On hesitant multi-fuzzy soft topology

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ABSTRACT
The aim of this paper is to introduce and study the concept of hesitant multi-fuzzy soft topological space and some of its structural properties, such as the neighbourhood of hesitant multi-fuzzy soft sets and interior hesitant multi-fuzzy soft sets, the hesitant multi-fuzzy soft basis and hesitant multi-fuzzy soft subspace topology. Additionally, we introduce and define the hesitant multi-fuzzy soft cover, hesitant multi-fuzzy soft open cover, and hesitant multi-fuzzy soft compactness, and some important results on them are presented.

1. Introduction

In real life situations, problems in economics, engineering, social sciences, medical science, etc. do not always involve crisp data. Therefore, we cannot successfully use traditional methods because of various types of uncertainties presented in these problems. To manage these uncertainties, theories were developed such as the theory of fuzzy sets, intuitionistic fuzzy sets, rough sets, and bipolar fuzzy sets, which we can use as mathematical tools for dealing with uncertainties. However, all these theories have inherent difficulties. Because of these difficulties, Molodtsov initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties.

Maji et al. (Maji et al., 2002) gave the first practical application of soft sets in decision making problems. They also introduced the concept of fuzzy soft sets, a more generalized concept, which is a combination of the fuzzy set and soft set, and also studied some of its properties. Ali et al. (Ali et al., 2009) introduced some new operations in soft set theory. After that Aygunoglu and Aygun (Aygunoglu and Aygun, 2012a) gave some interesting results on soft topological spaces. Also, in Aygunoglu and Aygun (Aygunoglu and Aygun, 2012b) introduced the concept of fuzzy soft groups. Majumdar and Samanta (Majumdar and Samanta, 2010) discussed the concept of generalised fuzzy soft sets. Also, Roy and Maji (Roy and Maji, 2007) applied soft set in decision making problems. Varol et al. (Varol et al., 2014) gave the neighborhood structures of fuzzy soft topological spaces. The concept of multi-fuzzy subgroups is discussed by Dey and Pal (Dey and Pal, 2016a). Jun (Jun, 2008) applied soft set in BCK/BCI-algebra. Torra (Torra, 2010) first introduced the concept of hesitant fuzzy sets, which permits the membership to have a set of possible values and defines some of its basic operations in expressing uncertainty and vagueness. Liao and Xu (Liao and Xu, 2014) introduced subtraction and division operations over hesitant fuzzy sets. Also, Verma and Sharma (Verma and Sharma, 2013) discussed some new operations over hesitant fuzzy sets.

A new type of fuzzy set (multi-fuzzy set) was introduced in a paper by Sebastian and Ramakrishnan (Sebastian and Ramakrishnan, 2011a) using the ordered sequences of membership function. The notion of multi-fuzzy sets provides a new method to represent some problems that are difficult to explain using other extensions of fuzzy set theory, such as the colour of pixels. Sebastian and Ramakrishnan (Sebastian and Ramakrishnan, 2010, 2011b, 2011c) discussed multi-fuzzy extensions of functions and multi-fuzzy subgroups. The notion of multi-fuzzy complex numbers and multi-fuzzy complex sets are introduced for the first time by Dey and Pal (Dey and Pal, 2014b). Using these concepts, Dey and Pal (Dey and Pal, 2014b) introduced the concept of hesitant multi-fuzzy soft groups. Majumdar and Samanta (Majumdar and Samanta, 2010) discussed the concept of generalised fuzzy soft sets. Also, Roy and Maji (Roy and Maji, 2007) applied soft set in decision making problems. Varol et al. (Varol et al., 2014) gave the neighborhood structures of fuzzy soft topological spaces. The concept of multi-fuzzy subgroups is discussed by Dey and Pal (Dey and Pal, 2016a). Jun (Jun, 2008) applied soft set in BCK/BCI-algebra. Torra (Torra, 2010) first introduced the concept of hesitant fuzzy sets, which permits the membership to have a set of possible values and defines some of its basic operations in expressing uncertainty and vagueness. Liao and Xu (Liao and Xu, 2014) introduced subtraction and division operations over hesitant fuzzy sets. Also, Verma and Sharma (Verma and Sharma, 2013) discussed some new operations over hesitant fuzzy sets.
Let $k$ be a positive integer. A multi-fuzzy set and the development of the hesitant multi-fuzzy soft set and hesitant multi-fuzzy soft compactness are introduced. In Section 5, some conclusions are noted.

