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An interpretation of skew-elliptical distributions in terms of fuzzy events

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Abstract In this note, we introduce a fuzzy method for producing family of univariate and multivariate skew-elliptical distributions based on fuzzy conditional events. We illustrate special cases of interest, such as skew-normal distribution. Furthermore, we use the idea of fuzzy events for calculating tail conditional expectations for elliptical and skew-elliptical distributions.

1. Introduction

The class of elliptically-contoured distributions (or elliptical distributions, for short) provides a generalization of the multivariate normal distributions. This class contains many non-normal multivariate distributions such as the multivariate Student-t, Cauchy, logistic, Laplace and symmetric stable. Skewed distributions have recently received a great deal of attention in the literature, since many data encountered in practice display a great deal of skewness. The class of skew-normal distributions was given its name by Azzalini [1] in 1985 and generalized to the multivariate case by Azzalini and DallaValle [2]. Moreover, Azzalini and Capitanio [3] proposed a generalization of skew Student-t distributions using a perturbation of symmetry; Arellano-Valle and Genton [4] discuss various generalizations and multivariate forms of skew-normal and skew-elliptical distributions. Also, Wang et al. [5] studied skew-symmetric distributions.

The class of skew-elliptical distributions can be obtained by several stochastic mechanisms, for example see Azzalini [6], Azzalini & Dalla Valle [2], Azzalini & Capitanio [3] and Branco & Dey [7]. Our motivation for this note is to provide a mechanism for producing a family of skew-elliptical distributions via fuzzy conditional events. In Section 2, we give a brief introduction on skew-elliptical distributions and the probability measure of fuzzy events. In Section 3, we show how to construct a skew-elliptical distribution using fuzzy events. Finally, in Section 4, we state the idea of fuzzy events for calculating tail conditional expectations and derive explicit expressions for normal and skew-normal distributions.

2. Preliminaries

In this section, we review definitions and basic properties of skew-elliptical distributions as well as the concepts of the probability measure of fuzzy events [8] that will be used in this paper. For more comprehensive review and characterization elliptical distributions see [4,9–11].

2.1. The family of the skew-elliptical distributions

An m-dimensional random vector $\mathbf{X}$ is said to have an elliptical distribution with location vector $\boldsymbol{\mu} \in \mathbb{R}^m$, dispersion matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times m}$, density generator $h^{(m)}$, if the probability density function (pdf) of $\mathbf{X}$ has the form:

$$\mathbf{f}_{\mathbf{EC}_m}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, h^{(m)}) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}} h^{(m)} \left[(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right], \quad (1)$$

which is denoted by $\mathbf{X} \sim \mathbf{EC}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}; h^{(m)})$.

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Now, let $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$ be two random vectors and

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim \mathbf{EC}_{m+n}\left(\begin{pmatrix} \boldsymbol{\gamma} \\ \delta \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Delta}^T \\ \boldsymbol{\Delta} & \boldsymbol{\Sigma} \end{pmatrix}; h^{(m+n)}\right), \tag{2}$$

where $\boldsymbol{\gamma} \in \mathbb{R}^m$, $\boldsymbol{\delta} \in \mathbb{R}^n$, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$, $\boldsymbol{\Gamma} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{\Delta} \in \mathbb{R}^{n\times m}$. The random vector $\mathbf{U} = (\mathbf{Y}|\mathbf{X} \in \mathbf{C})$, where $\mathbf{C} = \{x \in \mathbb{R}^m | \boldsymbol{x} > \mathbf{0}\}$, is said to have the unified multivariate skew-elliptical distribution (Arellano-Vallea & Genton [4]), denoted by $\mathbf{U} \sim \mathbf{SEC}_{n,m}(\boldsymbol{\theta}, h^{(m+n)})$, where $\boldsymbol{\theta} = (\boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})$, if the pdf of $\mathbf{U}$ has the form:

$$\mathbf{f}_{\mathbf{SEC}_{n,m}}\left(\mathbf{u}; \boldsymbol{\theta}, h^{(m+n)}\right) = \frac{\mathbf{f}_{\mathbf{EC}_n}\left(\mathbf{u}; \delta, \boldsymbol{\Sigma}, h^{(n)}\right)\mathbf{F}_{\mathbf{EC}_m}\left(\boldsymbol{\gamma} + \boldsymbol{\Delta}^T\boldsymbol{\Sigma}^{-1}(\mathbf{u}-\delta); \boldsymbol{\Lambda}, h^{(m)}_{\omega(\mathbf{u})}\right)}{\mathbf{F}_{\mathbf{EC}_m}\left(\boldsymbol{\gamma}; \boldsymbol{\Gamma}, h^{(m)}\right)},$$

