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Weak representations of quantified hyperspace structures

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Abstract

It is the aim of this note, to show that several results from Beer (1993), Beer et al. (1992) and Beer and Lucchetti (1993) about the description of some hypertopologies as weak or initial topologies can be generalized to the quantitative setting of approach hyperspace structures as introduced by Lowen and Sioen (1996, 1998). © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the study of topologies on the hyperspace of all non-empty closed subsets of a metric space, an important interest has been taken during the last decade in looking for descriptions of many of those well-investigated hypertopologies as weak or initial topologies, as can be seen, e.g., from Beer [2], Beer, Lechicki, Levi and Naimpally [3] and Beer and Lucchetti [4]. Here one detects two main types of results: on the one hand those describing a particular hyperspace topology as the supremum of a family of other hyperspace topologies and on the other hand, those providing a description of a hyperspace topology as being the initial topology for a source of $[0,\infty]$-valued functionals on the hyperspace.

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It is our aim in this paper to generalize some of these results from [2–4] to the broader, quantified setting of approach hyperstructures as presented by Lowen and Sioen [6,7], which were introduced to remedy the loss of quantitative information when passing from a metric space to its hyperspace endowed with one of the hypertopologies and which represent exactly the canonical numerical information, compatible with the studied hypertopology, we can retain under this transition.

2. Some weak representations

In the sequel, (X, d) will be an arbitrary metric space and we will write $CL(X)$ for the hyperspace of all non-empty closed subsets. Throughout the rest of the paper, we will use the notations $\mathcal{T}_{W_d}$ respectively $\mathcal{T}_{AW_d}$, $\mathcal{T}_{prox(d)}$, $\mathcal{T}_{b\text{-}prox(d)}$, $\mathcal{T}_V$ and $\mathcal{T}_{locfin}$ for the Wijsman, respectively the Attouch–Wets, the d-proximal, the bounded d-proximal, the Vietoris and the locally finite topology on $CL(X)$, as defined in [2–4] and we refer hereto for more information. For any $x \in X$ and $A, B \in 2^X$, we write $d(x, A) \doteq \inf_{y\in A} d(x, y)$, respectively $D_d(A, B) \doteq \inf_{x\in A} d(x, B)$, $e_d(A, B) \doteq \sup_{x\in A} d(x, B)$ and $h_d(A, B) \doteq e_d(A, B) \vee e_d(B, A)$ for the 'distance from x to A', respectively the 'gap between A and B', the 'excess of A over B' and the 'Hausdorff distance between A and B'. We will use $\mathcal{E}(d)$ respectively $\mathcal{E}_u(d)$ and $\mathcal{E}_u^b(d)$ to denote the set of all metrics on X which are equivalent to d, respectively which are uniformly equivalent to d, respectively which are uniformly equivalent to and determine the same bounded subsets as d. We will also denote the set of all (respectively all non-empty, all finite and all non-empty finite) subsets of X by 2^X, (respectively 2_0^X, $2^{(X)}$ and $2_0^{(X)}$) and if $A \in 2^X$ and $\varepsilon > 0$ we put

$$S_\varepsilon(A) \doteq \{y \in X \mid d(y, A) < \varepsilon\}.$$

The Euclidean metric on $[0, \infty[$ is denoted by d_E and when working in $[0, \infty]$, we adopt the conventions $\infty - \infty \doteq 0$, $0 \cdot \infty \doteq \infty \cdot 0 = 0$. If X is a set and A is a subset of X, θ_A (respectively 1_A) stands for the function on X taking the value 0 on A and ∞ on $X \setminus A$ (respectively 1 on A and 0 on $X \setminus A$).

For any terminology, notations or information about approach spaces, we refer to Lowen [5], so we will restrict ourselves to recalling some definitions about the approach hyperstructures introduced by Lowen and Sioen [6,7]. A non-empty subset of 2^X is called a tiling of X if $\emptyset \notin \Sigma$, the members of Σ cover X and Σ is closed for the formation of finite unions. If Σ is a tiling of X, we define for every $F \in \Sigma$

$$d_F : CL(X) \times CL(X) \to [0, \infty] : (A, B) \to \sup_{x\in F} \bigl| d(x, A) - d(x, B) \bigr|.$$

Then $\{d_F \mid F \in \Sigma\}$ is a collection of ∞ p-metrics on $CL(X)$ and hence defines a uniform approach distance, which we will call the 'distance of Σ uniform convergence' and which we will denote by $\delta_{\Sigma,d}$, as follows:

$$\delta_{\Sigma,d} : CL(X) \times 2^{CL(X)} \to [0, \infty] : (A, \mathcal{A}) \to \sup_{F\in\Sigma} \inf_{B\in\mathcal{A}} d_F(A, B).$$

Now one can prove that the ∞ p-metric coreflection of $(CL(X), \delta_{\Sigma,d})$ is $(CL(X), h_d)$ and that the topological coreflection of $(CL(X), \delta_{\Sigma,d})$ is exactly $(CL(X), \mathcal{T}_{\Sigma,d})$, where $\mathcal{T}_{\Sigma,d}$ is the topology of uniform convergence on members of Σ under the identification $CL(X)$ with the set of distance functionals $\{d(\cdot, A) \mid A \in CL(X)\}$. Note that $2_0^{(X)}$ is a tiling and that $\mathcal{T}_{2_0^{(X)},d} = \mathcal{T}_{W_d}$. We will therefore call $\delta_{2_0^{(X)},d}$ the 'Wijsman distance' and we will denote it by δ_{W_d}. The collection of all non-empty bounded subsets is a tiling too, the hyperdistance corresponding to which is denoted by δ_{AW_d} and will be called the 'Attouch–Wets' distance.