In Section 3, the notion of hesitant multi-fuzzy soft topology and hesitant multi-fuzzy soft set, interior hesitant multi-fuzzy soft sets, multi-fuzzy soft sets and hesitant fuzzy sets in Section 2. In Section 3, the notion of hesitant multi-fuzzy soft topology and some of its structural properties, such as neighbourhood of a hesitant multi-fuzzy soft set, hesitant multi-fuzzy soft basis and hesitant multi-fuzzy soft sub-space topology, are investigated. In Section 4, the concepts of the hesitant multi-fuzzy soft cover, hesitant multi-fuzzy soft open cover, and hesitant multi-fuzzy soft compactness are introduced. In Section 5, some conclusions are noted.

2. Preliminaries

From ready references, we present the following definitions for the development of the hesitant multi-fuzzy soft set and hesitant multi-fuzzy soft topology. Throughout this paper, $U$ refers to an initial universal set, $E$ is a set of parameters, $P(U)$ is the power set of $U$ and $A \subseteq E$.

Definition 1. (Molodtsov, 1999) (Soft sets) A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \rightarrow P(U)$.

In other words, a soft set over $U$ is a mapping from parameters to $P(U)$, and it is not a set but a parameterized family of subsets of $U$.

Now, we elaborate the definition of soft set by the following example.

Example 1. Let $U = \{b_1, b_2, b_3, b_4, b_5\}$ be a set of bikes under consideration. Let $A = \{e_1, e_2, e_3\}$ be a set of parameters, where $e_1$ is expensive, $e_2$ is beautiful and $e_3$ is good mileage. Suppose that $F(e_1) = \{b_2, b_4\}$, $F(e_2) = \{b_3, b_5\}$, $F(e_3) = \{b_2, b_3, b_5\}$. The soft set $(F, A)$ describes the “attractiveness of the bikes”. $F(e_1)$ means “bikes (expensive) whose function value is the set $\{b_2, b_4\}$, $F(e_2)$ means “bikes (beautiful) whose function value is the set $\{b_3, b_5\}$ and $F(e_3)$ means “bikes (good mileage)” whose function value is the set $\{b_2, b_3, b_5\}$.

Definition 2. (Maji et al., 2001) (Fuzzy soft sets) Let $P(U)$ be all fuzzy subsets of $U$. A pair $(F, A)$ is called fuzzy soft set over $U$, where $F$ is a mapping given by $F : A \rightarrow P(U)$.

Example 2. Let us consider Example 1.

The fuzzy soft set $(F, A)$ can describe the “attractiveness of the bikes” under fuzzy circumstances. $F(e_1) = \{b_1/0.3, b_2/0.8, b_3/0.4, b_4/0.7, b_5/0.5\}$, $F(e_2) = \{b_1/0.7, b_2/0.2, b_3/0.4, b_4/0.8, b_5/0.9\}$, $F(e_3) = \{b_1/0.6, b_2/0.4, b_3/0.7, b_4/0.2, b_5/0.1\}$.

Now, we introduce the concept of a multi-fuzzy set.

Definition 3. (Sebastian and Ramakrishnan, 2011a) (Multi-fuzzy sets) Let $k$ be a positive integer. A multi-fuzzy set $\tilde{A}$ in $U$ is a set of ordered sequences $\tilde{A} = \{(\mu_1(u), \mu_2(u), ..., \mu_k(u)) : u \in U\}$, where $\mu_i \in P(U)$, $i = 1, 2, ..., k$.

The function $\mu_k = (\mu_1(u), \mu_2(u), ..., \mu_k(u))$ is called the multi-membership function of multi-fuzzy set $\tilde{A}$, and $k$ is called dimension of $A$. The set of all multi-fuzzy sets of dimension $k$ in $U$ is denoted by $M^kFS(U)$.

Note 1 A multi-fuzzy set of dimension $1$ is a Zadeh’s fuzzy set, and a multi-fuzzy set of dimension $2$ with $\mu_1(u) + \mu_2(u) \leq 1$ is the Atanassov’s intuitionistic fuzzy set.