$$\mathbf{u} \in \mathbb{R}^n, \tag{3}$$

where $\boldsymbol{\Lambda} = \boldsymbol{\Gamma} - \boldsymbol{\Delta}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}$ and $\omega(\mathbf{u}) = (\mathbf{u}-\delta)^T\boldsymbol{\Sigma}^{-1}(\mathbf{u}-\delta)$. Also, $\mathbf{F}_{\mathbf{EC}_m}(\boldsymbol{\gamma} + \boldsymbol{\Delta}^T\boldsymbol{\Sigma}^{-1}(\mathbf{u}-\delta); \boldsymbol{\Lambda}, h^{(m)}_{\omega(\mathbf{x})})$ and $\mathbf{F}_{\mathbf{EC}_m}(\boldsymbol{\gamma}; \boldsymbol{\Gamma}, h^{(m)})$ denote $\Pr(\mathbf{Y} \in \mathbf{C})$ where $\mathbf{Y} \sim \mathbf{EC}_m(\boldsymbol{\gamma} + \boldsymbol{\Delta}^T\boldsymbol{\Sigma}^{-1}(\mathbf{u}-\delta), \boldsymbol{\Lambda}, h^{(m)}_{\omega(\mathbf{x})})$ and $\mathbf{Y} \sim \mathbf{EC}_m(\boldsymbol{\gamma}, \boldsymbol{\Gamma}; h^{(m)})$, respectively.

Note that the unified skew-elliptical distributions are reduced to the skew-elliptical distributions when $\boldsymbol{\gamma} = \mathbf{0}$. The family of the skew-elliptical distributions consist of many asymmetric distributions where the skew-normal, the skew-t, the skew-logistic and the skew-Laplace are its familiar examples. In this paper, we only study the normal case, i.e. when:

$$h^{(m+n)}(u) = (2\pi)^{-\frac{(m+n)}{2}} \exp\left(-\frac{u}{2}\right).$$

In this case, we obtain the unified multivariate skew-normal distribution, denoted by $\mathbf{U} \sim \mathbf{SN}_{n,m}(\boldsymbol{\theta})$, with the pdf

$$\boldsymbol{\phi}_{\mathrm{SN}_{n,m}}(\mathbf{u}; \boldsymbol{\theta}) = \frac{\boldsymbol{\phi}_n(\mathbf{u}; \delta, \boldsymbol{\Sigma})\,\boldsymbol{\Phi}_m\left(\boldsymbol{\gamma} + \boldsymbol{\Delta}^T\boldsymbol{\Sigma}^{-1}(\mathbf{u}-\delta); \boldsymbol{\Lambda}\right)}{\boldsymbol{\Phi}_m(\boldsymbol{\gamma}; \boldsymbol{\Gamma})},$$

$$\mathbf{u} \in \mathbb{R}^n, \tag{4}$$

where

- $\phi_n(\cdot; \delta, \boldsymbol{\Sigma})$ is the pdf of $\mathbf{N}_n(\delta, \boldsymbol{\Sigma})$;
- $\boldsymbol{\Phi}_m(\cdot; \boldsymbol{\Lambda})$ is the cumulative density function (cdf) of $\mathbf{N}_m(\mathbf{0}, \boldsymbol{\Lambda})$;
- $-\boldsymbol{\Phi}_m(\cdot; \boldsymbol{\Gamma})$ is the cdf of $\mathbf{N}_m(\mathbf{0}, \boldsymbol{\Gamma})$.

We can readily obtain the other skew-elliptical distributions in Eq. (3), in a manner similar to the above.

2.2. Concepts on fuzzy sets

Probability theory and fuzzy logic are the principal components of an array of methodologies for dealing with problems in which uncertainty and imprecision play important roles. In this subsection, we have collected together the basic ideas from fuzzy sets and probability of fuzzy events which are needed in this paper.