If on the other hand, for each $\Gamma \in 2_0^{(CL(X))}$ we define

$$d^{\Gamma}: CL(X) \times CL(X) \to [0,\infty]: (A, B) \to \sup_{D\in\Gamma} \left|D_d(A, D) - D_d(B, D)\right|,$$

the collection of p-metrics $\{d^{\Gamma} \mid \Gamma \in 2_0^{(CL(X))}\}$ generates a uniform approach distance $\delta_{prox(d)}$ on $CL(X)$, given by

$$\delta_{prox(d)}: CL(X) \times 2^{CL(X)} \to [0,\infty]: (A, \mathcal{A}) \to \sup_{\Gamma\in 2_0^{(CL(X))}} \inf_{B\in\mathcal{A}} d^{\Gamma}(A, B).$$

It can be shown that the ∞p-metric and the topological coreflection of $(CL(X), \delta_{prox(d)})$ are $(CL(X), h_d)$, respectively $(CL(X), \mathcal{T}_{prox(d)})$ and therefore, $\delta_{prox(d)}$ is called the 'd-proximal distance'. With

$$CLB(X) \doteqdot \left\{B \in CL(X) \mid B \text{ bounded}\right\},$$

the collection $\{d^{\Gamma} \mid \Gamma \in 2_0^{(CLB(X))}\}$ generates the bounded d-proximal distance $\delta_{b\text{-}prox(d)}$ in the same way.

The following proposition generalizes some results of Beer, Levi, Lechicki, Lucchetti and Naimpally (see [2–4]) concerning the representation of given hypertopologies as suprema to our framework.

Proposition 2.1. *For every metric space, the following equalities hold*

$$\delta_{\mathcal{T}_{prox(d)}} = \bigvee_{\rho\in\mathcal{E}_u(d)} \delta_{W_\rho}, \qquad \delta_{\mathcal{T}_{b-prox(d)}} = \bigvee_{\rho\in\mathcal{E}_u^b(d)} \delta_{W_\rho}, \qquad \delta_{\mathcal{T}_V} = \bigvee_{\rho\in\mathcal{E}(d)} \delta_{W_\rho},$$

$$\delta_{\mathcal{T}_V} = \bigvee_{\rho\in\mathcal{E}(d)} \delta_{prox(\rho)}, \qquad \delta_{\mathcal{T}_{locfin}} = \bigvee_{\rho\in\mathcal{E}(d)} \delta_{h_\rho}$$

where all the suprema are taken in AP.

Next we generalize a result from Beer and Lucchetti (see [4]) stating that the d-proximal topology $\mathcal{T}_{prox(d)}$ is the initial topology for a source of $[0,\infty]$-valued excess functionals, by showing that an analogous result holds for the overlaying "proximal" distance $\delta_{prox(d)}$. If we define $\delta_E: [0,\infty] \times 2^{[0,\infty]} \to [0,\infty]$ by

$$\delta_E(x, A) \doteqdot \begin{cases} \infty & A = \emptyset, \\ 0 & x = \infty,\ \sup A = \infty, \\ \infty & x = \infty,\ \sup A < \infty, \\ \delta_{d_E}(x, A \cap \mathbb{R}^+) & x < \infty, \end{cases}$$

then δ_E can be proved to be a distance on $[0,\infty]$ and one can verify that the topological coreflection of $([0,\infty],\delta_E)$ is $([0,\infty],\mathcal{T}_E^*)$, where $\mathcal{T}_E^*$ denotes the topology of the Alexandroff one-point compactification of $([0,\infty[,\mathcal{T}_{d_E})$. It is also easy to see that the ∞p-metric coreflection of $([0,\infty],\delta_E)$ is $([0,\infty],d_E^e)$ with

$$d_E^e:[0,\infty]\times[0,\infty]\to[0,\infty]:(x,y)\to\begin{cases}0 & x=y=\infty,\\ \infty & x\neq y,\ \infty\in\{x,y\},\\ d_E(x,y) & x,y<\infty.\end{cases}$$