Note 2 If $\sum_{i=1}^k \mu_i(u) = 0$, for all $u \in U$, then the multi-fuzzy set of dimension $k$ is called a normalized multi-fuzzy set. If $\sum_{i=1}^k \mu_i(u) = 1$ for some $u \in U$, we redefine the multi-membership degree $(\mu_1(u), \mu_2(u), ..., \mu_k(u))$ as $\frac{1}{k}(\mu_1(u), \mu_2(u), ..., \mu_k(u))$, then the non-normalized multi-fuzzy set can be changed into a normalized multi-fuzzy set.

Definition 4. (Sebastian and Ramakrishnan, 2011a) Let $\tilde{A} \in M^kFS(U)$. If $\tilde{A} = \{(u/0, 0, ..., 0) : u \in U\}$, then $\tilde{A}$ is called the null multi-fuzzy set of dimension $k$, denoted by $\Phi_k$. If $\tilde{A} = \{(u/1, 1, 1, ..., 1) : u \in U\}$, then $\tilde{A}$ is called the universal multi-fuzzy set of dimension $k$, denoted by $1_k$.

An example can be used to illustrate the concept of multi-fuzzy sets.

Example 3. Suppose a colour image is approximated by an $m \times n$ matrix of pixels. Let $U$ be the set of all pixels of the colour image. For any pixel $u$ in $U$, the membership values $\mu_1(u), \mu_2(u), \mu_3(u)$ being the normalized red value, green value and blue value of the pixel $u$, respectively. Therefore, the colour image can be approximated by the collection of pixels with the multi-membership function $(\mu_1(u), \mu_2(u), \mu_3(u))$, and it can be represented as a multi-fuzzy set $\tilde{A} = \{(u/\mu_1(u), \mu_2(u), \mu_3(u)) : u \in U\}$. In a two-dimensional image, the colour of the pixels cannot be characterized by a membership function of an ordinary fuzzy set, but it can be characterized by a three-dimensional membership function $(\mu_1(u), \mu_2(u), \mu_3(u))$. In fact, a multi-fuzzy set can be understood to be a more general fuzzy set using ordinary fuzzy sets as its building blocks.

Definition 5. (Sebastian and Ramakrishnan, 2011a) Let $\tilde{A} = \{(u/\mu_1(u), \mu_2(u), ..., \mu_k(u)) : u \in U\}$ and $\tilde{B} = \{(u/r_1(u), r_2(u), ..., r_k(u)) : u \in U\}$ be two multi-fuzzy sets of dimension $k$ in $U$. We define the following relations and operations:

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_i(u) \leq r_i(u), \forall u \in U$ and $1 \leq i \leq k$.
2. $\tilde{A} \bar{\subseteq} \tilde{B}$ if and only if $\mu_i(u) = r_i(u), \forall u \in U$ and $1 \leq i \leq k$.
3. $\tilde{A} \cup \tilde{B} = \{(u/\mu_1(u) \lor r_1(u), \mu_2(u) \lor r_2(u), ..., \mu_k(u) \lor r_k(u)) : u \in U\}$.
4. $\tilde{A} \cap \tilde{B} = \{(u/\mu_1(u) \land r_1(u), \mu_2(u) \land r_2(u), ..., \mu_k(u) \land r_k(u)) : u \in U\}$.
5. $\tilde{A}^c = \{(u/\mu_1(u), \mu_2(u), ..., \mu_k(u)) : u \in U\}$.

The relationships between multi-fuzzy set and soft set can be further discussed as follows:

Definition 6. (Yang et al., 2013) (Multi-fuzzy soft sets) A pair $(\tilde{F}, A)$ is called a multi-fuzzy soft set of dimension $k$ over $U$, where $\tilde{F}$ is a mapping given by $F : A \rightarrow M^kFS(U)$.

A multi-fuzzy soft set is a mapping from parameters to $M^kFS(U)$. It is a parameterized family of multi-fuzzy subsets of $U$. For $e \in A$, $F(e)$ may be considered as the set of $e$-approximate elements of the multi-fuzzy soft set $(\tilde{F}, A)$.

We illustrate this definition by an example.

