Weakly semirecursive sets and r.e. orderings

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Abstract


Weakly semirecursive sets have been introduced by Jockusch and Owings (1990). In the present paper their investigation is pushed forward by utilizing r.e. partial orderings, which turn out to be instrumental for the study of degrees of subclasses of weakly semirecursive sets.

0. Introduction

Semirecursive sets, introduced by Jockusch [4], have played an important role in various recursion-theoretic investigations. Recently, Jockusch and Owings [5] introduced the notions weakly semirecursive and semi-r.e. as common extensions of semirecursive and r.e. They showed that weakly semirecursive equals size-recursive, the latter property being motivated by recent research on ‘Bounded Query Classes’ (see e.g. [8, III.5.9], [1]), as many verbose sets are size-recursive. Jockusch and Owings also began to study the degrees of subclasses of weakly semirecursive sets: They proved that in hyperimmune-free degrees weakly semirecursive equals semirecursive, the converse was left open. Their only result in the other direction is the construction of a weakly semirecursive set which is neither semi-r.e. nor co-semi-r.e., in any nonrecursive r.e. degree. The question was posed of whether these are the only degrees with this property.

In the present paper we will continue the investigation of Jockusch and Owings.
utilizing a fruitful connection between weakly semirecursive sets and \( r.e. \) partial orderings. To this end we generalize the classical result of Appel and McLaughlin (see \([8,III.5.4]\)) that a set is semirecursive iff it is an initial segment of a recursive linear ordering: We prove that a set is weakly semirecursive iff it is an initial segment of an \( r.e. \) partial ordering. A similar characterization is obtained for semi-r.e. sets, and for sets satisfying the strong consistency property which was defined in \([5]\). The general framework of \( r.e. \) partial tree orderings is introduced to investigate the degrees of several subclasses of weakly semirecursive sets. In particular, we show that any hyperimmune degree contains a weakly semirecursive set which is not semirecursive, and that any degree containing a semi-r.e. nonsemirecursive set is \( r.e. \) in \( \emptyset' \). Furthermore, we construct in any degree which is \( r.e. \) in and above \( \emptyset' \) a semi-r.e. nonsemirecursive set and a weakly semirecursive set which is neither semi-r.e. nor co-semi-r.e. Additional results concern the relationship with regressive sets, the Boolean algebra generated by the \( r.e. \) sets, and recursive model theory.

1. Notation and definitions

With one exception we are using standard notation, cf. \([8], [14]\).

\( \{\varphi_e\}_{e<\omega} \) denotes here a numbering of all partial recursive functions of two arguments.

\( M \subseteq \omega \) is semirecursive iff there exists a total recursive function \( \psi \) of two variables such that for all \( x, y \in \omega \):

\[
((x \in M \land y \notin M) \lor (x \notin M \land y \in M)) \rightarrow \psi(x, y) \in \{x, y\} \cap M.
\]

The following definitions have been introduced by Jockusch and Owings \([5]\).

\( M \subseteq \omega \) is semi-r.e. iff there exists a partial recursive function \( \psi \) of two variables such that for all \( x, y \in \omega \): \( x \in M \lor y \in M \rightarrow \psi(x, y) \in \{x, y\} \cap M \).

\( M \subseteq \omega \) is weakly semirecursive iff there exists a partial recursive function \( \psi \) of two variables such that for all \( x, y \in \omega \):

\[
((x \in M \land y \notin M) \lor (x \notin M \land y \in M)) \rightarrow \psi(x, y) \in \{x, y\} \cap M.
\]

In any of these cases we say that \( M \) is semirecursive, or semi-r.e., or weakly semirecursive \textit{via} \( \psi \), respectively.

\( M \subseteq \omega \) has the \textit{consistency property} (CP) iff there exists a recursive approximation \( M(x, s) \) of \( M \) such that:

\[
\forall s \left( \{x \mid M(x, s) = 1 \land x \leq s\} \subseteq M \lor \{x \mid M(x, s) = 0 \land x \leq s\} \subseteq \overline{M} \right).
\]

In \([5]\) it is proved that \( M \) has CP iff \( M \leq_T K \) and \( M \) is weakly semirecursive.

\( M \subseteq \omega \) has the \textit{strong consistency property} (SCP) iff there exists a uniformly recursive sequence \( \{M_k\}_{k<\omega} \) of recursive sets such that:

1. \( \forall k [M_k \subseteq M \lor M \subseteq M_k] \),
2. \( \forall x (M(x) = \lim_{k \to \omega} M_k(x)) \).
2. A related definition

In a paper by Rozinas and Solon [13], which apparently escaped Jockusch and Owings, the term ‘weakly semirecursive’ was already used, in a related way. In order to avoid confusion we call the notion of Rozinas and Solon ‘wsr*’. 

$M \subseteq \omega$ is called wsr* iff there exists a partial recursive function $\psi$ of two variables such that for all $x, y \in \omega$:

1. $\psi(x, y) \downarrow \iff (x, y) \in \{x, y\}$,
2. $x \in M \land y \in M \rightarrow \psi(x, y) \downarrow$,
3. $\psi(x, y) \downarrow \land (x, y) \in M \rightarrow x \in M \land y \in M$.

Clearly any semirecursive set is wsr*. Rozinas and Solon proved that the pc-degree of any wsr* set consists of a single pm-degree. This answers Question 8.13 in [2].

Let us now consider the relationship between semi-r.e., weakly semirecursive, and wsr*. It is clarified by the following two observations:

1. $M \subseteq \omega$ is semi-r.e. iff $M$ is both weakly semirecursive and wsr*.

$(\Rightarrow)$ is obvious. $(\Leftarrow)$: Suppose that $A$ is weakly semirecursive via $\psi_1$ and wsr* via $\psi_2$. We define $\psi(x, y)$, $x \neq y$, as follows: Compute in parallel $\psi_1(x, y)$ and $\psi_2(x, y)$. If $\psi_1(x, y)$ converges first, let $\psi(x, y) = \psi_1(x, y)$. If $\psi_2(x, y)$ converges first, let $\{\psi(x, y)\} = \{x, y\} - \{\psi_2(x, y)\}$. Let $\psi(x, x) = x$. Then $M$ is semi-r.e. via $\psi$.

2. Any regressive set is wsr*.

This follows immediately from the definition of ‘regressive’ (see [8, II.6.2]).

Degtev [3, Proposition 6] proved that any semirecursive regressive set is r.e. or co-r.e. In fact, his proof shows that any weakly semirecursive regressive set is r.e. or co-r.e. It is well known (cf. [8, II.6.13]) that any nonrecursive degree contains a regressive set which is neither r.e. nor co-r.e. It follows that any nonrecursive degree contains a wsr* set which is not weakly semirecursive. For additional information see the remark at the end of Section 5.

3. Weakly semirecursive sets and the Boolean algebra generated by the r.e. sets

Jockusch and Owings proved that if $M$ is semirecursive and $M$ is a finite Boolean combination of r.e. sets then $m$ is r.e. or co-r.e. They asked whether this property generalizes to semi-r.e. sets.

Recall that a set $M$ is $k$-r.e. iff there exists a recursive approximation $M(x, s)$ of $M$ such that

$$\forall x [M(x, 0) = 0 \land |\{s: M(x, s) \neq M(x, s + 1)\}| \leq k].$$
It is well known that $M$ is a finite Boolean combination of r.e. sets iff $M$ is $k$-r.e. for some $k$.