Often continuity of a $[0,\infty]$-valued functional is shown by separately proving that it is lower and upper semicontinuous and this technique is also used in the proof of the topological result we want to generalize. We will therefore 'split' δ_E into two halves which will allow us to use a similar argument to show that a given function is a contraction. Let

$$\delta_E^+:[0,\infty]\times 2^{[0,\infty]}\to[0,\infty]:(x,A)\to\begin{cases}\infty & A=\emptyset,\\ (x-\sup A)\vee 0 & A\neq\emptyset\end{cases}$$

and

$$\delta_E^-:[0,\infty]\times 2^{[0,\infty]}\to[0,\infty]:(x,A)\to\begin{cases}\infty & A=\emptyset,\\ (\inf A-x)\vee 0 & A\neq\emptyset.\end{cases}$$

It can be verified that δ_E^+ and δ_E^- are distances on $[0,\infty]$, where a basis for the approach system of δ_E, respectively δ_E^+ and δ_E^- is given by:

$$\mathcal{B}_{\delta_E}(x)\doteqdot\begin{cases}\{d_E^e(x,\cdot)\} & x\in\mathbb{R}^+,\\ \{\theta_{]n,\infty]}\mid n\in\mathbb{N}\} & x=\infty,\end{cases}$$

respectively

$$\mathcal{B}_{\delta_E^+}(x)\doteqdot\begin{cases}\{d_E^e(x,\cdot)1_{[0,x]}\} & x\in\mathbb{R}^+,\\ \{\theta_{]n,\infty]}\mid n\in\mathbb{N}\} & x=\infty,\end{cases}$$

and

$$\mathcal{B}_{\delta_E^-}(x)\doteqdot\begin{cases}\{d_E^e(x,\cdot)1_{[x,\infty]}\} & x\in\mathbb{R}^+,\\ \{0\} & x=\infty.\end{cases}$$

This shows that δ_E is the supremum of δ_E^+ and δ_E^- in AP, so if (X,δ) is an approach space and $f:X\to[0,\infty]$ is a function $f:(X,\delta)\to([0,\infty],\delta_E)$ is a contraction if and only if both $f:(X,\delta)\to([0,\infty],\delta_E^+)$ and $f:(X,\delta)\to([0,\infty],\delta_E^-)$ are contractions. We now come to the actual theorem.

Theorem 2.2. *Let (X,d) be a metric space. Then we have that $\delta_{prox(d)}$ is the initial distance on $CL(X)$ for the source*

$$\big(e_d(\cdot,F):CL(X)\to\big([0,\infty],\delta_E\big):A\to e_d(A,F)\big)_{F\in CL(X)}.$$

Proof. To simplify notations, we will denote the initial distance on $CL(X)$ for the source

$$\big(e_d(\cdot,F):CL(X)\to\big([0,\infty],\delta_E\big):A\to e_d(A,F)\big)_{F\in CL(X)}$$

by δ. As δ is the coarsest distance on $CL(X)$ which makes all functions of this source into contractions, it suffices to verify that

$$e_d(\cdot, F) : \big(CL(X), \delta_{prox(d)}\big) \to \big([0, \infty], \delta_E\big) : A \to e_d(A, F)$$

is a contraction for every $F \in CL(X)$, in order to show that $\delta_{prox(d)} \geqslant \delta$. To do so, fix $F \in CL(X)$. We start by proving that

$$e_d(\cdot, F) : \big(CL(X), \delta_{prox(d)}\big) \to \big([0, \infty], \delta_E^-\big) : A \to e_d(A, F)$$

is a contraction. Take $A \in CL(X)$ and $\varphi \in \mathcal{B}_{\delta_E^-}(e_d(A, F))$. If $e_d(A, F) = \infty$, then $\varphi = 0$, yielding that $\varphi \circ e_d(\cdot, F) = 0 \in \mathcal{A}_{\delta_{prox(d)}}(A)$, so we are done. If $e_d(A, F) < \infty$, $\varphi = d_E^e(e_d(A, F), \cdot)1_{[e_d(A,F),\infty]}$. We now intend to verify that $\varphi \circ e_d(\cdot, F) \in \mathcal{A}_{\delta_{prox(d)}}(A)$, so fix $\omega \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}_0^+$. We now have to construct $\psi^{\omega,\varepsilon} \in \mathcal{A}_{\delta_{prox(d)}}(A)$ such that

$$\varphi \circ e_d(\cdot, F) \wedge \omega \leqslant \psi^{\omega,\varepsilon} + \varepsilon.$$

If $\omega \leqslant \varepsilon$ we can take $\psi^{\omega,\varepsilon}$ to be the constant zero-functional on $CL(X)$, so we may assume without loss of generality that $\omega > \varepsilon$. Take $n \in \mathbb{N} \setminus \{0, 1\}$ minimal such that $n\varepsilon \geqslant \omega$, define for every $k \in \{1, \ldots, n-1\}$

$$D_k^{\omega,\varepsilon} \doteq \begin{cases} (S_{k\varepsilon}(A))^c & (S_{k\varepsilon}(A))^c \neq \emptyset, \\ X & (S_{k\varepsilon}(A))^c = \emptyset, \end{cases}$$

and let $\Gamma^{\omega,\varepsilon} \doteq \{D_1^{\omega,\varepsilon}, \ldots, D_{n-1}^{\omega,\varepsilon}\}$ and $\psi^{\omega,\varepsilon} \doteq d^{\Gamma^{\omega,\varepsilon}}(A, \cdot)$. Then surely $\psi^{\omega,\varepsilon} \in \mathcal{A}_{\delta_{prox(d)}}(A)$ and as stated above, we are done in this case if we prove the following claim:

$$\varphi \circ e_d(\cdot, F) \wedge \omega \leqslant d^{\Gamma^{\omega,\varepsilon}}(A, \cdot) + \varepsilon.$$