Example 4. Suppose that $U = \{c_1, c_2, c_3, c_4, c_5\}$ is the set of cell phones under consideration, and $A = \{e_1, e_2, e_3\}$ is the set of parameters, where $e_1$ stands for the parameter colour which consists of red, green and blue, $e_2$ stands for the parameter ingredient which is made from plastic, liquid crystal and metal, and $e_3$ stands for the parameter price which can be various: high, medium or low. We define a multi-fuzzy soft set of dimension 3 as follows:
\[ F(\varepsilon_1) = \{ c_1 / (0.4, 0.2, 0.3), c_2 / (0.2, 0.1, 0.6), c_3 / (0.1, 0.3, 0.4), c_4 / (0.3, 0.1, 0.3), c_5 / (0.7, 0.1, 0.2) \}. \]

\[ F(\varepsilon_2) = \{ c_1 / (0.1, 0.2, 0.6), c_2 / (0.3, 0.2, 0.4), c_3 / (0.5, 0.3, 0.1), c_4 / (0.6, 0.1, 0.3), c_5 / (0.6, 0.2, 0.1) \}. \]

\[ F(\varepsilon_3) = \{ c_1 / (0.3, 0.4, 0.1), c_2 / (0.4, 0.1, 0.2), c_3 / (0.2, 0.2, 0.5), c_4 / (0.7, 0.1, 0.2), c_5 / (0.5, 0.2, 0.3) \}. \]

Definition 7. (Torra, 2010) (Hesitant fuzzy sets) A hesitant fuzzy set (HFS) on a universe of discourse (U) is defined by \( A = \{ (u, h_A(u)) \mid u \in U \} \) and is defined in terms of \( h_A(u) \) when applied to \( U \) and \( h_B(u) \) is a set of some different values in \([0,1]\), indicating the measurable membership degrees of the elements \( u \in U \) to the set \( A \).

For convenience, we call \( h_A(u) \) a hesitant fuzzy element (HFE).

Example 5. Let \( U = \{ x_1, x_2, x_3, x_4 \} \) and \( h_B(x_1) = \{ 0.1, 0.3, 0.7 \} \), \( h_B(x_2) = \{ 0.6, 0.8 \} \), \( h_B(x_3) = \{ 0.2, 0.4, 0.6, 0.9 \} \), \( h_B(x_4) = \{ 0.5, 0.7, 0.8 \} \) be the membership degree sets of \( x_1, x_2, x_3, x_4 \) respectively. Then, the hesitant fuzzy set \( A = \{ (x_1, (0.1, 0.3, 0.7)), (x_2, (0.6, 0.8)), (x_3, (0.2, 0.4, 0.6, 0.9)), (x_4, (0.5, 0.7, 0.8)) \} \) is denoted by \( A \) (Hesitant fuzzy set).

Definition 8. (Dey and Pal, 2016b) (Hesitant multi-fuzzy set) Let \( k \) be a positive integer. A hesitant multi-fuzzy set \( A \) of dimension \( k \) in \( U \) is a set

\[ A = \{ (u, h_A(u)) \mid u \in U \} , \]

where \( h_A(u) = \{ h_A^1(u), h_A^2(u), \ldots, h_A^k(u) \} \) and \( h_A^i(u) \) is a hesitant fuzzy element for \( i = 1, 2, \ldots, k \).

The function \( h_A(u) = \{ h_A^1(u), h_A^2(u), \ldots, h_A^k(u) \} \) is denoting the possible multi-membership degrees of the element \( u \in U \) to the set \( A \). For convenience, we call \( h_A(u) \) a hesitant multi-fuzzy element.

Here, \( k \) is called the dimension of the hesitant multi-fuzzy set \( A \). The set of all hesitant multi-fuzzy sets of dimension \( k \) in \( U \) is denoted by \( HM^kFS(U) \).

Note 3 A hesitant multi-fuzzy set of dimension 1 is a Torra's hesitant fuzzy set.

We now give the concept of hesitant multi-fuzzy soft set.

Definition 9. (Dey and Pal, 2016b) (Hesitant multi-fuzzy soft set) Let \( U = \{ x_1, x_2, \ldots, x_n \} \) be an initial universe, \( E = \{ e_1, e_2, \ldots, e_m \} \) be the universal set of parameters and \( A \subseteq E \). Additionally, let \( HM^kFS(U) \) be the set of all hesitant multi-fuzzy sets of dimension \( k \) in \( U \).