**Theorem 3.1.** If $M$ is $k$-r.e. and weakly semirecursive then $M$ is r.e. or co-r.e.

**Proof.** We use induction on $k$. If $k = 0$, 1, then $M$ is r.e. Suppose the theorem holds for $k \geq 1$. Let $M$ be $(k + 1)$-r.e. and weakly semirecursive. There exists a recursive approximation $M_d(n, s)$ to $M$ such that $M_d(n, 0) = 0$, and $\{s : M_d(n, s) \neq M_d(n, s + 1)\}$ is $k + 1$ for all $n$. From the proof of Theorem 3 in [5] it follows that there exists a strictly increasing, total recursive function $g$, such that $g(0) = 0$, and $M_d(n, s) := M_d(n, g(s))$ is a recursive approximation to $M$ with CP.

Let $h(n) = \{s : M_d(n, s) \neq M_d(n, s + 1)\}$. Clearly, $h(n) \leq k + 1$ for all $n$.

$B := \{n : h(n) = k + 1\}$ is r.e. If $k$ is odd then $B \subseteq M$, else $B \subseteq M$.

If $B$ is recursive then

$$M_2(n, s) := \begin{cases} M_d(n, s) & \text{if } n \notin B, \\ 1 & \text{if } s \geq 1 \land n \in B \land k \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

is a recursive approximation to $M$ witnessing that $M$ is $k$-r.e. From the induction hypothesis it follows that $M$ is r.e. or co-r.e.

Suppose $B$ is nonrecursive. Then $C := \{t : \exists n [n \in B \land n < t \land M_d(n, t) \neq M(n)]\}$ must be an infinite r.e. set. Let $b = (k + 1) \mod 2$. Since the $(1 - b)$'s are incorrect at stage $t \in C$, it follows by CP, that the $b$'s are correct at any stage $t \in C$ (for numbers less than $t$). Using the fact that $M_d$ is an approximation to $M$, we find that: $D = M$, if $k$ is even, and $D = M$, if $k$ is odd, for the r.e. set $D := \{m : \exists t \in C [m < t \land M_d(m, t) = b]\}$. \square

4. R.e. partial orderings

We are considering partial orderings (p.o.) $\sqsubseteq$ on $\omega$ the set of all nonnegative integers (i.e., $\sqsubseteq$ is an irreflexive, transitive relation on $\omega \times \omega$).

Let $a \equiv b$ iff $\forall z [(a \sqsubseteq z \iff b \sqsubseteq z) \land (z \sqsubseteq a \iff z \sqsubseteq b)]$. $\equiv$ is an equivalence relation on $\omega$, compatible with $\sqsubseteq$. For $A, B \subseteq \omega$ we write $A \sqsubseteq B$ iff $\forall x \in A \forall y \in B [x \sqsubseteq y]$.

$M \subseteq \omega$ is an initial segment of the p.o. $\sqsubseteq$ iff $M \sqsubseteq M$. Note that initial segments are closed under $\equiv$ and linearly ordered by inclusion. $M$ is an end segment of $\sqsubseteq$ iff $M$ is an initial segment of the reverse ordering $\sqsubseteq' := \{(y, x) \mid x \sqsubseteq y\}$.

A p.o. $\sqsubseteq$ is called almost linear iff $\sqsubseteq/\equiv$ is a linear ordering.

A p.o. $\sqsubseteq$ is r.e. iff $\{(n, m) \mid n \sqsubseteq m\}$ is r.e.

$M$ is IAL iff there exists an r.e. almost linear ordering, and $M$ is an initial segment of it. Note that for any almost linear ordering $\sqsubseteq$ the relation $\equiv$ is co-r.e.
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Remarks. (1) R.e. linear orderings are considered in [9], their order types coincide with the recursive order types. Also the linear order types of $\mathcal{C}/\equiv$, where $\mathcal{C}$ is an r.e. almost linear ordering coincide with the order types of recursive linear orderings.

(2) Manaster and Rosenstein [7] constructed an r.e. two-dimensional partial ordering which is not isomorphic to any recursive partial ordering, cf. also [12].

(3) In [10] the notion of an ‘r.e. presented linear order’ is introduced. There, equality is r.e., so this notion is different both from r.e. linear order and r.e. almost linear order. In fact, there are r.e. presented linear order types which are not recursive [11].

Theorem 4.1. $M$ is weakly semirecursive iff $M$ is an initial segment of an r.e. partial ordering.

Proof. ($\Rightarrow$) Let $M$ be weakly semirecursive. W.l.o.g. we may choose $f \in P_2$ such that

1. $\forall x, y f(x, y) = f(y, x)$, and
2. $\forall x \in M \forall y \notin M f(x, y) = x$.

The set $\{(x, y) \mid f(x, y) = f(x, y)\}$ is r.e. Let $(x_1, y_1), (x_2, y_2), \ldots$ be a recursive enumeration of this set. Uniformly in $n$ we define a recursive partial ordering $\mathcal{C}_n$, such that $\mathcal{C}_n$ is finite and $\mathcal{C}_{n+1}$ is an extension of $\mathcal{C}_n$. Let $\mathcal{C}_0 := \emptyset$. $\mathcal{C}_{n+1}$ is defined inductively:

1. If $y_n \subseteq x_n$ then $\mathcal{C}_{n+1} := \mathcal{C}_n$.
2. Otherwise: $\mathcal{C}_{n+1} :=$ transitive closure of $\mathcal{C}_n \cup \{(x_n, y_n)\}$.

Finally, $\mathcal{C} := \bigcup \{\mathcal{C}_n \mid n \in \omega\}$. Clearly, $\mathcal{C}$ is an r.e. partial ordering.

$M$ is an initial segment of $\mathcal{C}$: Suppose for a contradiction that there exists $x \in M$, $y \notin M$ such that $\forall x \subseteq y$. $f(x, y) = x$, so there exists an $n$ such that $x_n = x$, $y_n = y$. By definition of $\mathcal{C}_{n+1}$ we must have $y \subseteq x$. Thus, there exists a sequence $u_1, u_2, \ldots, u_m$ such that $u_i = y$, $u_m = x$, and $f(u_i, u_{i+1}) = u_i$, for $1 \leq i < m$. Let $k$ be the greatest index $i$ such that $u_i \notin M$. Note that $k$ exists and $1 = k < m$, since $u_1 \notin M$, $u_m \in M$. Then, $f(u_{k+1}, u_k) = f(u_k, u_{k+1}) = u_k$, $u_k \notin M$, $u_{k+1} \in M$, contradicting property (2) of $f$.

($\Leftarrow$) If $M$ is an initial segment of the r.e. partial ordering $\mathcal{C}$, then

$$f(x, y) := \begin{cases} x & \text{if } x \subseteq y, \\ y & \text{if } y \subseteq x, \\ \uparrow & \text{otherwise.} \end{cases}$$

is a partial recursive function witnessing that $M$ is weakly semirecursive.

