1. Introduction

All groups considered are finite. It is well known that the intersection of two subnormal subgroups of a group $G$ is a subnormal subgroup of $G$. As a result, the subnormal closure of a subgroup $H$ of a group $G$, i.e., the intersection of all subnormal subgroups of $G$ that contain $H$, is the unique smallest subnormal subgroup of $G$ containing $H$.

In [3], Bartels obtained the following description of the subnormal closure of a subgroup $H$ of a group $G$.

**Theorem.** The subnormal closure of a subgroup $H$ of a group $G$ is the subgroup

$$\langle H^g \mid g \in \langle H, H^g \rangle \rangle.$$ 

Let $\mathcal{F}$ be a saturated formation of finite groups. A maximal subgroup $M$ of a group $G$ is called $\mathcal{F}$-normal in $G$ if $G/\text{core}_G(M) \in \mathcal{F}$; otherwise, $M$ is called $\mathcal{F}$-abnormal in $G$. A subgroup $H$ of a group $G$ is called $\mathcal{F}$-subnormal in $G$ if $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ for some subgroups $\{H_i : 0 \leq i \leq n\}$ of $G$ such that $H_i$ is $\mathcal{F}$-normal maximal subgroup in $H_{i+1}$ for $0 \leq i < n$.

Feldman in [5] has obtained that the intersection of two $\mathcal{F}$-subnormal subgroups of a soluble group $G$ is an $\mathcal{F}$-subnormal subgroup of $G$, where $\mathcal{F}$ is a saturated formation of soluble groups, locally defined by $\{f(p)\}$, where for each prime $p$, $f(p)$ is a non-empty, subgroup-closed formation contained in $\mathcal{F}$.

As a result, if $H$ is a subgroup of a soluble group $G$, the $\mathcal{F}$-subnormal closure of $H$ in $G$; i.e., the intersection of all the $\mathcal{F}$-subnormal subgroups of $G$ that contain $H$, is the unique smallest $\mathcal{F}$-subnormal subgroup of $G$ containing $H$. 

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In [4], Doerk and Pérez-Ramos have obtained a criterion for $\mathcal{F}$-subnormality, where $\mathcal{F}$ is a formation under the same hypothesis as above.

K. Doerk has conjectured the following description for the $\mathcal{F}$-subnormal closure of a subgroup $H$ of a group $G$:

"Given a subgroup $H$ of a group $G$, the $\mathcal{F}$-subnormal closure of $H$ in $G$ is the subgroup $\langle g \in G \mid g \in H \langle H, H^x \rangle^\mathcal{F} \rangle".

In this paper we prove Doerk's conjecture for groups $G \in \mathcal{F}\mathcal{F}$, where $\mathcal{F}$ is the class of all soluble groups, and $\mathcal{F}$ is a subgroup-closed saturated formation of finite groups containing the formation of nilpotent groups. (Note that the formation $\mathcal{F}$ in [4, 5] is subgroup-closed and contains the formation of nilpotent groups.) Recall that $\mathcal{F}\mathcal{F}$ is the class of all groups $G$ with soluble $\mathcal{F}$-residual.

The first step to obtain it was to extend the main result in [4] to $\mathcal{F}\mathcal{F}$-groups.

Finally, we use the $\mathcal{F}$-subnormal closure to give a criterion of $\mathcal{F}$-pronormality for $\mathcal{F}\mathcal{F}$-groups which extends the Generalized Frattini Argument obtained in [5, Th. 2.10].

Both results are obtained by using some techniques which do not involve the arithmetical properties familiar in the soluble case.

Henceforth $\mathcal{F}$ will denote a subgroup-closed saturated formation of finite groups containing $\mathcal{F}^*$, the class of all nilpotent groups.

We begin with some preliminary results and definitions that are needed in the sequel:

**Lemma 1.** (P. Förster [6, Lemma 1.1]). If $H$ is $\mathcal{F}$-subnormal in $G$, and $H \leq U \leq G$, then $H$ is $\mathcal{F}$-subnormal in $U$.

**Definitions** (A. Ballester-Bolinches [2]). Let $U \leq G$. $U$ is $\mathcal{F}$-critical in $G$ if $U$ is $\mathcal{F}$-abnormal monolithic maximal subgroup of $G$ and $G = UF'(G)$, where $F'(G) = \text{Soc}(G \mod \Phi(G))$. A subgroup $D$ of $G$ is an $\mathcal{F}$-normalizer of $G$, if there exists a chain of subgroups:

$$D = H_n \leq H_{n-1} \leq \cdots \leq H_1 \leq H_0 = G$$

such that $H_i$ is an $\mathcal{F}$-critical subgroup of $H_{i-1}$ ($i = 1, \ldots, n$) and such that $H_n$ contains no $\mathcal{F}$-critical subgroup.