A pair \( (F, A) \) is called a hesitant multi-fuzzy soft set of dimension \( k \) over \( U \), where \( F \) is a mapping given by \( F: A \rightarrow HM^kFS(U) \).

A hesitant multi-fuzzy soft set is a mapping from parameters to \( HM^kFS(U) \), and it is a parameterized family of hesitant multi-fuzzy subsets of \( U \). We can consider, \( F(e) \) as the set of \( e \)-approximate element of \( (F, A) \).

Definition 10. (Dey and Pal, 2016b) Let \( A \subseteq E \) and \( (F, A), (G, B) \) be two hesitant multi-fuzzy soft sets of dimension \( k \) in \( U \). Then, \( (G, B) \) is said to be a hesitant multi-fuzzy soft subset of \( (F, A) \) if

1. \( B \subseteq A \), and
2. \( G(e) \subseteq F(e) \) for all \( e \in B \).

Here, we write \( (G, B) \subseteq (F, A) \).

Note 4 Two hesitant multi-fuzzy soft sets are equal if \( (G, B) \subseteq (F, A) \) and \( (F, A) \subseteq (G, B) \), and it is denoted by \( (F, A) = (G, B) \).

Definition 11. (Dey and Pal, 2016b) (Null hesitant multi-fuzzy soft sets) A hesitant multi-fuzzy soft set \( (F, A) \) of dimension \( k \) over \( U \) is called the null hesitant multi-fuzzy soft set if \( F(e) = \emptyset_k \) for all \( e \in A \), and it is denoted by \( \emptyset^k_A \).
3. Hesitant multi-fuzzy soft topology

Now we are ready to define the concept of hesitant multi-fuzzy soft topology. Let $U$ be an initial universe, $E$ be the set of parameters, multi-fuzzy soft topology and discrete hesitant multi-fuzzy soft topology, respectively, as called in point set topology.

**Example 6.** Let $(\tilde{F}, \tilde{X})$ be as $(\tilde{F}, A)$ in Example 6. Then, the subfamily

$$\mathcal{P}(U)$$

be the set of all subsets of $U$ and $\mathfrak{F}_k^h(U; E)$ be the family of all hesitant multi-fuzzy soft sets over $U$ via parameters in $E$.

**Definition 18.** Let $(\tilde{F}, \tilde{X})$ be an element of $\mathfrak{F}_k^h(U; E)$, $\mathcal{P}(\tilde{F}, X)$ be the set of all hesitant multi-fuzzy soft subsets of $(\tilde{F}, \tilde{X})$ and $\tilde{\tau}$ be a subfamily of $\mathcal{P}(\tilde{F}, X)$. Then, $\tilde{\tau}$ is called hesitant multi-fuzzy soft topology on $(\tilde{F}, \tilde{X})$ if the following conditions are satisfied:

1. $\Phi_k^h(\tilde{F}, X) \in \tilde{\tau}$
2. $G(\tilde{A}, G(B) \in \tau \Rightarrow (\tilde{G}(\tilde{A}), \tilde{G}(B) \in \tilde{\tau})$
3. $(\{G_k(A_k) : k \in K\} \in \tau \Rightarrow \bigcup_{k \in K} G_k(A_k) \in \tilde{\tau}$

The pair $(X_\tilde{\tau}, \tilde{\tau})$ is called a hesitant multi-fuzzy soft topological space. Each member of $\tilde{\tau}$ is called a $\tilde{\tau}$-open hesitant multi-fuzzy soft set. A hesitant multi-fuzzy soft set is called $\tilde{\tau}$-closed if its complement is $\tilde{\tau}$-open. We shall call a $\tilde{\tau}$-open ($\tilde{\tau}$-closed) hesitant multi-fuzzy soft set simply open (closed) set.

Here, $(\Phi_k^h(\tilde{F}, X))$ and $\mathfrak{F}(\tilde{F}, X)$ are two examples for hesitant multi-fuzzy soft topology on $(\tilde{F}, \tilde{X})$ and are called indiscr-
Then, $\tau_1$ is hesitant multi-fuzzy soft finer than $\tau_2$.