5. A characterization of semi-r.e. sets

If $M$ is semi-r.e. via $f$, we may assume that: (0) $f(x, y) \downarrow \rightarrow f(x, y) \in \{x, y\}$. Now consider the construction from the first part of the proof of Theorem 4.1,
supposing \( M \) semi-r.e. and \( f \) satisfying properties (0), (1), (2), and (3) \( \forall x, y \in M \ [f(x, y)] \). Then any two \( x, y \in M \) are comparable. On the other hand, if \( M \) is a linearly ordered initial segment of an r.e. partial ordering, then \( M \) is semi-r.e., by the second part of the previous proof. Thus, we get the next result:

**Theorem 5.1.** \( M \) is semi-r.e. iff \( M \) is a linearly ordered initial segment of an r.e. partial ordering.

**Theorem 5.2.** If \( M \) is semi-r.e. then \( M \) is semirecursive or \( M \) is co-r.e. in \( K \).

**Proof.** Suppose that \( M \) is semi-r.e. By Theorem 5.1 there exists an r.e. partial ordering \( \sqsubseteq \) such that \( M \) is some initial segment of \( \sqsubseteq \). We distinguish two cases:

1. There exists \( x \notin M \) such that:
   - (a) \( \forall y \ [x \sqsubseteq y \lor x = y \lor y \sqsubseteq x] \), and
   - (b) \( \{z \mid z \sqsubseteq x\} \) is linearly ordered by \( \sqsubseteq \).

   Let
   
   \[
   f(a, b) := \begin{cases} 
   a & \text{if } a = b \lor x \sqsubseteq b \lor a \sqsubseteq b \sqsubseteq x, \\
   b & \text{if } b \sqsubseteq x \sqsubseteq a \lor b \sqsubseteq a \sqsubseteq x.
   \end{cases}
   \]

   \( f \) is total recursive, and \( M \) is semirecursive via \( f \).

2. Otherwise, i.e., for every \( x \notin M \):
   - (a') \( \exists y \ [\neg x \sqsubseteq y \land \neg x = y \land \neg y \sqsubseteq x] \), or
   - (b') \( \exists y, z \ [y, z \sqsubseteq x \land \neg y \sqsubseteq z \land \neg y = z \land \neg y \sqsubseteq z] \).

   As \( M \) is a linearly ordered initial segment of \( \sqsubseteq \), it follows that the condition ‘(a’) or (b’)) characterizes the elements \( x \notin M \). Since the set of all \( x \) satisfying ‘(a’) or (b’)) is r.e. in \( K \), \( M \) is co-r.e. in \( K \).

**Corollary 5.3.** If \( M \) is semi-r.e. and \( \text{dg}(M) \) is not r.e. in \( \emptyset' \) then \( M \) is semirecursive.

**Remark.** Wsr\(^*\) sets also admit an order-theoretic characterization: Call a set \( M \) a *branch* of the p.o. \( \sqsubseteq \) iff \( M \) is linearly ordered and closed downwards. It is easy to see that \( M \) is wsr\(^*\) iff \( M \) is a branch of an r.e. partial ordering.

If \( M \) is wsr\(^*\) and co-wsr\(^*\) then there exists an r.e. p.o. \( \sqsubseteq \) such that \( M \) is a branch of \( \sqsubseteq \) and \( M \) is a branch of \( \sqsubseteq' \). If \( M \) is not semirecursive then there exist \( x \in M, y \notin M \) such that \( x \) and \( y \) are incomparable, and it follows that \( M = \{z \mid z \sqsubseteq x \lor (x \sqsubseteq z \land y \text{ and } z \text{ are incomparable})\} \), so \( M \) is recursive in \( K \) (in fact, \( M \) is 2-r.e.). A somewhat closer analysis of this case gives the following characterization: \( M \) is wsr\(^*\) and co-wsr\(^*\) iff \( M \) is semirecursive or there exist a recursive set \( A \), and r.e. semirecursive sets \( B, C \) such that \( M = (A \cap B) \cup (A - C) \).
6. On initial segments of almost linear orderings

The following definition generalizes the SCP-property to sets which are not necessarily recursive in $K$.

**Definition.** $M \subseteq \omega$ has the **strong-inclusion-property** (SIP), iff there exists a uniformly recursive sequence $\{A_k\}_{k \in \omega}$ such that:

1. $\forall k \ [A_k \subseteq M \lor M \subseteq A_k]$, and
2. $\forall x \in M \exists k \ [x \in A_k \land A_k \subseteq M] \lor \forall x \notin M \exists k \ [x \notin A_k \land M \subseteq A_k]$.

**Lemma 6.1.** Given an r.e. almost linear ordering $\sqsubset$, and $x$, $y$ such that $D_x \sqsubset D_y$. Then uniformly (in $x$, $y$) a recursive initial segment $S(D_x, D_y)$ of $\sqsubset$ can be enumerated such that $D_x \subseteq S(D_x, D_y) \land S(D_x, D_y) \cap D_y = \emptyset$.

**Proof.** Suppose that $\sqsubset$ is an r.e. almost linear ordering. Then for any two sets $A$, $B$ such that $A \sqsubset B$, and any $z$, either $A \sqsubset \{z\}$ or $\{z\} \sqsubset B$. Given $x$, $y$ such that $D_x \sqsubset D_y$ we define $A_0 := D_x$, $B_0 := D_y$, and in stage $n + 1$: We enumerate $\sqsubset$ until we discover that (1) $A_n \sqsubset \{n\}$ or (2) $\{n\} \sqsubset B_n$. If (1) occurs first, let $A_{n+1} := A_n$; $B_{n+1} := B_n \cup \{n\}$. If (2) occurs first, let $A_{n+1} := A_n \cup \{n\}$; $B_{n+1} := B_n$. Note that $A_n \sqsubset B_n$, for all $n$. Let $S(D_x, D_y) := \bigcup \{A_n : n \in \omega\}$. $S(D_x, D_y)$ and its complement are r.e., thus $S(D_x, D_y)$ is recursive. \(\square\)

**Theorem 6.2.** $M$ has SIP iff $M$ is IAL.

**Proof.** The theorem clearly holds if $M$ is recursive.

$(\Rightarrow)$ Suppose that $M$ is nonrecursive and has SIP witnessed by a uniformly recursive sequence $\{A_k\}_{k \in \omega}$ of recursive sets. We suppose that $\forall x \in M \exists k \ [x \in A_k \land A_k \subseteq M]$ (by hypothesis $A_k \subseteq M \rightarrow A_k \subseteq M$), the other case is proved analogously.

For each $x$ we define the real number $r(x) := \sum_{n : x \in A_n(x)} 3^{-n}$.

Define an almost linear ordering $\sqsubset$ by: $x \sqsubset y := r(x) > r(y)$. Note that

$$(x, y) : x \sqsubset y = \left\{ (x, y) : \exists k \left[ \sum_{n < k} A_n(x) \cdot 3^{-n} > \sum_{n < k} A_n(y) \cdot 3^{-n} \right] \right\},$$

thus $\sqsubset$ is r.e. Let $r := \sum_{n : M \subseteq A_n} 3^{-n}$. If $x \in M$ then $\{n : M \subseteq A_n\} \subseteq \{n : x \in A_n\}$, thus, $r < r(x)$ if $x \notin M$ then $r(x) \leq r$. Therefore, $M$ is the lower cut of $\sqsubset$ determined by $r$.

$(\Leftarrow)$ Let $\sqsubset$ be an r.e. almost linear ordering, and let $M$ be a nonrecursive initial segment of $\sqsubset \neq \emptyset$. Let $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle, \ldots$ be a recursive enumeration of $\sqsubset$. Using Lemma 6.1 we define $A_k := S(\{a_k\}, \{b_k\})$, for all $k$. $\{A_k\}_{k \in \omega}$ is uniformly recursive. Condition (1) in the definition of SIP is satisfied since $M$, $A_k$ are initial segments.
We verify that condition (2) is satisfied, by distinguishing two cases:

1. \( \exists w \{ M = \{ v : v \sqsubset w \} \} \). (Note that in this case \( M \) is r.e.)