2.2.1. The membership function

The concept of fuzzy set was initiated by Zadeh [12] in 1965. Let Ω be a universe of discourse and $\tilde{A}$ a fuzzy subset of Ω. If, for all $x \in \Omega$, there is a number $\mu_{\tilde{A}}(x) \in [0, 1]$ assigned to represent the membership of x to $\tilde{A}$, then $\mu_{\tilde{A}}$ is called the membership function of $\tilde{A}$. In many cases, the membership functions take on specific functional forms like triangular, trapezoidal, S-functions, Pi-functions, sigmoid, and even Gaussian for convenience in representation and computation. Alternately, membership functions can be estimated from training data, much as probability density and cumulative distribution functions are estimated. A good overview of the various interpretations of membership functions in fuzzy set theory can be found in [13]. The focus in this paper is on skew-membership functions and cumulative distribution functions (cdfs) that provide a method for constructing skew distributions in Section 3.

2.2.2. Probability of fuzzy events

Let (Ω, A, P) be a probability space where A is the σ-field of Borel sets on the sample space Ω and P is a probability measure over Ω. A fuzzy event in Ω is a fuzzy set $\tilde{A}$ in Ω whose membership function, $\mu_{\tilde{A}} : \Omega \longrightarrow [0, 1]$, is Borel measurable. A fuzzy-logical treatment for the probability of classical events has been widely studied in the last years. In particular, starting from the basic ideas exposed by Zadeh [8], unconditional and conditional probability can be studied using various kinds of modal-fuzzy logics [14]. In the case where we have some probability distributions on the universe Ω, it seems to be more natural to view a vague concept as providing additional information according to which we may be able to conditionally update this distribution. In the context of fuzzy set theory, definitions for the probability and conditional probability of fuzzy sets are required. Zadeh [8] defined the probability of a fuzzy event as the expected value of its membership function.

Let (Ω, A, P) be a probability space and $\mathbf{X} \in \mathbb{R}^n$ be a random vector in the continuous sample space Ω. If $f_{\mathbf{X}}$ is the pdf of $\mathbf{X}$, then probability of the fuzzy event $\tilde{A}$ in Ω is defined by:

$$P\left(\tilde{A}\right) = \int_{\Omega} \mu_{\tilde{A}}(\mathbf{x})\mathbf{f}_{\mathbf{X}}(\mathbf{x})d\mathbf{x}. \tag{5}$$

Based on the formula for the probability of a fuzzy set in Eq. (5), he also proposed the following definition for conditional distribution of $\mathbf{X}$ given the fuzzy event $\tilde{A}$.

Definition 2.1. Let $\mathbf{X} \in \mathbb{R}^n$ be a n-dimensional random vector in $\boldsymbol{\Omega}$ and $\tilde{A}$ be a fuzzy event defined over $\boldsymbol{\Omega}$ with the membership function $\mu_{\tilde{A}}(\mathbf{x})$, then the conditional distribution of $\mathbf{X}$ given the fuzzy constraint $\tilde{A}$ is:

$$\mathbf{f}(\mathbf{x}|\tilde{A}) = \frac{\mu_{\tilde{A}}(\mathbf{x})\mathbf{f}_{\mathbf{X}}(\mathbf{x})}{\int_{\mathbf{x}\in\boldsymbol{\Omega}} \mu_{\tilde{A}}(\mathbf{x})\mathbf{f}_{\mathbf{X}}(\mathbf{x})d\mathbf{x}}, \quad \mathbf{x} \in \boldsymbol{\Omega}. \quad \square \tag{6}$$

Indeed, Definition 2.1 can be viewed as a version of Bayes theorem, provided we interpret the value of membership function $\mu_{\tilde{A}}$ as the likelihood of fuzzy event $\tilde{A}$ given the value $\mathbf{x}$ [15]. Also see Example 83 of [14].

3. Construction of skew-elliptical distributions via fuzzy events

Information imprecision and uncertainty exist in real-world applications. It can be due to human errors in collecting data or some unexpected situations. Therefore, the fuzzy set theory naturally provides an appropriate tool in modeling the imprecise concepts. Also, sometimes, the vagueness in statistical data effects to distort the symmetry of the symmetric distributions.

This leads to the idea of studying skew distribution via fuzzy data. In this section, we use the concept of the vagueness in data for constructing skew-elliptical distributions via fuzzy events for two cases as follows:

(I) Firstly, we suppose the membership function of the fuzzy event depends on a cdf. This approach is based on the following general result by Azzalini [1,3,6]:

Lemma 3.1. *If $\mathbf{g}_n$ is a n-dimensional pdf such that $\mathbf{g}_n(\mathbf{x}) = \mathbf{g}_n(-\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{H}$ is an m-dimensional differentiable cdf such that the corresponding pdf of $\mathbf{H}$ is symmetric about $\mathbf{0}$ and $w_k(.)$ is a polynomial of order k, such that $w_k(-\mathbf{x}) = -w_k(\mathbf{x})$ for all $x \in \mathbb{R}^n$, then:*

$$\mathbf{f}_n(\mathbf{x}) = 2\mathbf{g}_n(\mathbf{x})\mathbf{H}_m\left\{w_k(\mathbf{x})\right\}, \tag{7}$$

is a pdf on $\mathbb{R}^n$.