**Definition 23.** Let $\tilde{\tau}$ be a hesitant multi-fuzzy soft topology on $(\tilde{F}, X)$ and $\tilde{\tau}$ be a subfamily of $\tilde{\tau}$. If every element of $\tilde{\tau}$ is written as arbitrary hesitant multi-fuzzy soft union of some elements of $\tilde{\tau}$, then $\tilde{\tau}$ is called a hesitant multi-fuzzy soft basis for the hesitant multi-fuzzy soft topology $\tilde{\tau}$.

**Example 9.** Let us consider the hesitant multi-fuzzy soft topology $\tilde{\tau}_2$ as in Example 16. Then, the subfamily

$$\tilde{\tau}_2 = \{\tilde{F}_A, \tilde{F}_B, \tilde{F}_C, \tilde{F}_D\}$$

is a basis of $\tilde{\tau}_2$.

The next two results can be proved using the previous results.

**Proposition 1.** Let $\tilde{\tau}_1, \tilde{\tau}_2$ be two hesitant multi-fuzzy soft topologies on $F$, and $\tilde{\tau}_1 \subseteq \tilde{\tau}_2$, then $\tilde{\tau}_1$ is hesitant multi-fuzzy soft finer than $\tilde{\tau}_2$.

**Proposition 2.** Let $\tilde{\tau}$ be a hesitant multi-fuzzy soft topology on $F$ and $\tilde{\tau}$ be a hesitant multi-fuzzy soft basis for $\tilde{\tau}$. Then, $\tilde{\tau}$ equals to the collection of hesitant multi-fuzzy soft unions of the elements of $\tilde{\tau}$.

We are now ready to show:

**Theorem 5.** Let $(X, \tilde{\tau})$ be a hesitant multi-fuzzy soft topological space and $(\tilde{G}, A) \subseteq \tilde{\tau}(F, X)$. Then, the collection $\tilde{\tau}_{(\tilde{G}, A)} = \{(\tilde{G}, A) \cap \tilde{\tau}_{(\tilde{G}, A)} \cap \tilde{\tau}_{(\tilde{G}, A)} : (\tilde{G}, A) \subseteq \tilde{\tau}_{(\tilde{G}, A)} \cap \tilde{\tau}_{(\tilde{G}, A)} \}$ is a hesitant multi-fuzzy soft topology on the hesitant multi-fuzzy soft subsets $(\tilde{G}, A)$ relative to the parameter set $A$.

**Proof.**

(i) Since $\tilde{\tau}_{(\tilde{G}, A)} \subseteq \tilde{\tau}_{(\tilde{G}, A)}$, $\tilde{\tau}_{(\tilde{G}, A)} \cap \tilde{\tau}_{(\tilde{G}, A)}$ is a hesitant multi-fuzzy soft topology on the hesitant multi-fuzzy soft subsets $(\tilde{G}, A)$ relative to the parameter set $A$.

Thus, $\tilde{\tau}_{(\tilde{G}, A)} = \tilde{\tau}_{(\tilde{G}, A)} \cap \tilde{\tau}_{(\tilde{G}, A)}$.

(ii) The proof is obvious.

**Definition 22.** Let $(X_1, \tilde{\tau}_1), (X_2, \tilde{\tau}_2)$ be two hesitant multi-fuzzy soft topological spaces. If each $(G, A) \in \tilde{\tau}_1$ implies $(G, A) \in \tilde{\tau}_2$, then $\tilde{\tau}_2$ is called hesitant multi soft finer than $\tilde{\tau}_1$, or equivalently $\tilde{\tau}_1$ is hesitant multi-fuzzy soft coarser than $\tilde{\tau}_2$.
(ii) Let \((G_{11}, A_1), (G_{12}, A_2) \in \tau\).

Therefore, there exist \((G_{21}, B_1), (G_{22}, B_2) \in \tau\) such that \((G_{11}, A_1) = (G_{11}, A_1) \cap (G_{21}, B_1)\) and \((G_{12}, A_2) = (G_{12}, A_2) \cap (G_{22}, B_2)\). Therefore, \((G_{11}, A_1) \cap (G_{12}, A_2) = (G_{11}, A_1) \cap (G_{21}, B_1) \cap (G_{12}, A_2) \cap (G_{22}, B_2)\).