Then, for each \( x \in M \) there exists \( k \) such that \( (x, w) = (a_k, b_k) \), i.e., \( x \in A_k \land A_k \subseteq M \).

2. Otherwise, i.e., there exists a descending sequence \( \{ w_m \}_{m \in \omega} \) \( w_{m+1} \sqsubset w_m \) such that \( M = \{ v : \forall m [v \sqsubset w_m] \} \).

Since \( \sqsubset \) is almost linear, for each \( x \notin M \) there exists \( m \) such that \( w_m \sqsubset x \), thus, there exists \( k \) such that \( (w_m, x) = (a_k, b_k) \), i.e., \( x \notin A_k \land M \subseteq A_k \).

**Theorem 6.3.** \( M \) has SCP iff \( M \leq_T K \) and \( M \) is IAL.

**Proof.** \((\rightarrow)\) This follows immediately from the definition of SCP and Theorem 6.2.

\((\Leftarrow)\) Suppose that \( M \leq_T K \) and \( M \) is an initial segment of the r.e. almost linear ordering \( \sqsubset \). Since any recursive set has SCP we may assume that \( M \) is nonrecursive. There exists a recursive approximation \( M(x, s) \) of \( M \), such that for all \( x: M(x) = \lim_{s \to \infty} M(x, s) \).

Let \( \{ \sqsubset \}_{s \in \omega} \) denote a recursive enumeration of \( \sqsubset \). For each \( k \) let:

\[ f(k) := \mu s \geq k. \forall x, y \leq k [M(x, s) = 1 \land M(y, s) = 0 \to x \sqsubset y] \]

Note that \( f \) is a total recursive function, such that for any \( x \) and all sufficiently large \( k \): \( M(x) = M(x, f(k)) \). By Lemma 6.1 we define a uniformly recursive sequence \( \{ A_k \}_{k \in \omega} \) such that

\[ A_k := S(\{ x : x \leq k \land M(x, f(k)) = 1 \}, \{ x : x \leq k \land M(x, f(k)) = 0 \}) \]

This sequence satisfies conditions (1), (2) in the definition of SCP. \( \Box \)

From the fact that the \( A_k \)'s constructed in the proofs of Theorems 6.2, 6.3 are initial segments of the p.o., we conclude:

**Corollary 6.4.** If \( M \) has SIP (SCP) then there exists a uniformly recursive sequence \( \{ A_k \}_{k \in \omega} \) such that \( \forall n, m [A_n \subseteq A_m \lor A_m \subseteq A_n] \), and the conditions in the definition of SIP (SCP) are satisfied for \( M \) and \( \{ A_k \}_{k \in \omega} \).

If \( M \) is not r.e. then, in addition, \( \forall x \notin M \exists k [x \notin A_k \land M \subseteq A_k] \).

**Remark.** In [5] Jockusch and Owings construct a weakly semirecursive set \( M \leq_T K \) which is neither semi-r.e. nor co-semi-r.e. In the light of r.e. partial orderings this construction can be visualized as follows: \( M \) is an initial segment of an r.e. almost linear ordering \( \sqsubset \) whose equivalence classes modulo \( \sim \) have cardinality at most two. \( \sqsubset /\sim \) has order type \( \omega + \omega^* \) and \( M \) is the ‘lower part’, i.e. the \( \omega \)-part, of this ordering. To satisfy the requirement that \( M \) is not semi-r.e. via \( q_e \) two witnesses \( a, b \) are put into the lower part of the ordering. If \( q_e(a, b) \uparrow \) then \( a, b \) are incomparable. If \( q_e(a, b) = a \) then \( b \sqsubset a \), and \( a \) is put into the ‘upper’
part, i.e., we try to put almost all numbers into the interval \([b, a]\). If \(q_\alpha(a, b) = b\) then \(a \sqsubset b\), and \(b\) is put into the upper part. The different requirements are combined in a finite injury construction. From Theorem 6.3 we conclude that the set \(M\), constructed by Jockusch and Owings, has SCP.

Jockusch and Owings asked whether any weakly semirecursive set \(M \leq_T K\) had SCP. This will be answered negatively in Section 10 below.

7. R.e. partial tree orderings

Let \(\Lambda\) be a countable set. A tree \(T\) (over \(\Lambda\)) is a subset of \(\Lambda^{\omega}\) (the set of all finite strings of elements from \(\Lambda\)) which is closed under initial segments. \(f \in \Lambda^{\omega}\) is called an infinite path through \(T\) iff \(f \upharpoonright n \in T\), for all \(n \in \omega\). Strings are denoted by \(\alpha, \beta, \gamma, \ldots\), elements of \(\Lambda\) by \(a, b, c, \ldots\). Let \(\alpha \sqsubseteq \beta\) (\(\alpha \sqsubset \beta\)) denote that string \(\beta\) extends (properly extends) \(\alpha\). Let \(\lambda\) denote the empty string. Let \(\alpha^* a\) denote the extension of \(\alpha\) by \(a\). \((T, \sqsubseteq)\) is a partially ordered tree iff \(T\) is a tree and \(\sqsubseteq\) is a partial ordering of \(T\).

A partial ordering \(<\) of the sons of each node of \(T\) (i.e., \(<\) is the union of partial orderings on \(\text{Sons}(\alpha) := \{\alpha^* a: a \in \Lambda, \alpha^* a \in T\}\), for all \(\alpha \in T\)) is called a preorder.

A preorder \(<\) induces the partial ordering \(\sqsubseteq\) defined as follows:
\[
\alpha \sqsubseteq \beta :\iff \alpha \sqsubseteq \beta \text{ or there exist } a, b \in \Lambda, \gamma \sqsubseteq \alpha, \beta \text{ such that:}
\begin{align*}
\gamma^* a &< \gamma^* b, \\
\gamma^* a &\subseteq \alpha, \text{ and } \gamma^* b \subseteq \beta.
\end{align*}
\]

\((T, \sqsubseteq)\) is an r.e. partially ordered tree iff \((T, \sqsubseteq)\) is a partially ordered tree, \(T\) is r.e., and \(\sqsubseteq\) is r.e. If \(T\) is r.e. and \(<\) is an r.e. preorder then the induced partially ordered tree is r.e.

Convention. If \((T, \sqsubseteq)\) is an r.e. partially ordered tree and \(T\) is infinite then there exists a recursive bijection \(f: \omega \to T\), and we obtain an induced r.e. partial ordering \(\sqsubseteq'\) on \(\omega\): \(n \sqsubseteq' m :\iff f(n) \sqsubseteq f(m)\). For simplicity we will identify in the following sections nodes \(\alpha\) and numbers \(f^{-1}(\alpha)\), for some \(f\) as above. In particular, we will write \(q_\alpha(\alpha, \beta)\) for \(q_\alpha(f^{-1}(\alpha), f^{-1}(\beta))\), etc. We also identify \(\sqsubseteq\) and \(\sqsubseteq'\).

8. Degrees of weakly semirecursive sets which are not semirecursive

Recall that a degree \(a\) is hyperimmune iff there exists a total function \(f\) recursive in \(a\) such that \(f\) is not dominated by any total recursive function.