Therefore, fix $B \in CL(X)$. If $e_d(B, F) \leqslant e_d(A, F)$, we see that $\varphi(e_d(B, F)) = 0$, so there is nothing to prove. Assume that $e_d(B, F) > e_d(A, F)$. If $d_E^e(e_d(A, F), e_d(B, F)) \geqslant \omega$, we have that $\varphi(e_d(B, F)) \wedge \omega = \omega$. On the other hand, it then follows that

$$e_d(B, A) \geqslant d_E^e\big(e_d(A, F), e_d(B, F)\big) \geqslant \omega > (n-1)\varepsilon,$$

which yields that $B \cap (S_{(n-1)\varepsilon}(A))^c \neq \emptyset$. This implies that $D_{n-1}^{\omega,\varepsilon} = (S_{(n-1)\varepsilon}(A))^c$ and that $D_d(B, D_{n-1}^{\omega,\varepsilon}) = 0$. On the other hand it is obvious that $D_d(A, D_{n-1}^{\omega,\varepsilon}) \geqslant (n-1)\varepsilon$, so it follows that

$$\begin{aligned} d^{\Gamma^{\omega,\varepsilon}}(A, B) + \varepsilon &\geqslant \big|D_d(A, D_{n-1}^{\omega,\varepsilon}) - D_d(B, D_{n-1}^{\omega,\varepsilon})\big| + \varepsilon \\ &\geqslant n\varepsilon \geqslant \omega \geqslant \varphi\big(e_d(B, F)\big) \wedge \omega. \end{aligned}$$

Next we treat the case where $d_E^e(e_d(A, F), e_d(B, F)) < \omega$. If moreover $d_E^e(e_d(A, F), e_d(B, F)) \leqslant \varepsilon$ we have nothing to prove, so we only need to consider the case where $d_E^e(e_d(A, F), e_d(B, F)) > \varepsilon$. Then we have that

$$k\varepsilon \geqslant d_E^e\big(e_d(A, F), e_d(B, F)\big) > (k-1)\varepsilon$$

for some $k \in \{2, \ldots, n\}$, whence

$$e_d(B, A) \geqslant d_E^e\big(e_d(A, F), e_d(B, F)\big) > (k-1)\varepsilon,$$

yielding that $B \cap (S_{(k-1)\varepsilon}(A))^c \neq \emptyset$. Then $D^{\omega,\varepsilon}_{k-1} = (S_{(k-1)\varepsilon}(A))^c$, and $D_d(B, D^{\omega,\varepsilon}_{k-1}) = 0$. Since $D_d(A, D^{\omega,\varepsilon}_{k-1}) \geqslant (k-1)\varepsilon$, it now follows in the same way as above that

$$d^{\Gamma^{\omega}_{\varepsilon}}(A, B) + \varepsilon \geqslant k\varepsilon \geqslant \varphi\big(e_d(B, F)\big) \wedge \omega,$$

which completes this part of the proof. Our next step is to show that

$$e_d(\cdot, F) : \big(CL(X), \delta_{prox(d)}\big) \to \big([0, \infty], \delta^+_E\big)$$

is a contraction. Fix $A \in CL(X)$ and $\varphi \in \mathcal{B}_{\delta^+_E}(e_d(A, F))$. Again we will have to prove that $\varphi \circ e_d(\cdot, F) \in \mathcal{A}_{\delta_{prox(d)}}(A)$, and we will proceed in the same way as we did higher up. If $e_d(A, F) = \infty$, then $\varphi = \theta_{]m,\infty]}$ for some $m \in \mathbb{N}$. Fix $\omega \in \mathbb{R}^+$. Since $e_d(A, F) = \infty$ implies that $e_d(A, F^{(m)}) = \infty$, there exists $a^\omega \in A$ such that $d(a^\omega, F^{(m)}) \geqslant \omega$. It suffices now to prove that

$$\varphi \circ e_d(\cdot, F) \wedge \omega \leqslant d^{\{\{a^\omega\}\}}(A, \cdot),$$

in order to complete this part of the proof. Therefore take $B \in CL(X)$ arbitrary. In case that $e_d(B, F) > m$, $\varphi(e_d(B, F)) = 0$, so there is nothing to prove. If $e_d(B, F) \leqslant m$ it follows that $B \subset F^{(m)}$ yielding that

$$d^{\{\{a^\omega\}\}}(A, B) = d(a^\omega, B) \geqslant d(a^\omega, F^{(m)}) \geqslant \omega = \varphi\big(e_d(B, F)\big) \wedge \omega,$$