Now, let $\mathbf{X} \sim \mathbf{EC}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; h^{(n)})$ and $\tilde{A}$ be a fuzzy event over Ω. According to the properties of cdf and membership functions, we can simply show that for $\alpha, \beta \in \mathbb{R}^+$

$$\mathbf{F}^{\alpha}_{\mathbf{EC}_n}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, h^{(n)}) \times \mathbf{H}^{\beta}_m(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{8}$$

and:

$$\mathbf{F}^{\alpha}_{\mathbf{EC}_n}\left(P_k\left(\frac{x_1-\mu_1}{\sigma_1}\right), \ldots, P_k\left(\frac{x_n-\mu_n}{\sigma_n}\right), h^{(n)}\right), \tag{9}$$

are membership functions for $\tilde{A}$, where $P_k(\frac{x_i-\mu i}{\sigma_i})$ is an odd polynomial of order k, μ_i is the location parameters, and σ_i^2 is the diagonal entries of $\boldsymbol{\Sigma}$ for $i = 1, \ldots, n$. For example, in normal case, $\mathbf{X} \sim \mathbf{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then according to Eq. (9):

$$\boldsymbol{\Phi}^{\alpha}_n\left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\mu})\right), \tag{10}$$

and:

$$\boldsymbol{\Phi}\left(\lambda_1\left(\frac{x-\mu_1}{\sigma_1}\right) + \lambda_2\left(\frac{x-\mu_1}{\sigma_1}\right)^3\right), \tag{11}$$

are membership functions, where $\boldsymbol{\lambda}^T = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and the constants μ_i and σ_i^2 are the mean and variance of X_i, respectively, for $i = 1, \ldots, n$.

Using the following theorem, we can construct the skew-elliptical distributions via fuzzy events when the membership functions depend on a cdf.

Theorem 3.2. *Let (Ω, A, P) be a probability space, $\mathbf{X} \sim \mathbf{EC}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; h^{(n)})$ and $\tilde{A}$ be a fuzzy event over $\boldsymbol{\Omega}$ with the membership function $\mathbf{H}_m(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\mu}))$, $\boldsymbol{\lambda} \in \mathbb{R}^n$, where $\mathbf{H}_m$ is defined in Lemma 3.1. Then, the conditional random vector, $\mathbf{Y} = \mathbf{X}|\tilde{A}$, generates a family of multivariate skew-elliptical distributions.*

Proof. Let $\mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}|\tilde{A})$ denote the pdfs of $\mathbf{X}$ and $\mathbf{Y} = \mathbf{X}|\tilde{A}$, respectively. Using the notion of probability measure of fuzzy event given in Definition 2.1 and Eq. (3), we see that $\mathbf{f}(\mathbf{x}|\tilde{A}) = 2\mathbf{f}_{\mathbf{EC}_n}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, h^{(n)})\mathbf{H}_m(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\mu}))$ is a multivariate skew elliptical distribution, for any fixed $\mathbf{x}$. □

Example 3.3 (*Multivariate Skew-Normal Distribution*)**.** If $\mathbf{X} \sim \mathbf{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\tilde{A}$ is a fuzzy event with the membership function $\boldsymbol{\Phi}(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\mu}))$, then by Theorem 3.2, $\mathbf{Y} = \mathbf{X}|\tilde{A}$ has the multivariate skew-normal distribution [2], denoted by $\mathbf{Y} \sim \mathbf{SN}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; \boldsymbol{\lambda})$, with the pdf:

$$\boldsymbol{\phi}_{\mathrm{SN}}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}) = 2\boldsymbol{\phi}_n(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\boldsymbol{\Phi}\left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{x}-\boldsymbol{\mu})\right). \tag{12}$$

If $\mathbf{Y}$ is the standard variable, i.e. $\boldsymbol{\mu} = \mathbf{0}$, $\boldsymbol{\Sigma} = \mathbf{I}$, where $\mathbf{I} \in \mathbb{R}^{n\times n}$ is the identity matrix, then we write $\mathbf{Y} \sim \mathbf{SN}_n(\boldsymbol{\lambda})$. The parameter $\boldsymbol{\lambda}$ regulates the skewness where $\boldsymbol{\lambda} = 0$ corresponds to the standard normal case. □

Theorem 3.2 states that if $\mathbf{X} \sim \mathbf{EC}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; h^{(n)})$, then one can generate the family of skew-elliptical distributions based on $\tilde{A}$. In the next theorem, the family of skew-elliptical distributions is generated based on $\tilde{A}$ when the distribution is also skew-elliptical.