If $G \in \mathcal{F}$, we interpret the definition to mean $D = G$. The condition on $H_n$ is equivalent to $D \in \mathcal{F}$.

(For the existence of these subgroups see [2].)

For the sake of completeness we include the proof of the following lemma.
**Lemma 2.** (A. Ballester-Bolinches [1, Lemma II.2.5]). Let $G \in \mathcal{V}_F$, where $\mathcal{V}$ denotes the class of all nilpotent groups, and let $E$ be an $\mathcal{F}$-maximal subgroup of $G$ satisfying $G = EF(G)$. Then $E$ is an $\mathcal{F}$-normalizer of $G$.

**Proof.** We use induction on $|G|$. Clearly, we may assume $E < G$. Let $M$ be a maximal subgroup of $G$ such that $E \leq M$. Since $M = EF(M)$ and $E$ is $\mathcal{F}$-maximal in $M$, then $E$ is an $\mathcal{F}$-normalizer of $M$ by induction. Now, $M$ is $\mathcal{F}$-critical in $G$. Consequently, $E$ is an $\mathcal{F}$-normalizer of $G$.

**Lemma 3.** (A. Ballester-Bolinches [2]). Let $G$ be a group such that $G^p$ is abelian. Then $G^p$ has a unique conjugacy class of complements in $G$. Moreover, the complements of $G^p$ in $G$ are the $\mathcal{F}$-normalizers of $G$.

**2. A Criterion for $\mathcal{F}$-Subnormality**

**Theorem 1.** For a subgroup $H$ of a group $G \in \mathcal{F}$, the following conditions are pairwise equivalent:

1. $H$ is $\mathcal{F}$-subnormal in $G$.
2. $H$ is $\mathcal{F}$-subnormal in $\langle H, x \rangle$, for every $x \in G$.
3. $H$ is $\mathcal{F}$-subnormal in $\langle H, H' \rangle$, for every $x \in G$.
4. If $T$ is a subgroup of $G$ such that $T$ is contained in $\langle H, T \rangle^p$, then $I \leq H$.

5. From $x \in G$ and $x \in \langle H, x \rangle^p$, it follows that $x \in H$.
6. From $x \in G$ and $x \in \langle H, H' \rangle^p$, it follows that $x \in H$.

**Proof.** (3) implies (1). We argue by induction on $|G|$. We can assume $G^p \neq 1$. Let $N$ be a minimal normal subgroup of $G$ such that $N \leq G^p$. By induction, $HN/N$ is $\mathcal{F}$-subnormal in $G/N$. Moreover, if $HN$ is a proper subgroup of $G$, then $H$ is $\mathcal{F}$-subnormal in $HN$, by induction. Consequently, $H$ is $\mathcal{F}$-subnormal in $G$ and we are done. So, we can suppose $G = HN$ and $G \neq H$. Since $N$ is soluble, $H$ is a maximal subgroup of $G$. If $H$ is a normal subgroup of $G$, then $H$ is $\mathcal{F}$-subnormal in $G$. If $H$ is not a normal subgroup of $G$, then there exists an element $x \in G$, such that $H \neq H'^x$. Then, $G = \langle H, H'^x \rangle$. By (3), $H$ is $\mathcal{F}$-subnormal in $G$.

By Lemma 1.1, (1) implies (2) and (2) implies (3). Consequently, (1), (2), and (3) are pairwise equivalent.

It is clear that (4) implies (5) and (5) implies (6).