Since \((G_{21}, B_1), (G_{22}, B_2) \in \tau\), therefore, \((G_{11}, A_1) \cap (G_{12}, A_2) \in \tau\).

(iii) Let \((G_{i}, B_i) : k \in K\) be a subfamily of \(\tau\).

Thus, for each \(k \in K\), there is a hesitant multi-fuzzy soft set \((H_k, C_k)\) of \(\tau\) such that \((G_{i}, B_i) = (G_{i}, A_i) \cap (H_k, C_k)\).

Now, \(\bigcup_{k \in K} (G_{i}, B_i) = \bigcup_{k \in K} (G_{i}, A_i) \cap (H_k, C_k) = (G_{i}, A_i) \cap \bigcup_{k \in K} (H_k, C_k)\).

Since \(\bigcup_{k \in K} (H_k, C_k) \in \tau\), therefore, \(\bigcup_{k \in K} (G_{i}, B_i) \in \tau\).

Definition 24. Let \((X_{\tau}, \tau)\) be a hesitant multi-fuzzy soft topological space and \((G_{i}, A_i) \in \tau(X_{\tau})\). Then, the hesitant multi-fuzzy soft topology \(\tau\) is given in Theorem 5 is called hesitant multi-fuzzy soft subspace topology, and this topology is called a hesitant multi-fuzzy soft subspace of \((X_{\tau}, \tau)\).

The following result follows from the previous results.

Theorem 6. Let \((X_{\tau}, \tau)\) be a hesitant multi-fuzzy soft topological space on \((X, \tau)\). \(B\) be a hesitant multi-fuzzy soft basis for \(\tau\) and \((G_{i}, A_i) \in \tau(X_{\tau})\). Then, the collection \(B = \{ (G_{i}, A_i) \cap (H_k, C_k) : (G_{i}, B) \in \tau\}\) is a hesitant multi-fuzzy soft basis for the hesitant multi-fuzzy soft subspace topology \(\tau\) of \((X_{\tau}, \tau)\).

4. Compactness in hesitant multi-fuzzy soft topological spaces

In this section, we introduce hesitant multi-fuzzy soft cover, hesitant multi-fuzzy soft open cover, hesitant multi-fuzzy soft compactness and the theorem of these concepts.

Let \(U\) be an initial universe, \(E\) be the set of parameters, \(P(U)\) be the set of all subsets of \(U\) and \(\mathcal{W}^*_{\tau} (U, E)\) be the family of all hesitant multi-fuzzy soft set of dimension \(k\) over \(U\) via parameters in \(E\).

Definition 25. Let \((F, \tau)\) be an element of \(\mathcal{W}^*_{\tau} (U, E)\). A family \(\{ (F_{k}, A_k) : k \in K\} \in \mathcal{W}^*_{\tau} (U, E)\) of hesitant multi-fuzzy soft sets is a cover of \((F, \tau)\) if \((F, \tau) \in \bigcup_{k \in K} (F_{k}, A_k)\).

If each member of the family \(\{ (F_{k}, A_k) : k \in K\}\) is a hesitant multi-fuzzy soft open set, then it is called a hesitant multi-fuzzy soft open cover of \((F, \tau)\).

If a subfamily of \(\{ (F_{k}, A_k) : k \in K\}\) is also cover of \((F, \tau)\), then it is called a subcover.

Definition 26. Let \((F, \tau)\) be an element of \(\mathcal{W}^*_{\tau} (U, E)\). Then, \((F, \tau)\) is said to be a hesitant multi-fuzzy soft compact if each hesitant multi-fuzzy soft open cover of \((F, \tau)\) has a finite subcover.

Additionally, a hesitant multi-fuzzy soft topological space \((X_{\tau}, \tau)\) is called compact if each hesitant multi-fuzzy soft open cover of \(U_{X}\) has a finite subcover.

Note 5 Let \((X_{\tau}, \tau_1), (X_{\tau}, \tau_2)\) be two hesitant multi-fuzzy soft topological spaces and \(\tau_2\) be hesitant multi-fuzzy soft finer than \(\tau_1\). Hence, \((X_{\tau}, \tau_2)\) is compact, then \((X_{\tau}, \tau_1)\) is compact.