Theorem 3.4. *Let $(\boldsymbol{\Omega}, A, P)$ be a probability space, $\mathbf{X} \sim \mathbf{SEC}_{n,m}(\boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta}, h^{(m+n)})$ with the cdf $\mathbf{F}_{\mathbf{SEC}_{n,m}}$ and $\tilde{A}$ be a fuzzy event over $\boldsymbol{\Omega}$ with membership function $\mathbf{F}_{\mathbf{SEC}_{n,m}}(\boldsymbol{x}; \boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta}, h^{(m+n)})$. Then, the pdf of $\mathbf{Y} = \mathbf{X}|\tilde{A}$, belongs to the family of skew-elliptical distributions.*

Proof. By Definition 2.1, we have:

$$\mathbf{f}(\mathbf{x}|\tilde{A}) \propto \mathbf{f}_{\mathbf{EC}_n}(\mathbf{x}; \boldsymbol{\delta}, \boldsymbol{\Sigma}, h^{(n)})\mathbf{H}_{n,m}(\mathbf{x}), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

where $\mathbf{H}_{n,m}(\mathbf{x}) = \mathbf{F}_{\mathbf{EC}_m}(\boldsymbol{\gamma} + \boldsymbol{\Delta}^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\delta}); \boldsymbol{\Lambda}, h^{(m)}_{\omega(\mathbf{u})}) \times \mathbf{F}_{\mathbf{SEC}_{n,m}}(\mathbf{x}; \boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta}, h^{(m+n)})$. But according to Eq. (8), $\mathbf{H}_{n,m}$ is a membership function. So, the rest of the proof is similar to the Theorem 3.2. □

Example 3.5 (*Univariate Balakrishnan Skew-Normal Distribution*)**.** Let the random variable $X \sim \mathrm{SN}(\lambda)$ and $\tilde{A}$ be a fuzzy event with the membership function $\boldsymbol{\Phi}^{n-1}(\lambda x)$, for $n > 1$, then by Theorem 3.4, the random variable $Y = X|\tilde{A}$ follows the univariate Balakrishnan skew-normal distribution [16], with the pdf:

$$\phi_{\mathrm{BSN}}(x; \lambda) = c_n(\lambda)\phi(x)\Phi^n(\lambda x), \quad x, \lambda \in \mathbb{R}, \tag{13}$$

where $c_n(\lambda)$ is the normalizing constant. □

Similarly, other skew elliptical distributions such as skew-t and skew-Cauchy [3,7] can be represented with appropriate choice of the membership functions in Theorem 3.2. Also, we can obtain an extension of the above results, such as the skew Cauchy-normal and the skew Cauchy-t [17], by suggesting that one takes $\mathbf{f}_{\mathbf{EC}_n}$ and $\mathbf{H}_m$ in Theorem 3.2 to belong to different families.

(II) Secondly, we present some generalizations of the skew-elliptical distribution via fuzzy event based on skew membership functions.

Note that any continuous skew-function π can be written as:

$$\pi(\mathbf{x}) = H\{w_k(\mathbf{x})\}, \quad x \in \mathbb{R}^n, \tag{14}$$

where $H : \mathbb{R} \to [0, 1]$ is the cdf of a continuous random variable symmetric around 0 and $w_k : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary odd polynomial of order k. By Eq. (14), Yanyuan and Genton [18] introduced the family of multivariate flexible skew-symmetric distributions with the pdf:

$$\mathbf{f}_n(\mathbf{x}) = 2\mathbf{f}_{EC_n}(\mathbf{x}-\boldsymbol{\xi})H\{w_k(\mathbf{x}-\boldsymbol{\xi})\}, \tag{15}$$

where $\boldsymbol{\xi} \in \boldsymbol{R}^n$, H and w_k are given in Eq. (14). Genton and Loperfido [11] introduced a result similar to Eq. (15), class of univariate generalized skew-elliptical distributions, defined by densities of the form:

$$h(x) = 2f_{EC}(x)\pi(x), \tag{16}$$

where $f_{EC} : \mathbb{R} \to \mathbb{R}^+$ is univariate elliptical pdf and $\pi : \mathbb{R} \to [0, 1]$ is a skew-function such that:

$$0 \le \pi(x) \le 1, \qquad \pi(-x) + \pi(x) = 1. \tag{17}$$

Also, Wang et al. [5] introduced the family of multivariate skew-symmetric distributions with the pdf:

$$h_n(\mathbf{x}) = 2\mathbf{f}_{EC_n}(\mathbf{x} - \boldsymbol{\xi})\boldsymbol{\pi}(\mathbf{x} - \boldsymbol{\xi}), \tag{18}$$

where $\boldsymbol{\xi} \in \boldsymbol{R}^n$, $\mathbf{f}_{EC_n} : \mathbb{R}^n \to \mathbb{R}^+$ is multivariate elliptical pdf and $\boldsymbol{\pi} : \mathbb{R}^n \to [0, 1]$ is a skew-function with given properties in Eq. (17).

The following theorem can be used for constructing generalized skew-elliptical distributions like skew-symmetric and flexible skew-symmetric distributions via fuzzy events.

Theorem 3.6. *Let (Ω, A, P) be a probability space, the random vector $\mathbf{X} \sim E_n(\mathbf{0}, \boldsymbol{\Sigma}; h^{(n)})$ and $\tilde{A}$ be a fuzzy event over Ω with the skew-membership function $\mu_{\tilde{A}}(\mathbf{x}) = \boldsymbol{\pi}(\mathbf{x})$ defined in Eq. (17). Then, the conditional random vector $\mathbf{X}|\tilde{A}$, has a generalized skew-elliptical distribution.*

Proof. The result can be easily obtained using Definition 2.1 and Eq. (15). □

Example 3.7 (*Flexible Generalized Skew-Normal Distribution*)**.** Let the random vector $\mathbf{X} = (X_1, X_2)^T \sim \mathbf{N}_2(\mathbf{0}, \mathbf{I})$, where $\mathbf{I} \in \mathbb{R}^{2\times 2}$ is the identity matrix and $\tilde{A}$ is a fuzzy event. By Theorem 3.6 and Eq. (9):

$$f(x_1) = 2\phi(x_1)\Phi(\alpha_1 x_1 + \beta_1 x_1^3), \tag{19}$$

and:

$$\mathbf{f}(x_1, x_2) = 2\boldsymbol{\phi}_2(x_1, x_2)\Phi\left(\alpha_1 x_1 + \alpha_2 x_2 + \beta_1 x_1^3 + \beta_2 x_2^3 + \beta_3 x_1^2 x_2 + \beta_4 x_1 x_2^2\right), \tag{20}$$

can be the pdfs [10] for the flexible generalized skew-normal random variable $Y = X_1|\tilde{A}$ and the random vector $\mathbf{Z} = \mathbf{X}|\tilde{A}$, respectively. □

4. The tail conditional expectation with fuzzy conditions

A risk measure is a mapping from random variables representing the risks to the real line. Its purpose is to give a single value for the degree of risk or uncertainty associated with the random variables. Examples of such risk measures are the standard deviation and the quantile of a distribution, also called Value-at-Risk (VaR). As one of the most commonly used risk measures,VaR suffers serious deficiencies if the losses are not normally distributed, Artzner et al. [19] introduced the notion of coherent risk measure, i.e. a risk measure which fulfills the following properties: monotonicity, subadditivity, positive homogeneity and translation invariance. They proved that the VaR is not coherent. A coherent alternative risk measure is the Tail Conditional Expectation (TCE).

Let the random variable X be the amount of claims on an insurance portfolio or the loss on an investment portfolio. The conditional expectation of X given that $X > x_q$, is denoted by:

$$\mathrm{TCE}_X(x_q) = E(X|X > x_q), \tag{21}$$

is called the tail conditional expectation of X at x_q, where x_q is the qth quantile of the distribution of X, for $0 < q < 1$. If $F_X(x)$ is cdf of the random variable X, then x_q is defined as:

$$x_q = \inf(x|F_X(x) \ge q). \tag{22}$$

So, the TCE risk measure is the conditional expectation of the random variable X given that X exceeds a specified quantile. Panjer [20] developed the TCE for the normal family, while Landsman and Valdez [21] deduced expressions for this risk measure for the class of elliptical distributions. Also Vernic [22] extended the *TCE* for the skew-normal distributions. In this section, we extend TCE formulas for elliptical and skew-elliptical distributions via fuzzy events. In particular, we obtain new TCE formulas for normal and skew-nomal distributions.