Theorem 8.1. Any hyperimmune degree contains a weakly semirecursive set which is not semirecursive.
Proof. \( A := \{a, b, c, 0, 1\} \). Let \( T \) be the least subset \( S \) of \( A^{\omega} \) such that for all \( \alpha \):

1. \( \lambda \in S \),
2. \( \alpha \in S \) and \( |\alpha| \) even \( \rightarrow \alpha^{0} \in S \) (coding nodes),
3. \( \alpha \in S \) and \( |\alpha| \) odd \( \rightarrow \alpha^{a}, \alpha^{b}, \alpha^{c} \in S \) (diagonalization nodes).

\( T \) is a recursive tree. Now we define an r.e. preordering \( < \) as follows: If \( |\alpha| \) is even then \( \alpha^{0} < \alpha^{-1} \). If \( |\alpha| = 2(e, i) + 1 \) then \( \alpha^{-a} < \alpha^{-b} \); furthermore: if \( \varphi_{e}(\alpha^{a}, \alpha^{b}) = \alpha^{a} \) then \( \alpha^{b} < \alpha^{-a} \), if \( \varphi_{e}(\alpha^{a}, \alpha^{b}) = \alpha^{b} \) then \( \alpha^{-a} < \alpha^{-b} \).

Let \( \preceq \) be the induced r.e. partial ordering of \( T \).

Let \( a \) be a hyperimmune degree and \( A \in a \). Since \( a \) is hyperimmune there exists a total function \( h : \omega \to \omega \), \( h \equiv_{T} A \), which is not dominated by any total recursive function.

We define an infinite path \( f \) through \( T \) by induction:

\[
\begin{align*}
f(2e) := 0, & \quad \text{if } e \in A; \quad f(2e) := 1, \quad \text{if } e \notin A. \\
\text{If } \alpha = f \upharpoonright (2(e, i) + 1) & \text{ then } \\
f(2\langle e, i \rangle + 1) := & \begin{cases} 
\begin{align*}
b & \text{if } \varphi_{e,h}(\alpha^{-a}, \alpha^{-b}) = \alpha^{-a}, \\
a & \text{if } \varphi_{e,h}(\alpha^{-a}, \alpha^{-b}) = \alpha^{-b}, \\
c & \text{otherwise.}
\end{align*}
\end{cases}
\end{align*}
\]

Note that \( f \) is recursive in \( A \). Define \( M := \{\alpha \in T : \exists n \ [\alpha \preceq f \upharpoonright n]\} \).

1. \( M \) is an initial segment of \( \preceq \): Suppose that \( \beta \notin M \), i.e., \( \forall n \ [\beta \preceq f \upharpoonright n] \). It suffices to show that \( \forall n \ [f \upharpoonright n \subset \beta] \). Fix \( n \), let \( \alpha := f \upharpoonright n \), and let \( \gamma \) be the maximal common prefix of \( \alpha \) and \( \beta \). Note that \( \gamma \neq \beta \). If \( \gamma = \alpha \subset \beta \) then \( \alpha \subset \beta \), as required. Otherwise:

If \( |\gamma| \) is even, then we must have \( \gamma^{-0} \subset \alpha, \gamma^{-1} \subset \beta \). Thus \( \alpha \subset \beta \).

If \( |\gamma| \) is odd, then either \( \gamma^{-a} \subset \alpha \) or \( \gamma^{-b} \subset \alpha \). In the first case, by definition of \( < \) and \( f \), we have \( \gamma^{-a} \subset \gamma^{-b} \subset \gamma^{-c} \), and \( \gamma^{-b} \subset \gamma^{-c} \), therefore \( \alpha \subset \beta \). The second case is symmetric to the first case.

2. \( M \equiv_{T} A \): We show how to compute \( M(\alpha) \) for \( \alpha \in T \), using an \( A \)-oracle. As \( f \) is recursive in \( A \) we can compute the maximal common prefix \( \gamma \) of \( \alpha \) and \( f \). If \( \gamma = \alpha \) then \( \alpha \in M \). Otherwise: If \( |\gamma| \) is even then \( \alpha \in M \) iff \( \gamma^{-0} \subset \alpha \). If \( |\gamma| \) is odd and \( \gamma^{-c} \subset f \) then \( \alpha \in M \); otherwise \( \alpha \notin M \), by definition of \( < \) and \( f \).

3. \( A \equiv_{T} M \): It is easy to check that \( f \) can be computed recursively, using an \( M \)-oracle. As \( A \) is recursive in \( f \), \( A \) is recursive in \( M \), too.

4. \( M \) is not semirecursive: Suppose for a contradiction that \( M \) is semirecursive, and let \( \varphi_{e} \) be a total recursive function such that for any \( x \in M \), \( y \notin M \): \( \varphi_{e}(x, y) = x \). W.l.o.g. we assume that \( \varphi_{e}(x, y) = \varphi_{e}(y, x) \in \{x, y\} \), for all \( x, y \). Define the total recursive function \( g \) as follows:

\[
g(i) := \mu s. \forall \alpha, \beta \in T \ [(|\alpha| = |\beta| = 2\langle e, i \rangle + 2 \rightarrow \varphi_{e,\alpha}(\alpha, \beta) = 1].
\]

As \( h \) is not dominated by \( g \) there exists an \( i \) such that \( g(i) < h(i) \). Consider the
definition of $f(2\langle e, i \rangle + 1)$, $\alpha := f \uparrow 2\langle e, i \rangle + 1$.

If $q_e(\alpha^\sim a, \alpha^\sim b) = \alpha^\sim a$ then $q_{e,h(0)}(\alpha^\sim a, \alpha^\sim b) = \alpha^\sim a$. Then $f(2\langle e, i \rangle + 1) = b$, and $\alpha^\sim b < \alpha^\sim a$. Thus $\alpha^\sim b \in M$, $\alpha^\sim a \notin M$, contradicting the properties of $\varphi_e$. The other case is symmetric.

We conclude that $M \in (2, 3)$, $M$ is weakly semirecursive (1., Theorem 4.1), and $M$ is not semirecursive (4.).

Jockusch and Owings [5] proved that weakly semirecursive equals semirecursive in any hyperimmune-free degree. In fact, their result can be extended as follows: Call a set $M$ weakly $K$-semirecursive iff there exists a partial function $\psi \leq_T K$ such that for all $x, y \in \omega$: $[x \in M \land y \notin M] \lor [x \notin M \land y \in M] \rightarrow \psi(x, y) \in \{x, y\} \cap M$.

Every weakly semirecursive set is weakly $K$-semirecursive, but not conversely.

**Proposition 8.2.** If $M$ is weakly $K$-semirecursive and $\text{dg}(M)$ is hyperimmune-free, then $M$ is semirecursive.

**Proof** (A similar idea was used in the proof of Theorem 1.2 in [6]). Let $M$ be weakly $K$-semirecursive witnessed by $\psi \leq_T K$, and suppose that $\text{dg}(M)$ is hyperimmune-free. By the Limit Lemma there exists a total recursive function $f(x, y, s)$ such that for all $x, y$: $\psi(x, y) \downarrow \psi(x, y) = \lim_{n \rightarrow \omega} f(x, y, s)$. W.l.o.g. $f(x, y, s) \in \{x, y\}$.