The main result of this section is:

Theorem 7. Let \((G, \tau)\) be a hesitant multi-fuzzy soft closed set in a hesitant multi-fuzzy soft compact topological space \((X_{\tau}, \tau)\). Then, \((G, \tau)\) is also compact.

Proof.

Let \(\{ (F_{k}, A_k) : k \in K\}\) be any hesitant multi-fuzzy soft open cover of \((G, \tau)\).

Then, \(U_{X} = \bigcup_{k \in K} (F_{k}, A_k)\). Hence, \(\bigcup_{k \in K} (F_{k}, A_k) \in \tau(G, \tau)\).

Thus, \(\{ (F_{k}, A_k) : k \in K\}\) together with hesitant multi-fuzzy soft open set \((G, \tau)\) is a hesitant multi-fuzzy soft open cover of \(U_{X}\). Since, \((X_{\tau}, \tau)\) is compact, there exists a finite subcover.

Therefore, \(\{ (F_{k}, A_k) : k \in K\}\) be the finite subcover.

This implies \((G, \tau) \in \bigcup_{k \in K} (F_{k}, A_k)\).

Hence, \((G, \tau)\) is compact.

Definition 27. Let \((X_{\tau}, \tau)\) be a hesitant multi-fuzzy soft topology space and \(e_1 e_2 \in X\) with \(e_1 \neq e_2\). If there exists two hesitant multi-fuzzy soft open sets \((F, \tau)\), \((G, \tau)\) such that \((F(e_1) \in (F, \tau))\), \((G(e_2) \in (G, \tau))\) and \(e_1 e_2 \subseteq (F, \tau)\), then \((X_{\tau}, \tau)\) is called a hesitant multi-fuzzy soft Hausdorff space.

Theorem 8. Let \((G, \tau)\) be a hesitant multi-fuzzy soft compact set in a hesitant multi-fuzzy soft Hausdorff space \((X_{\tau}, \tau)\). Then, \((G, \tau)\) is hesitant multi-fuzzy soft closed set.

Proof.

Let \(e_1 \in X\) with \(G(e_1)\), but does not belong to \((G, \tau)\), i.e., \(G(e_1) \notin (G, \tau)\).

Let \(G(e_2) \in (G, \tau)\).

Therefore, \(e_1 \neq e_2\) and \(e_1 e_2 \in X\).

Thus, by the Hausdorff property of \((X_{\tau}, \tau)\) there exist two disjoint hesitant multi-fuzzy soft open sets \((H(e_1), A)\), \((G(e_2), A)\), such that \(H(e_1) \in (H(e_1), A)\) and \((G(e_2), A) \in (G(e_2), A)\).

Thus, the collection \(\{ (G(e_1), A) : e_2 \in X, \ G(e_2) \in (G, \tau) \}\) is a hesitant multi-fuzzy soft open cover of \((G, \tau)\).

Since \((G, \tau)\) is a hesitant multi-fuzzy soft compact set in \((X_{\tau}, \tau)\), therefore, there exists a finite subcover for \((G, \tau)\).

Let \(\{ (G_{k}, A), (G_{k}, A), ..., (G_{k}, A)\} \) be the finite subcover.

Thus, \(\bigcap_{k=1}^{n} (H(e_1), A) \subseteq (G_{k}, A)\) is a hesitant multi-fuzzy soft open set contained in \((G, \tau)\).

Therefore, \((G, \tau)\) is a multi-fuzzy soft open set.

Hence, \((G, \tau)\) is a multi-fuzzy soft closed set.

5. Conclusion

The purpose of this paper is to discuss some important properties of hesitant multi-fuzzy soft topological spaces. We introduce the neighbourhood of a hesitant multi-fuzzy soft sets and interior hesitant multi-fuzzy soft sets. We also introduce the hesitant multi-fuzzy soft basis and hesitant multi-fuzzy soft subspace topology and have established several interesting properties. To extend this work, one could study connectedness or other interesting properties for hesitant multi-fuzzy soft topological spaces. We hope that this work will help enhance the study of hesitant multi-fuzzy soft topological spaces for researchers.

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References