and we are done. We now consider the case where $e_d(A, F) < \infty$, so we have that $\varphi = d^e_E(e_d(A, F), \cdot)1_{[0,e_d(A,F)]}$. Fix $\varepsilon \in \mathbb{R}^+_0$. First note that if $e_d(A, F) > \varepsilon$ we have that $A \cap (S_{e_d(A,F)-\varepsilon}(F))^c \neq \emptyset$, which implies that $(S_{e_d(A,F)-\varepsilon}(F))^c \neq \emptyset$. Let

$$D^\varepsilon \doteqdot \begin{cases} S_{e_d(A,F)-\varepsilon}(F))^c & e_d(A, F) > \varepsilon, \\ X & e_d(A, F) \leqslant \varepsilon. \end{cases}$$

We now only have to verify that

$$\varphi \circ e_d(\cdot, F) \leqslant d^{\{D^\varepsilon\}}(A) + \varepsilon.$$

Take $B \in CL(X)$. Note that when $e_d(B, F) > e_d(A, F)$, $\varphi(e_d(B, F)) = 0$, so there is nothing to prove. We therefore may assume that $e_d(B, F) \leqslant e_d(A, F)$. On the one hand, if $e_d(A, F) \leqslant \varepsilon$, we see that

$$\varphi\big(e_d(B, F)\big) = e_d(A, F) - e_d(B, F) \leqslant \varepsilon = d^{\{D^\varepsilon\}}(A, B) + \varepsilon.$$

If, on the other hand $e_d(A, F) > \varepsilon$ it follows that $D^\varepsilon \doteqdot (S_{e_d(A,F)-\varepsilon}(F))^c$ and that $D_d(A, D^\varepsilon) = 0$. Suppose

$$D_d(B, D^\varepsilon) + \varepsilon < e_d(A, F) - e_d(B, F)$$

would hold true. Then there would exist $x \in D^\varepsilon$ and $y \in B$ such that

$$d(x, y) + \varepsilon < e_d(A, F) - e_d(B, F),$$

which would imply that

$$d(x, F) \leqslant d(x, y) + d(y, F) \leqslant d(x, y) + e_d(B, F) < e_d(A, F) - \varepsilon,$$

yielding a contradiction. We therefore may conclude that

$$d^{\{D^\varepsilon\}}(A,B)+\varepsilon = D_d(B,D^\varepsilon)+\varepsilon \geqslant \varphi\bigl(e_d(B,F)\bigr),$$

which also in this case completes the verification. The second part of the proof consists in showing that $\delta_{prox(d)} \leqslant \delta$. To do this, it suffices to show that

$$\forall A, D \in CL(X)\colon\ d^{\{D\}}(A,\cdot) \in \mathcal{A}_\delta(A).$$

Take $A, D \in CL(X)$ arbitrary. Fix $\omega \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}_0^+$. The proof will be completed if we construct $\psi^{\omega,\varepsilon} \in \mathcal{A}_\delta(A)$ such that

$$d^{\{D\}}(A,\cdot) \wedge \omega \leqslant \psi^{\omega,\varepsilon} + \varepsilon.$$

If $\omega \leqslant \varepsilon$, we can take the constant zero-functional on $CL(X)$ for $\psi^{\omega,\varepsilon}$ so we may assume that $\omega > \varepsilon$ for the rest of the proof. Then we can take $n \in \mathbb{N} \setminus \{0,1\}$ minimal such that $n\varepsilon \geqslant \omega$. Let

$$F_0^{\omega,\varepsilon} \doteqdot A,$$

and for every $k \in \{1,\ldots,n-1\}$ define

$$F_k^{\omega,\varepsilon} \doteqdot \begin{cases} (S_{D_d(A,D)+k\varepsilon}(D))^c & (S_{D_d(A,D)+k\varepsilon}(D))^c \neq \emptyset, \\ X & (S_{D_d(A,D)+k\varepsilon}(D))^c = \emptyset. \end{cases}$$

For every $k \in \{0,\ldots,n-1\}$ we put

$$\rho_k^{\omega,\varepsilon} \doteqdot \begin{cases} d_E^e(e_d(A,F_k^{\omega,\varepsilon}),\cdot) & e_d(A,F_k^{\omega,\varepsilon}) < \infty, \\ \theta_{]0,\infty]} & e_d(A,F_k^{\omega,\varepsilon}) = \infty, \end{cases}$$

and we define

$$\psi^{\omega,\varepsilon} \doteqdot \sup_{k=0}^{n-1} \rho_k^{\omega,\varepsilon} \circ e_d(\cdot, F_k^{\omega,\varepsilon}).$$