We will now turn our attention to TCE of the random variable X when the conditional event in Eq. (21), i.e. $A = \{X > x_q\}$ is fuzzy.

Remark 4.1. Let the random variable X has a elliptical or skew-elliptical distribution and $\tilde{A}$ is a fuzzy event, denoted by $\tilde{A} = \{X \gtrsim x_q\}$, then the distribution $Y = X|\tilde{A}$ can be skew-elliptical. So we can find TCE formulas for elliptical and skew-elliptical distributions via fuzzy events, according to Theorems 3.2, 3.4 and 3.6

Example 4.2 (*TCE for Normal Distribution*)**.** We first recall the result obtained by Panjer [20], for $X \sim N(\mu, \sigma^2)$, i.e.:

$$\mathrm{TCE}_X(x_q) = \mu + \frac{\phi(z_q)}{1 - \Phi(z_q)}\sigma, \tag{23}$$

where $z_q = (x_q - \mu)\sigma^{-1}$. Now, let $H(x) = \frac{x - x_q + a}{a+b}$ is the cdf of a uniform random variable on the interval $[x_q - a, x_q + b]$, where $a, b \in \mathbb{R}^+$. According to Definition 2.1 and Theorem 3.2, if $\tilde{A} = \{X \gtrsim x_q\}$ is a fuzzy event with the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < x_q - a \\ \dfrac{x - x_q + a}{a+b} & x_q - a \le x < x_q + b, \\ 1 & x \ge x_q + b \end{cases} \quad x \in \mathbb{R}, \tag{24}$$

then, the pdf of the random variable $Y = X|\tilde{A}$ is:

$$f(x|\tilde{A}) = \frac{1}{\alpha}\begin{cases} 0 & x < x_q - a \\ \left(\dfrac{x - x_q + a}{a+b}\right)\phi\left(x; \mu, \sigma^2\right) & x_q - a \le x < x_q + b, \\ \phi(x; \mu, \sigma^2) & x \ge x_q + b \end{cases} \quad x \in \mathbb{R}. \tag{25}$$

According to properties of pdf, the normalizing constant α is:

$$\alpha = \frac{\sigma}{a+b}(\phi(\upsilon_1) - \phi(\upsilon_2)) + \frac{\sigma\upsilon_1}{a+b}(\Phi(\upsilon_1) - \Phi(\upsilon_2)) + \Phi(-\upsilon_2),$$

where $\upsilon_1 = \frac{x_q - a - \mu}{\sigma}$ and $\upsilon_2 = \frac{x_q + b - \mu}{\sigma}$. So the fuzzy TCE of Y is defined as:

$$\mathrm{TCE}_Y(x_q) = \frac{1}{\alpha}\left(\int_{x_q - a}^{x_q + b} x\left(\frac{x - x_q + a}{a+b}\right)\phi(x; \mu, \sigma^2)dx + \int_{x_q + b}^{+\infty} x\phi(x; \mu, \sigma^2)dx\right).$$

Upon integrating by parts and after some simple algebraic calculations, TCE of the random variable Y is obtained as:

$$\mathrm{TCE}_Y(x_q) = \frac{1}{\alpha}\left(\frac{\sigma^2}{a+b}(\upsilon_1\phi(\upsilon_1) - \upsilon_2\phi(\upsilon_2)) + \frac{\sigma(\mu-\sigma\upsilon_1)}{a+b}(\phi(\upsilon_1)-\phi(\upsilon_2)) - \frac{\sigma(\sigma-\mu\upsilon_1)}{a+b}(\Phi(\upsilon_1)-\Phi(\upsilon_2)) + \sigma\phi(\upsilon_2) + \mu\Phi(-\upsilon_2)\right). \quad \square \tag{26}$$

Example 4.3. Let the daily profit of a company in a particular industry (X), is normally distributed with a mean of $1300^{\$}$ and a standard deviation of $250^{\$}$. We want to find the expected value of X, given that this profit already has been defined approximately more than $1190^{\$}$ with the membership function (24), when $a = b = 1$. We know that in this normal distribution 1190 is the 33th quantile of the distribution of X. Thus according to Eq. (26), we have

$$E(X|X \gtrsim 1190) = \mathrm{TCE}_X(1190) = 1435.2^{\$},$$

whereas the value of TCE for crisp condition is $1435.1^{\$}$. Table 1 shows the values of TCE for different quantiles for fuzzy and crisp conditions in normal distribution with mean $1300^{\$}$ and standard deviation $250^{\$}$.

