix},\right. \\ &\qquad \left. \zeta_3 \cdot \begin{pmatrix} -\zeta_{60}^{10} & 2\zeta_{60}^{10} \\ 0 & \zeta_{60}^{10}-1 \end{pmatrix}, \zeta_3 \cdot \begin{pmatrix} \zeta_{60}^{10}-1 & 0 \\ -\zeta_{60}^{10}+1 & -\zeta_{60}^{10} \end{pmatrix}\right). \end{aligned}$$

By [6, Cor. 5.9], one can show that the elements B_i, $i = 1, 2, 3$, generate a subgroup which is conjugate to a subgroup of $\mathrm{SU}_{1,1}(\mathbb{R})$. The latter group is well known to be conjugate to the group $\mathrm{SL}_2(\mathbb{R})$ inside the group $\mathrm{GL}_2(\mathbb{C})$.

Consider the element

$$\tilde{B} := B_1 B_2 B_1 = \begin{pmatrix} 2\zeta_{60}^{10}-1 & -4\zeta_{60}^{10}-2\zeta_{60}^{8}+2\zeta_{60}^{2} \\ -\zeta_{60}^{14}+\zeta_{60}^{8}+\zeta_{60}^{6}+\zeta_{60}^{4}-\zeta_{60}^{2}-1 & 2\zeta_{60}^{14}-2\zeta_{60}^{10}-2\zeta_{60}^{6}-2\zeta_{60}^{4}+3 \end{pmatrix}.$$

It is an elliptic element since the absolute value of its trace is $|-\sqrt{5}+1| < 2$. Moreover, the order of $\tilde{B}$ is infinite. (The eigenvalues of $\tilde{B}$ are roots of $f := X^4 - 2X^3 - 2X^2 - 2X + 1$ with $\mathrm{Gal}(f) = D_8$, thus the eigenvalues cannot be roots of unity.) Thus, the monodromy tuple $\mathrm{MC}_{\zeta_6}(A)$ of H generates a non-discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})$. Hence, the Fuchsian system H does not admit a correspondence to a uniformizing differential equation of a Fuchsian lattice.

Let G be a Gauss' hypergeometric differential equation and assume that $\phi(t)$ be a rational function of degree N such that the pullback of G along $\phi(t)$ has the same monodromy representation as the Fuchsian system H. Let a, b and c denote the orders of the local projective monodromy of G at the singularities 0, 1 and ∞ (resp.). Since the orders of the local projective monodromy of H is 3 at $0, 1, 81, \infty$ we get the following conditions for the ramification indices: Over 0 we have $r + r_1$ points ($r, r_1 \geqslant 0$), where at the first r points have trivial monodromy and the last r_1 points have local monodromy of order 3. The ramification indices at these points can be written as follows (resp.):

$$(ax_1, \ldots, ax_r, \alpha_1, \ldots, \alpha_{r_1}), \quad \text{with } x_i \in \mathbb{N}, \quad a \nmid \alpha_i, \quad \text{and} \quad \frac{a}{3} \mid \alpha_i \tag{8}$$

(we obtain only a local monodromy of order 3 if $3 \mid a$ and $\frac{a}{3} \mid \alpha_i$). Similarly, we get the ramification indices for the points lying over 1 and ∞ (resp.):

$$(by_1, \dots, by_s, \beta_1, \dots, \beta_{s_1}), (cz_1, \dots, cz_t, \gamma_1, \dots, \gamma_{t_1}), \tag{9}$$

where $y_j, z_k \in \mathbb{N}$, $b \nmid \beta_j$, $c \nmid \gamma_k$, $\frac{b}{3} \mid \beta_j$, $\frac{c}{3} \mid \gamma_k$. Note that the sum over the ramification indices at all points lying over $0, 1, \infty$ (resp.) is N. Since H has only 4 singularities we get that $r_1 + s_1 + t_1 = 4$. The Riemann–Hurwitz formula implies therefore that

$$-1 \geqslant (-N) + \frac{1}{2}\bigl(\bigl(N - (r + r_1)\bigr) + \bigl(N - (s + s_1)\bigr) + \bigl(N - (t + t_1)\bigr)\bigr) = \frac{1}{2}\bigl(N - (r + s + t + 4)\bigr).$$