Then clearly $\psi^{\omega,\varepsilon} \in \mathcal{A}_\delta(A)$. Now take $B \in CL(X)$ arbitrary. If $D_d(B,D) \leqslant D_d(A,D)$, we have that

$$d^{\{D\}}(A,B) \leqslant e_d(B,A) = d_E^e\bigl(e_d(A,A), e_d(B,A)\bigr) \leqslant \psi^{\omega,\varepsilon}(B),$$

so we are done. Now assume that $D_d(B,D) > D_d(A,D)$. If $D_d(B,D) - D_d(A,D) \leqslant \varepsilon$ there is nothing to prove, so we can assume that $D_d(B,D) - D_d(A,D) > \varepsilon$ without loss of generality. We first consider the case that $D_d(B,D) - D_d(A,D) \geqslant \omega$. Then we have that

$$D_d(B,D) > D_d(A,D) + (n-1)\varepsilon,$$

which implies that $B \subset (S_{D_d(A,D)+(n-1)\varepsilon}(D))^c$ and therefore

$$F_{n-1}^{\omega,\varepsilon} = \bigl(S_{D_d(A,D)+(n-1)\varepsilon}(D)\bigr)^c.$$

Suppose there would exist $\gamma \in \mathbb{R}_0^+$ with $e_d(A, F_{n-1}^{\omega,\varepsilon}) < (n-1)\varepsilon - \gamma$. Then for every $a \in A$ there would exist $x_a \in F_{n-1}^{\omega,\varepsilon}$ such that $d(a,x_a) < (n-1)\varepsilon - \gamma$, but this would imply that for every $a \in A$

$$D_d(A,D)+(n-1)\varepsilon \leqslant d(x_a,D) \leqslant d(x_a,a)+d(a,D)$$
$$< (n-1)\varepsilon-\gamma+d(a,D),$$

yielding a contradiction. So we have that $e_d(A,F^{\omega,\varepsilon}_{n-1}) \geqslant (n-1)\varepsilon$. If $e_d(A,F^{\omega,\varepsilon}_{n-1}) = \infty$ we have that

$$\rho^{\omega,\varepsilon}_{n-1}\big(e_d(B,F^{\omega,\varepsilon}_{n-1})\big)=\infty$$

so there is nothing to prove in this case. If on the other hand $e_d(A,F^{\omega,\varepsilon}_{n-1})<\infty$, it follows that

$$d^{\{D\}}(A,B)\wedge\omega=\omega\leqslant n\varepsilon\leqslant d^e_E\big(e_d(A,F^{\omega,\varepsilon}_{n-1}),e_d(B,F^{\omega,\varepsilon}_{n-1})\big)+\varepsilon$$
$$\leqslant \psi^{\omega,\varepsilon}(B)+\varepsilon.$$

Now assume that $D_d(B,D)-D_d(A,D)<\omega$. Then we have that

$$k\varepsilon \geqslant D_d(B,D)-D_d(A,D)>(k-1)\varepsilon$$

for some $k\in\{2,\dots,n\}$. Then following the same way as above, we find that $F^{\omega,\varepsilon}_{k-1} = (S_{D_d(A,D)+(k-1)\varepsilon}(D))^c$, $B\subset F^{\omega,\varepsilon}_{k-1}$ and that $e_d(A,F^{\omega,\varepsilon}_{k-1})\geqslant(k-1)\varepsilon$. If $e_d(A,F^{\omega,\varepsilon}_{k-1})=\infty$, we see that

$$\rho^{\omega,\varepsilon}_{k-1}\big(e_d(B,F^{\omega,\varepsilon}_{k-1})\big)=\infty$$

and we are done, where in the case that $e_d(A,F^{\omega,\varepsilon}_{k-1})<\infty$ we find that

$$d^{\{D\}}(A,B)\wedge\omega=d^{\{D\}}(A,B)\leqslant k\varepsilon$$
$$\leqslant d^e_E\big(e_d(A,F^{\omega,\varepsilon}_{k-1}),e_d(B,F^{\omega,\varepsilon}_{k-1})\big)+\varepsilon\leqslant\psi^{\omega,\varepsilon}(B)+\varepsilon,$$

which completes the proof. □

The topological result from Beer and Lucchetti now can be obtained as a corollary

Corollary 2.3. *Let (X,d) be a metric space. Then we have that $\mathcal{T}_{prox(d)}$ is the initial topology on $CL(X)$ for the source*

$$\big(e_d(\cdot,F):CL(X)\to\big([0,\infty],\mathcal{T}^*_E\big):A\to e_d(A,F)\big)_{F\in CL(X)}.$$

Proposition 2.4. *For every metric space (X,d) and every tiling Σ of X, we have that $\delta_{\Sigma,d}$ is the initial approach structure on $CL(X)$ for the source*

$$\big(d_F(G,\cdot):CL(X)\to\big([0,\infty],\delta_E\big):A\to d_F(G,A)\big)_{(F,G)\in\Sigma\times CL(X)}.$$

Proof. Fix $A\in CL(X)$ and $\mathcal{A}\subset CL(X)$. If we denote the initial approach distance on $CL(X)$ by δ, we have that $\delta(A,\mathcal{A})$ equals

$$\sup_{\Gamma\in 2^{(\Sigma\times CL(X))}}\ \sup_{\varphi\in\prod_{(F,G)\in\Gamma}\mathcal{B}_{\delta_E}(d_F(G,A))}\ \inf_{B\in\mathcal{A}}\ \sup_{(F,G)\in\Gamma}\varphi\big((F,G)\big)\big(d_F(G,B)\big).$$