Define a total function $h \leq_T M$:

$$h(x, y, s) := \begin{cases} \mu k. f(x, y, s) \neq f(x, y, s + k) & \text{if } \{x, y\} \cap M \neq \emptyset \land f(x, y, s) \notin M, \\ 0 & \text{otherwise.} \end{cases}$$

As $\text{dg}(M)$ is hyperimmune-free there exists a total recursive function $g$ which dominates $h$. Set $m(x, y) := \mu s. \forall k. f(x, y, s) \land f(x, y, s + k)$. $m(x, y)$ is partial recursive. $M$ is weakly semirecursive via $\lambda x, y. f(x, y, m(x, y))$. Thus, $M$ is semirecursive by the result of Jockusch and Owings.

The preordering $<$ from the proof of Theorem 8.1 also induces a coarser r.e. partial ordering $\sqsubseteq'$ defined as follows:

$$\alpha \sqsubseteq' \beta \iff \alpha \sqsubseteq \beta \text{ and for any } \gamma \subseteq \alpha, \text{ such that } |\gamma| \text{ is odd:}$$

$$(\gamma^\sim a \subseteq \alpha, \beta \lor \gamma^\sim b \subseteq \alpha, \beta) \rightarrow (\gamma^\sim a < \gamma^\sim b \lor \gamma^\sim b < \gamma^\sim a).$$

$\sqsubseteq'$ is almost linear. Any initial segment of $\sqsubseteq$ as defined in the proof of Theorem 8.1 is also an initial segment of $\sqsubseteq'$. Therefore, in the proof of Theorem 8.1 $M$ can be chosen as an IAL.
Corollary 8.3. The following degree classes are identical:
(a) \( \{ a : a \text{ is hyperimmune} \} \),
(b) \( \{ a : a \text{ contains a weakly semirecursive set which is not semirecursive} \} \),
(c) \( \{ a : a \text{ contains a weakly K-semirecursive set which is not semirecursive} \} \),
(d) \( \{ a : a \text{ contains an IAL which is not semirecursive} \} \).

9. Degrees of weakly semirecursive sets which are neither semi-r.e. nor co-semi-r.e.

Immediately from Corollary 5.3 we get:

Corollary 9.1. Any hyperimmune degree not r.e. in \( 0' \) contains a weakly semirecursive set which is neither semi-r.e. nor co-semi-r.e.

In view of Corollary 9.1 the nonrecursive degrees r.e. in \( 0' \) remain to be considered. Each such degree is hyperimmune [4, Corollary 5.9]. Jockusch and Owings [5] observed that their basic construction of a weakly semirecursive, non-semi-r.e., and non-co-semi-r.e. set can be performed in every nonrecursive r.e. degree. Our next theorem covers the degrees above \( 0' \).

Theorem 9.2. Any degree above \( 0' \) contains a weakly semirecursive set which is neither semi-r.e. nor co-semi-r.e.

Proof. This proof is quite similar to the proof of Theorem 8.1.
\[ A := \{ a, b, c, d, 0, 1 \} \text{. Let } T \text{ be the least subset } S \text{ of } A^{<\omega} \text{ such that for all } \alpha:\]
\[ \begin{align*}
(0) & \quad \lambda \in S, \\
(1) & \quad \alpha \in S \land |\alpha| = 3e \rightarrow \alpha^0, \alpha^1 \in S \\
(2) & \quad \alpha \in S \land |\alpha| = 3e + 1, 3e + 2 \rightarrow \alpha^a, \alpha^b, \alpha^c, \alpha^d \in S.
\end{align*} \]

\( T \) is a recursive tree. Now we define an r.e. preordering \(<\) as follows: If \( |\alpha| = 3e \) then \( \alpha^0 < \alpha^1 \). If \( |\alpha| = 3e + 1, 3e + 2 \) then \( \alpha^d < (\alpha^a, \alpha^b) < \alpha^c \), if \( \varphi_e(\alpha^a, \alpha^b) = \alpha^a \) then \( \alpha^b < \alpha^a \), if \( \varphi_e(\alpha^a, \alpha^b) = \alpha^b \) then \( \alpha^a < \alpha^b \).

Let \( r \) be the induced r.e. partial ordering of \( T \).

Let \( a \) be a degree above \( 0' \) and \( A \in a \).

We define an infinite path \( f \) through \( T \) by induction:
\[ f(3e) := 0, \quad \text{if } e \in A; \quad f(3e) := 1, \quad \text{if } e \notin A. \]

If \( \alpha - f \uparrow 3e + i, i \in \{ 1, 2 \} \) then
\[ f(3e + i) := \begin{cases} 
\text{b} & \quad \text{if } \varphi_e(\alpha^a, \alpha^b) = \alpha^a, \\
\text{a} & \quad \text{if } \varphi_e(\alpha^a, \alpha^b) = \alpha^b, \\
\text{c} & \quad \text{if } \varphi_e(\alpha^a, \alpha^b) \uparrow \text{ and } i = 1, \\
\text{d} & \quad \text{if } \varphi_e(\alpha^a, \alpha^b) \uparrow \text{ and } i = 2.
\end{cases} \]
Weakly semirecursive sets and r.e. orderings

Note that \( f \) is recursive in \( A \oplus K \in \mathfrak{a} \). Define \( M := \{ \alpha \in T : \exists n \ [\alpha \sqsubset f \uparrow n] \} \). As in the proof of Theorem 8.1 we have:

1. \( A \equiv_T M \).
2. \( M \) is an initial segment of \( \sqsubset \).
3. \( M \) is not semi-r.e.: Suppose for a contradiction that \( M \) is semi-r.e. via \( \varphi_e \).
   W.l.o.g. we assume that for all \( x, y \): \( \varphi_e(x, y) = \varphi_e(y, x) \in \{ x, y \} \).
   Consider the definition of \( f(3e + 1) \), \( \alpha := f \uparrow 3e + 1 \):
   If \( \varphi_e(\alpha^\sim a, \alpha^\sim b) = \alpha^\sim a \), then \( f(3e + 1) = b \), and \( \alpha^\sim b < \alpha^\sim a \). Thus, \( \alpha^\sim b \in M \), \( \alpha^\sim a \notin M \), contradicting the properties of \( \varphi_e \). The case, \( \varphi_e(\alpha^\sim a, \alpha^\sim b) = \alpha^\sim b \) is symmetric.
   If \( \varphi_e(\alpha^\sim a, \alpha^\sim b) \) is undefined, then \( f(3e + 1) = c \). Thus, \( \alpha^\sim a, \alpha^\sim b \in M \), and therefore \( \varphi_e(\alpha^\sim a, \alpha^\sim b) \) should be defined, contradiction.
4. \( M \) is not co-semi-r.e.: \( M \) is not co-semi-r.e. via \( \varphi_e \), follows by a similar argument as above: Consider \( f(3e + 2) \) instead of \( f(3e + 1) \).
   We conclude that \( M \in \mathfrak{a} \). \( M \) is weakly semirecursive, and \( M \) is neither semi-r.e. nor co-semi-r.e. \( \square \)

As in the previous section we obtain from the proof of Theorem 9.2:

**Corollary 9.3.** Any degree above \( 0' \) contains a set which is IAL and neither semi-r.e. nor co-semi-r.e.

10. Degrees of weakly semirecursive non-IAL sets

In the previous constructions recursive trees were used and we obtained almost linear orderings. For the construction of weakly semirecursive non-IAL sets we need r.e. nonrecursive trees.