Hence

$$N \leqslant r + s + t + 2. \tag{10}$$

(Note that we only get an inequality since ϕ can have a priori more than 3 ramification points.) Since the monodromy group of H is not finite we can assume by the classification of the finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ that $(a, b, c) \in M_1 \cup M_2$, where

$$M_1 = \bigl\{(2, 3, c) \bigm| c \geqslant 7\bigr\} \quad \text{and} \quad M_2 = \bigl\{(a, b, c) \bigm| 3 \leqslant a,\ a \leqslant b,\ 4 \leqslant c\bigr\}.$$

By the remark following (8) and (9) and since the ramification indices are $\geqslant 1$, we have

$$r \leqslant \sum_{i=1}^{r} x_i = \frac{N - \sum_i \alpha_i}{a}, \qquad s \leqslant \sum_{j=1}^{s} y_j = \frac{N - \sum_j \beta_j}{b}$$

and

$$t \leqslant \sum_{k=1}^{t} z_k = \frac{N - \sum_k \gamma_k}{c}.$$

Again by the conditions in (8), we know that $\frac{1}{3} \leqslant \frac{\alpha_i}{a}$ if $0 < r_1$. Thus

$$r + s + t \leqslant \sum x_i + \sum y_j + \sum z_k \leqslant \frac{N}{a} + \frac{N}{b} + \frac{N}{c} - \frac{4}{3} < N - \frac{4}{3}. \tag{11}$$

(The last inequality follows from the fact that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$ if $(a, b, c) \in M_1 \cup M_2$.) From this and from (10) we obtain

$$r + s + t + \frac{4}{3} < N \leqslant r + s + t + 2.$$

Since r, s, t and N are integers, we get

$$N = r + s + t + 2. \tag{12}$$

It follows from this, from (11), and because all $x_i, y_i, z_i \in \mathbb{N}$, that $x_i = y_j = z_k = 1$ (if one of the x_i, y_i, z_i is $\geqslant 2$, then the first inequality in (11) is strict, which leads to a contradiction to (12)). It also follows from (11) and (12) that

$$N \leqslant \frac{N}{a} + \frac{N}{b} + \frac{N}{c} + \frac{2}{3}. \tag{13}$$

We have already shown that the monodromy group of H contains the element $B_1B_2B_1$ with trace equal to $-\sqrt{5}+1=-2(\zeta_5+\zeta_5{}^{-1})$. It is a well-known consequence of this that the last statement implies that the number 5 divides at least one of the numbers a, b, c (use the rigidity of the monodromy representation or [4, Lemma 12.5.1]). Since the orders of the local monodromy of H are prime to 5, we obtain $5 \mid N$. If $(a,b,c)\in M_1$ we get by (13)

$$N \leqslant \frac{N}{a}+\frac{N}{b}+\frac{N}{c}+\frac{2}{3}\leqslant \frac{N}{2}+\frac{N}{3}+\frac{N}{10}+\frac{2}{3}=\frac{14N}{15}+\frac{2}{3}.$$

Hence $N\leqslant 15\cdot\frac{2}{3}=10$. Since $a=2$ and $5\mid N$ the only possibility is $N=10$ with the ramification indices

$$(2,2,2,2,2),\quad (3,3,1,1,1,1),\quad (10).$$

Analogously, if $(a,b,c)\in M_2$ we get

$$N\leqslant \frac{N}{3}+\frac{N}{3}+\frac{N}{5}+\frac{2}{3}=\frac{13N}{15}+\frac{2}{3},$$

which gives $N=5$ and the ramification indices

$$(3,1,1),\quad (3,1,1),\quad (5).$$

In the first case the orders of the local monodromy of the Gauss hypergeometric differential equation (in $\mathrm{SL}_2(\mathbb{C})$) are $(4,3,20)$ (resp. $(4,6,20)$). But the pullback gives only rise to a tuple of nontrivial elements of order $(2,2,2,2,2,3,3,3,3,2)$ (resp. $(2,2,2,2,2,2,2,6,6,6,6,2)$) which gives a quadruple of elements of order 3 (after multiplication with elements of order 2, i.e. with $-E_2$). But this is contrary to the monodromy of H whose local monodromy orders are $(3,3,3,6)$. In the second case the orders of local monodromy in $\mathrm{SL}_2(\mathbb{C})$ have to be (w.l.o.g.) 3, 3 and 10 in order to obtain local monodromy orders $(3,3,3,6)$. But the monodromy group of this Gauss hypergeometric differential equation leaves a positive definite form invariant (see [4, Cor. 2.21]). (It can be easily shown that it even is the finite monodromy group $2.A_5$.) This shows that H is not a pullback of a hypergeometric differential equation. □

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