On the one hand, it follows by a simple consideration of cases that

$$\nu\big(d_F(G,B)\big)\leqslant d_F(A,B)$$

for every $B \in CL(X)$, each $(F, G) \in \Sigma \times CL(X)$ and every $\nu \in \mathcal{B}_{\delta_E}(d_F(G, A))$, whence

$$\delta(A, \mathcal{A}) \leqslant \sup_{\mathbb{F} \in 2^{(\Sigma)}} \inf_{B \in \mathcal{A}} \sup_{F \in \mathbb{F}} d_F(A, B) = \delta_{\Sigma,d}(A, \mathcal{A})$$

because Σ is closed with respect to taking finite unions. Because on the other hand

$$\delta(A, \mathcal{A}) \geqslant \sup_{F \in \Sigma} \sup_{\varphi \in \mathcal{B}_{\delta_E}(0)} \inf_{B \in \mathcal{A}} \varphi\big(d_F(A, B)\big) = \delta_{\Sigma,d}(A, \mathcal{A})$$

we are done. $\square$

Corollary 2.5. *For every metric space (X, d) and every tiling Σ of X, we have that $\mathcal{T}_{\Sigma,d}$ is the initial topology on $CL(X)$ for the source*

$$\big(d_F(G, \cdot) : CL(X) \to \big([0, \infty], \mathcal{T}_E^*\big) : A \to d_F(G, A)\big)_{(F,G) \in \Sigma \times CL(X)}.$$

We conclude by proving some generalizations of weak representations of $\mathcal{T}_{h_d}$ and $\mathcal{T}_{AW_d}$ to be found in [2,4].

Proposition 2.6. *If (X, d) is a metric space, δ_{h_d} is the initial distance on $CL(X)$ for the source formed by following set of functionals*

$$\big\{e_d(\cdot, F) : CL(X) \to \big([0, \infty], \delta_E\big) \mid F \in CL(X)\big\}$$
$$\cup \big\{e_d(F, \cdot) : CL(X) \to \big([0, \infty], \delta_E\big) \mid F \in CL(X)\big\}.$$

Proof. If we use the notation δ for the initial distance on $CL(X)$ with respect to the source above, we have that for each $A \in CL(X)$ and $\mathcal{A} \subset CL(X)$, that $\delta(A, \mathcal{A})$ equals

$$\sup_{\Gamma \in 2^{(CL(X) \times \{-1,1\})}} \sup_{\varphi \in \prod_{(F,\varepsilon) \in \Gamma} \mathcal{B}_{\delta_E}(e_d^\varepsilon(F, A))} \inf_{B \in \mathcal{A}} \sup_{(F,\varepsilon) \in \Gamma} \varphi\big((F, \varepsilon)\big)\big(e_d^\varepsilon(F, B)\big),$$

where for all $F, B \in CL(X)$, $e_d^1(F, B) \doteqdot e_d(F, B)$ and $e_d^{-1}(F, B) \doteqdot e_d(B, F)$. The inequality $\delta \leqslant \delta_{h_d}$ is proved by verifying that $\mu(e_d^\varepsilon(F, B)) \leqslant h_d(A, B)$ for every $B, F \in CL(X)$, $\varepsilon \in \{-1, 1\}$ and $\mu \in \mathcal{B}_{\delta_E}(e_d^\varepsilon(F, A))$, whereas the converse inequality is obvious since

$$\delta(A, \mathcal{A}) \geqslant \inf_{B \in \mathcal{A}} \big(d_E^e\big(0, e_d^1(A, B)\big) \vee d_E^e\big(0, e_d^{-1}(A, B)\big)\big)$$
$$= \inf_{B \in \mathcal{A}} h_d(A, B) = \delta_{h_d}(A, \mathcal{A}). \qquad \square$$

Again the topological result now can be obtained as a corollary.

Corollary 2.7. *If (X, d) is a metric space, $\mathcal{T}_{h_d}$ is the initial topology on $CL(X)$ for the source formed by following set of functionals*

$$\big\{e_d(\cdot, F) : CL(X) \to \big([0, \infty], \mathcal{T}_E^*\big) \mid F \in CL(X)\big\}$$
$$\cup \big\{e_d(F, \cdot) : CL(X) \to \big([0, \infty], \mathcal{T}_E^*\big) \mid F \in CL(X)\big\}.$$

Proposition 2.8. *For every metric space (X,d), we have that $\delta_{\mathcal{T}_{h_d}}$ is the initial approach structure on $CL(X)$ for the source*

$$\big(e_\rho(F,\cdot): CL(X) \to \big([0,\infty],\delta_E\big): A \to e_\rho(F,A)\big)_{(\rho,F)\in\mathcal{E}_u(d)\times CL(X)}.$$