Example 4.4 (*TCE for Skew-Normal Distribution*)**.** Let $X \sim \mathbf{SN}(\boldsymbol{\lambda})$, $\lambda \in \mathbb{R}$, with cdf Φ_{SN}. Vernic [22] obtained TCE of X as:

$$\mathrm{TCE}_X(x_q) = \frac{2}{1-\Phi_{\mathrm{SN}}(x_q;\lambda)}\left[\phi(x_q)\,\Phi\left(\frac{\gamma}{\sqrt{1-\gamma^2}}x_q\right) + \frac{\gamma}{\sqrt{2\pi}}\Phi\left(-\frac{1}{\sqrt{1-\gamma^2}}x_q\right)\right], \tag{27}$$

where $\gamma = \frac{\lambda}{\sqrt{1+\lambda^2}}$. Now, if $\tilde{A} = \{X \gtrsim x_q\}$ be a fuzzy event with the given membership function in Eq. (24), we can proceed, similar to Example 4.2, to derive explicit expressions for the pdf of $Y = X|\tilde{A}$, as:

$$f(x|\tilde{A}) = \frac{1}{\beta}\begin{cases} 0 & x < x_q - a \\ \left(\dfrac{x - x_q + a}{a+b}\right)\phi_{\mathrm{SN}}(x;\lambda) & x_q - a \le x < x_q + b \\ \phi_{\mathrm{SN}}(x;\lambda) & x \ge x_q + b \end{cases}, \quad x \in \mathbb{R}, \tag{28}$$

with:

$$\beta = \left(1 + \frac{1}{a+b}(\phi_{\mathrm{SN}}(\omega_1;\lambda) - \phi_{\mathrm{SN}}(\omega_2;\lambda)) + \frac{2\lambda}{(a+b)\sqrt{2\pi(1+\lambda^2)}}(\Phi(\omega_1)-\Phi(\omega_2)) + \frac{\omega_1}{a+b}(\Phi_{\mathrm{SN}}(\omega_1;\lambda) - \Phi_{\mathrm{SN}}(\omega_2;\lambda)) - \Phi_{\mathrm{SN}}(\omega_2;\lambda)\right),$$

where $\omega_1 = x_q - a$ and $\omega_2 = x_q + b$. Similarly, upon integrating by parts and some simple algebraic calculations, TCE of the skew-normal distribution is:

$$\mathrm{TCE}_Y(x_q) = \frac{1}{\beta}\left(1 + \frac{1}{a+b}(\omega_1\phi_{\mathrm{SN}}(\omega_1;\lambda) - \omega_2\phi_{\mathrm{SN}}(\omega_2;\lambda)) - \frac{1}{a+b}(\Phi_{\mathrm{SN}}(\omega_1;\lambda)) - (1-a-b)\Phi_{\mathrm{SN}}(\omega_2;\lambda) + \frac{2}{(a+b)\sqrt{2\pi(1+\lambda^2)}}\left(\lambda(\Phi(\omega_1)-\Phi(\omega_2)) - \frac{\omega_1}{\sqrt{1+\lambda^2}}(\phi(\omega_1)-\phi(\omega_2))\right)\right). \quad \square \tag{29}$$

Table 1: Some of values of TCE for $\mathbf{N}(1300, 250^2)$.

q	x_q	TCE (crisp)	TCE (fuzzy, $a = b = 1$)
0.0007	500	1300.6	1290
0.036	850	1320.5	1309.3
0.330	1190	1435.1	1435.2
0.421	1250	1468.8	1470.2
0.500	1300	1499.5	1499.5
0.788	1500	1641.8	1641.5
0.990	1890	1973.7	2000

Results as those in Examples 4.2 and 4.4 can be found for other elliptical and skew-elliptical distributions via fuzzy events. These results can be used by researcher in portfolio selection problem. That is how to determine an optimal portfolio which guarantees a maximal profit, the capital allocation and computing risk measure etc.

5. Conclusion

The class of skew-elliptical distributions provides a generalization of the multivariate skew normal distributions. These distributions can be obtained by several stochastic mechanisms. In this paper, we have proposed a fuzzy mechanism for producing the skew-elliptical distributions. The new method is also efficient for producing every skew distribution.

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