**Theorem 10.1.** Any degree above \( 0' \) contains a weakly semirecursive set which is not IAL.

**Proof.** \( A := \{ a, b, c, d, 0, 1 \} \). Let \( T \) be the least subset \( S \) of \( A^{<\omega} \) such that for all \( \alpha \):

\[
\begin{align*}
(0) & \quad \lambda \in S, \\
(1) & \quad \alpha \in S \land |\alpha| \text{ even } \rightarrow \alpha^\sim 0, \alpha^\sim 1 \in S, \\
(2) & \quad \alpha \in S \land |\alpha| \text{ odd } \rightarrow \alpha^\sim a, \alpha^\sim b, \alpha^\sim d \in S, \\
(3) & \quad \alpha \in S \land |\alpha| \text{ odd } \land \varphi_e(\alpha^\sim a, \alpha^\sim b) \downarrow \rightarrow \alpha^\sim c \in S.
\end{align*}
\]

\( T \) is an r.e. tree. Now we define an r.e. preordering \( < \) as follows: If \( |\alpha| = 2e \) then \( \alpha^\sim 0 < \alpha^\sim 1 \). If \( |\alpha| = 2e + 1 \) then \( \alpha^\sim d < \alpha^\sim a < \alpha^\sim b \), and if \( \varphi_e(\alpha^\sim a, \alpha^\sim b) \downarrow \),
then $\alpha^\sim d < \alpha^\sim c$, and

(i) $\exists \{q_{c,1}(\alpha^\sim a, \alpha^\sim c) \land q_{c,2}(\alpha^\sim a, \alpha^\sim b)\}$, then: $\alpha^\sim c < \alpha^\sim a$.

(ii) $\exists \{q_{c,1}(\alpha^\sim c, \alpha^\sim b) \land q_{c,2}(\alpha^\sim a, \alpha^\sim c)\}$, then: $\alpha^\sim b < \alpha^\sim c$.

Let $\sqsupset$ be the induced r.e. partial ordering of $T$, let $\mathbf{a}$ be a degree above $\emptyset'$, let $A \in \mathbf{a}$.

We define an infinite path $f$ through $T$ by induction:

$$f(2e) := 0, \quad \text{if } e \in A; \quad f(2e) := 1, \quad \text{if } e \notin A.$$  

If $\alpha = f \upharpoonright 2e + 1$ then

$$f(2e + 1) := \begin{cases}  
\alpha & \text{if } q_{\alpha}(\alpha^\sim a, \alpha^\sim b) \uparrow,  
\beta & \text{if } q_{\alpha}(\alpha^\sim a, \alpha^\sim b) \downarrow \text{ and } \alpha^\sim c < \alpha^\sim a,  
\gamma & \text{if } q_{\alpha}(\alpha^\sim a, \alpha^\sim b) \downarrow \text{ and } \alpha^\sim b < \alpha^\sim c,  
\delta & \text{otherwise.}  
\end{cases}$$

$f$ is recursive in $A \oplus K \in \mathbf{a}$. $M := \{\alpha \in T : \exists n [\alpha \sqsupset f \upharpoonright n]\}$.

As in the previous proofs it can be checked that $M$ is an initial segment of $\sqsupset$ and $M \in a$.

$M$ is not IAL: Suppose for a contradiction that $M$ is an initial segment of the r.e. almost linear ordering $\sqsupset'$ and let $\{(\alpha, \beta) : \alpha \sqsupset' \beta \} = \text{dom}(q_{\alpha})$. Consider the definition of $f(2e + 1)$, $\alpha := f \upharpoonright 2e + 1$:

If $q_{\alpha}(\alpha^\sim a, \alpha^\sim b) \uparrow$ then $\alpha^\sim a \in M$, $\alpha^\sim b \notin M$. If $q_{\alpha}(\alpha^\sim a, \alpha^\sim b) \downarrow$, i.e. $\alpha^\sim a \sqsupset' \alpha^\sim b$, then either (i) $\alpha^\sim a \sqsupset' \alpha^\sim c$, and $\alpha^\sim c < \alpha^\sim a$, i.e., $\alpha^\sim c \in M$, $\alpha^\sim a \notin M$ or (ii) $\alpha^\sim c \sqsupset' \alpha^\sim b$, and $\alpha^\sim b < \alpha^\sim c$, i.e., $\alpha^\sim b \in M$, $\alpha^\sim c \notin M$. In any case $M$ is not an initial segment of $\sqsupset'$, contradiction.

We conclude that $M \in \mathbf{a}$, $M$ is weakly semirecursive and $M$ is not IAL.

From Theorems 10.1 and 6.3 we obtain a negative answer to a question of Jockusch and Owings:

**Corollary 10.2.** $0'$ contains a weakly semirecursive set which does not have SCP.

**Remark.** It is possible to prove Corollary 10.2 by a recursive approximation construction in the style of the proof of Theorem 5 in [5]. In this way also a semi-r.e. set of degree $0'$ can be obtained which does not have SCP. It is, however, not clear whether this construction can be performed in every nonrecursive r.e. degree; in any case, the construction is combinable with the lowness requirement, and one can obtain a low r.e. degree containing a semi-r.e. set which does not have SCP.

Roy [12] observed that any recursive partial ordering can be extended to a recursive linear ordering, and he constructed an r.e. partial ordering which does not have such an extension. Note that no r.e. partial ordering which has a non-semirecursive initial segment can be extended to a recursive linear ordering.
From Theorems 4.1 and 10.1 we immediately obtain:

**Corollary 10.3.** There exists an r.e. partial ordering which does not have an r.e. almost linear extension.

### 11. Degrees of semi-r.e. non-semirecursive sets

By Corollary 5.3 any such degree must be r.e. in 0′. Any r.e. nonrecursive degree contains a (semi-) r.e. non-semirecursive set [4, Theorem 4.2].

**Theorem 11.1.** Any degree \( a \) which is r.e. in and above 0′ contains a semi-r.e. non-semirecursive set.

**Proof.** \( A := \omega \). We will define a recursive function \( m(\alpha, s) \geq 1 \) which is nondecreasing in \( s \). Then \( T \) is defined to be the least subtree \( S \) of \( \Lambda^\omega \) such that:

\[
\begin{align*}
(0) & \quad \lambda \in S, \\
(1) & \quad \alpha \in S \land \exists s \left[ m(\alpha, s) + 1 \geq k \right] \rightarrow \alpha \uparrow k \in S,
\end{align*}
\]

Remember that whenever we are referring to nodes \( \alpha \in T \) as arguments of recursive functions we are in fact referring to the coding number of \( \alpha \) w.r.t. a recursive bijection of \( \omega \) and \( T \).

Let \( a \) be r.e. and above 0′, choose \( A \in a \), such that \( A \in \Pi_2 \).

There exists a total recursive function \( g \) such that for all \( x: x \in A \Leftrightarrow W_{g(x)} \) is infinite. We will now define in stages \( m(\alpha, s) \), a preordering \( \prec_s \), and an approximation \( f_s \) of an infinite path \( f \) through \( T. \prec_s \) denotes the partial ordering induced by \( \prec \). We put \( M := \{ \alpha \in T: \forall n [ \alpha \in f \downarrow n ] \} \), and \( \tilde{M} \) will be our semi-r.e. non-semirecursive set. \( \subset \) is constructed by an application of \( \Pi_2 \)-guessing, a standard tool of the \( 0^\alpha \)-priority method (cf. [14, Ch. XIV]).

As usual we will follow the convention that for any variable \( p \) the value \( p(s + 1) \) equals \( p(s) \) unless \( p(s + 1) \) has been explicitly defined otherwise.