Proof. Because concrete coreflectors preserve initiality, it follows from [4], where it was shown that $\mathcal{T}_{h_d}$ is the initial topology for the source

$$\big(e_\rho(F,\cdot): CL(X) \to \big([0,\infty],\mathcal{T}_E^*\big): A \to e_\rho(F,A)\big)_{(\rho,F)\in\mathcal{E}_u(d)\times CL(X)},$$

that it suffices to prove that the initial distance δ for the AP source mentioned in the formulation of the proposition is topological, or equivalently, that δ can only take the values 0 and ∞. Therefore assume that $A \in CL(X)$ and $\mathcal{A} \subset CL(X)$ with $\delta(A,\mathcal{A}) > 0$. Because $\delta(A,\mathcal{A})$ equals

$$\sup_{\Gamma\in 2^{(\mathcal{E}_u(d)\times CL(X))}} \ \sup_{\varphi\in\prod_{(\rho,F)\in\Gamma}\mathcal{B}_{\delta_E}(e_\rho(F,A))} \ \inf_{B\in\mathcal{A}} \ \sup_{(\rho,F)\in\Gamma} \varphi\big((\rho,F)\big)\big(e_\rho(F,B)\big),$$

there exist $\Gamma_0 \in 2^{(\mathcal{E}_u(d)\times CL(X))}$ and $\varphi_0 \in \prod_{(\rho,F)\in\Gamma_0}\mathcal{B}_{\delta_E}(e_\rho(F,A))$ with

$$\alpha \doteqdot \inf_{B\in\mathcal{A}} \ \sup_{(\rho,F)\in\Gamma_0} \varphi_0\big((\rho,F)\big)\big(e_\rho(F,B)\big) > 0.$$

Note that for each $(\rho,F)\in\Gamma_0$, $\varphi_0((\rho,F)) = d_E^e(e_\rho(F,A),\cdot)$ if $e_\rho(F,A) < \infty$ and $\varphi_0((\rho,F)) = \theta_{]m((\rho,F)),\infty]}$ for some $m((\rho,F)) \in \mathbb{N}_0$ if $e_\rho(F,A) = \infty$. For every $k \in \mathbb{N}_0$ and $(\rho,F)\in\Gamma_0$ we define

$$\psi^k\big((\rho,F)\big) \doteqdot \begin{cases} d_E^e(e_{k\cdot\rho}(F,A),\cdot) & e_\rho(F,A) < \infty, \\ \theta_{]k\cdot m((\rho,F)),\infty]} & e_\rho(F,A) = \infty. \end{cases}$$

Then we obviously have that

$$\forall k\in\mathbb{N}_0\colon\ \psi^k \doteqdot \big(\psi^k((\rho,F))\big)_{(\rho,F)\in\Gamma_0} \in \prod_{(\rho,F)\in\Gamma_0} \mathcal{B}_{\delta_E}\big(e_{k\cdot\rho}(F,A)\big).$$

Because $k\cdot\rho \in \mathcal{E}_u(d)$ for all $k\in\mathbb{N}_0$ and $\rho\in\mathcal{E}_u(d)$, we now obtain that

$$\begin{aligned} \delta(A,\mathcal{A}) &\geqslant \sup_{k\in\mathbb{N}_0} \inf_{B\in\mathcal{A}} \sup_{(\rho,F)\in\Gamma_0} \psi^k\big((\rho,F)\big)\big(e_{k\cdot\rho}(F,B)\big) \\ &= \sup_{k\in\mathbb{N}_0} \inf_{B\in\mathcal{A}} \sup_{(\rho,F)\in\Gamma_0} \big(k\cdot\varphi_0\big((\rho,F)\big)\big(e_\rho(F,B)\big)\big) \\ &= \sup_{k\in\mathbb{N}_0} (k\cdot\alpha) = \infty. \qquad \square \end{aligned}$$

Proposition 2.9. *For every metric space (X,d), we have that $\delta_{\mathcal{T}_{AW_d}}$ is the initial approach structure on $CL(X)$ for the source*

$$\big(e_\rho(F,\cdot): CL(X) \to \big([0,\infty[,\delta_{d_E}\big): A \to e_\rho(F,A)\big)_{(\rho,F)\in\mathcal{E}_u^b(d)\times CLB(X)}.$$

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, J. Wiley and Sons, 1990.

[2] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer, Dordrecht, 1993.

[3] G. Beer, A. Lechicki, S. Levi, S. Naimpally, Distance functionals and suprema of hyperspace topologies, Annali di Matematica Pura ed Applicata CLXII (1992) 367–381.

[4] G. Beer, R. Lucchetti, Weak topologies for the closed subsets of a metrizable space, Trans. Amer. Math. Soc. 335 (1993) 805–822.

[5] R. Lowen, Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad, Oxford University Press, Oxford, 1997.

[6] R. Lowen, M. Sioen, The Wijsman and Attouch–Wets topologies on hyperspaces revisited, Topology Appl. 70 (1996) 179–197.

[7] R. Lowen, M. Sioen, Proximal hypertopologies revisited, Set-Valued Analysis 6 (1998) 1–19.