**Construction**

**Stage 0.** Let \( m(\alpha, 0) = 0 \) and \( \alpha \uparrow 0 \prec_0 \alpha \uparrow 1 \) for all \( \alpha \in \Lambda^\omega \) such that \( |\alpha| \) is even. Let \( m(\alpha, 0) = 1 \) and \( \alpha \uparrow 1 \), \( \alpha \uparrow 2 \prec_0 \alpha \uparrow 0 \) for all \( \alpha \in \Lambda^\omega \) such that \( |\alpha| \) is odd.

**Stage \( s + 1 \)**

(1) For \( 0 \leq i < s \), given \( \alpha := f_s \downarrow i \) define \( f_s(i) \) inductively as follows:

(1.1) If \( |\alpha| = 2e \), let

\[
f_s(2e) := \begin{cases} 0 & \text{if } |W_{g(\alpha)}(i+1)| > |W_{g(\alpha)}(i)| \text{ where } i \text{ is the greatest } \alpha \text{-stage } \leq s, \\ 1 & \text{otherwise.}
\end{cases}
\]

(1.2) If \( |\alpha| = 2e + 1 \): Let \( \beta = \alpha \uparrow m(\alpha, s), \gamma = \alpha \uparrow (m(\alpha, s) + 1) \).
If $\beta$ and $\gamma$ are incomparable w.r.t. $<_s$, and $\varphi_{e,s}(\beta, \gamma) \in \{\beta, \gamma\}$ then enlarge $<_s$ by:

$$\beta <_s \gamma, \text{ if } \varphi_{e,s}(\beta, \gamma) = \gamma, \text{ and } \gamma <_s \beta, \text{ if } \varphi_{e,s}(\beta, \gamma) = \beta.$$  

Define

$$f_s(2e + 1) \begin{cases} m(\alpha, s) & \text{if } \beta <_s \gamma, \\ m(\alpha, s) + 1 & \text{if } \gamma <_s \beta, \\ 0 & \text{otherwise}. \end{cases}$$

(2) Let $\sqsubseteq_e$ be the ordering induced by the transitive closure of $<_s$. Initialize all nodes $\alpha$, $|\alpha| < s$, $|\alpha|$ odd, such that $f_e \sqsubseteq_e \alpha$, i.e., do the following:

If $\alpha ^\prec m(\alpha, s)$ and $\alpha ^\prec (m(\alpha, s) + 1)$ are incomparable w.r.t. $<_s$ then enlarge $<_s$ by $\alpha ^\prec m(\alpha, s) <_s \alpha ^\prec (m(\alpha, s) + 1)$, and then let $m(\alpha, s + 1) = m(\alpha, s) + 2$. Add $\alpha ^\prec (m(\alpha, s) + 1) <_s (\alpha ^\prec m(\alpha, s + 1), \alpha ^\prec (m(\alpha, s + 1) + 1)) <_s \alpha ^\prec 0$.

(3) $<_s + 1 :=$ transitive closure of $<_s$. 

End of construction

Let $<$ denote the union of $<_s$ for all $s$. $<$ is a preordering of $T$. Let $\sqsubset$ denote the induced partial ordering of $T$. $\sqsubset$ is r.e. The path $f$ is defined inductively:

$$f(2e) := 0, \text{ if } e \in A, \quad f(2e) := 1, \text{ if } e \notin A.$$  

If $\alpha = f \uparrow 2e + 1$, and $m = \lim_{n \to s} m(\alpha, s)$ exists then

$$f(2e + 1) := \begin{cases} m & \text{if } \varphi_s(\alpha ^\prec m, \alpha ^\prec (m + 1)) = \alpha ^\prec (m + 1), \\ m + 1 & \text{if } \varphi_s(\alpha ^\prec m, \alpha ^\prec (m + 1)) = \alpha ^\prec m, \\ 0 & \text{otherwise}. \end{cases}$$

Each node $\alpha \sqsubseteq f$ is initialized only finitely often, i.e., $\lim_{n \to s} m(\alpha, s)$ exists. Thus $f$ is infinite. Each node $\alpha$ to the right of $f$ is initialized infinitely often, thus $<$ is a linear order of Sons$(\alpha)$. Let $M$ be defined as above. $M$ is an initial segment of $\sqsubset$. By the initialization action it is clear that the nodes of $M$, i.e. the nodes to the right of $f$, are linearly ordered by $\sqsubseteq$. Thus, by Theorem 5.1, $\tilde{M}$ is semi-r.e.

Note that $f$ is recursive in $A \oplus K \in \alpha$ (the $K$-oracle is used to decide $<$ and to compute the limit of $m(\alpha, s)$, for $\alpha \sqsubseteq f$). On the other hand $\tilde{f} x f(2x)$ is the characteristic function of $A$, thus $A$ is recursive in $f$. In turn $f$ is recursive in $M$: It suffices to show that $f(x)$ is computable from $\alpha = f \uparrow x$ with the help of an $M$-oracle. If $x$ is even, we test whether $\alpha ^\prec 0 \in M$. If $x$ is odd and $\alpha ^\prec 0 \in M$ then $f(x) = 0$, else we search $s$ such that $M(\alpha ^\prec m(\alpha, s)) \neq M(\alpha ^\prec (m(\alpha, s) + 1))$, and check whether $\varphi_s(\alpha ^\prec m(\alpha, s), \alpha ^\prec (m(\alpha, s) + 1))$ equals $\alpha ^\prec m(\alpha, s)$ or $\alpha ^\prec (m(\alpha, s) + 1)$, in the former case $f(x) = m(\alpha, s) + 1$, in the latter $f(x) = m(\alpha, s)$.

As in the previous proofs $M$ is recursive in $f$. It follows that $M \equiv_T A$.

$M$ is not semirecursive: Suppose for a contradiction that $M$ is semirecursive via $\varphi_e$. We may assume that $\varphi_e(x, y) = \varphi_e(y, x) \in \{x, y\}$, for all $x, y$. Let $\alpha =$
$f \uparrow 2e + 1$ and choose an $a$-stage $s$ large enough such that $m(a, s)$ has settled down. Let $\beta = \alpha^\sim m(a, s)$, $\gamma = \alpha^\sim (m(a, s) + 1)$. If $\varphi_e(\beta, \gamma) = \gamma$ then $\beta < \gamma$ and $\beta \in M$, $\gamma \notin M$, by definition of $f$, contradicting the property of $\varphi_e$. The other case is symmetric.

We conclude that $\tilde{M} \in a$, $\tilde{M}$ is semi-r.e. and not semirecursive. \(\square\)

**Corollary 11.2.** $0''$ is the greatest (w.r.t. $\leq_T$) degree containing a semi-r.e. nonsemirecursive set.

### 12. Conclusion

The previous results show the usefulness of r.e. partial orderings for constructions of degrees. We expect that they turn out to be useful in other branches of recursion theory, too. Almost all questions from [5] have been answered now. In the present paper some new problems turned up, however:

1. The classification in Sections 9 and 11 is incomplete because the non-r.e. degrees which are r.e. in $0'$ and not above $0'$ are not included. Any such degree is hyperimmune, and we conjecture that Theorems 9.2 and 11.1 can be extended to cover these cases. Note that by Theorem 8.1 any such degree either contains a weakly semirecursive set which is neither semi-r.e. nor co-semi-r.e., or contains a semi-r.e. non-semirecursive set.

2. It is open whether every hyperimmune degree contains a weakly semirecursive set which is not IAL.

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### References