(1) implies (4). Suppose that $H$ is $\mathcal{F}$-subnormal in $G$ and $T$ is a subgroup of $G$ such that $T \leq \langle H, T \rangle^p$. Then, $\langle H, T \rangle = H \langle H, T \rangle^p$. If $H$ is a proper subgroup of $\langle H, T \rangle$, there exists an $\mathcal{F}$-normal maximal subgroup $M$ of $\langle H, T \rangle$ such that $H \leq M$. Since $\langle H, T \rangle^p \leq M$, we have that $M = \langle H, T \rangle$, a contradiction. Thus, $H = \langle H, T \rangle$ and $T \leq H$. 
(6) implies (1). We use induction on $|G|$. Let $x$ be an element of $G$, and denote $\langle H, H' \rangle$ by $T$. If $T$ is a proper subgroup of $G$, then by induction $H$ is $\mathcal{F}$-subnormal in $T$. Since (3) is equivalent to (1), we can assume that $T = G$, for some $x \in G$. On the other hand, if $HG^p < G$, then $H$ is $\mathcal{F}$-subnormal in $HG^p$ by induction. This implies that $H$ is $\mathcal{F}$-subnormal in $G$. Consequently, we can suppose $G = \langle H, H' \rangle = HG^p = H\langle H, H' \rangle^p$. In particular, $x = ht$ for some $h \in H$ and $t \in <H, H'>^p = \langle H, H' \rangle^p$. But by hypothesis $t \in H$ and of course $x \in H$. Thus, $G - \langle H, H' \rangle = H$ and $H$ is $\mathcal{F}$-subnormal in $G$.

Note that $H^p \not< G^p$, for any subgroup $H$ of $G$, because $\mathcal{F}$ is subgroup-closed. Consequently, if $H_1$ and $H_2$ are $\mathcal{F}$-subnormal subgroups of a group $G \in \mathcal{F}$, then $H_1 \cap H_2$ is also $\mathcal{F}$-subnormal in $G$. Thus, for a subgroup $H$ of a group $G \in \mathcal{F}$, the $\mathcal{F}$-subnormal closure of $H$ in $G$, that is the intersection of all the $\mathcal{F}$-subnormal subgroups of $G$ that contain $H$, denoted by $S_G(H, \mathcal{F})$, is the unique smallest $\mathcal{F}$-subnormal subgroup of $G$ containing $H$.

One might wonder if the set of $\mathcal{F}$-subnormal subgroups of a group $G \in \mathcal{F}$ forms a sublattice of the subgroup lattice of $G$. The answer is in general negative. For instance, let $\mathcal{F}$ be the saturated formation of 2-nilpotent groups and $G = \text{Sym}(4)$, the symmetric group of degree 4. $G$ has an irreducible and faithful module $V$ over $GF(3)$. Let $R = [V] G$ the corresponding semidirect product. If $P$ is a Sylow 2-subgroup of $G$, then $PV$ is an $\mathcal{F}$-normal maximal subgroup of $G$. Since $PV$ is an $\mathcal{F}$-group, we have that $P$ is an $\mathcal{F}$-subnormal subgroup of $G$. However, if we take $x \in G \setminus N_R(P)$, then $G = \langle P, P' \rangle$ is not an $\mathcal{F}$-subnormal subgroup of $R$.

### 3. The $\mathcal{F}$-Subnormal Closure

**Lemma 1.** Let $H \leq G$ and denote $T_G(H, \mathcal{F}) = \langle x \in G \mid x \in H\langle H, H' \rangle^p \rangle$. If $N \leq G$, then $T_{G/N}(HN/N, \mathcal{F}) = T_G(H, \mathcal{F}) N/N$.

**Proof.** Denote with bars the images in $G = G/N$. Clearly, $T_G(H, \mathcal{F}) \leq T_G(H, \mathcal{F})$.

Consider now $\bar{g} \in \langle \bar{H}, \bar{H}' \rangle^p$, and $t \in G$ such that $\bar{t} = \bar{g}$, in particular $\bar{t} \in \langle \bar{H}, \bar{H}' \rangle^p$, and $\langle \bar{H}, \bar{H}' \rangle^p$ has minimal order. If $x \in \langle \bar{H}, \bar{H}' \rangle^p$ such that $\bar{x} = \bar{t}$ then $\bar{x} = \bar{t} = \bar{g}$, and $\langle \bar{H}, \bar{H}' \rangle^p \leq \langle \bar{H}, \bar{H}' \rangle^p$. By our choice of $t$, we have that $\langle H, H' \rangle^p = \langle H, H' \rangle^p$, and the result follows.

**Theorem 2.** Let $H \leq G \in \mathcal{F}$. Then the $\mathcal{F}$-subnormal closure of $H$ in $G$, $S_G(H, \mathcal{F})$, is the subgroup $\langle x \in G \mid x \in H\langle H, H' \rangle^p \rangle$. Moreover, $S_G(H, \mathcal{F}) = \langle T \leq G \mid T \leq H\langle H, T \rangle^p \rangle$. 

Proof. Denote by $S = T_0(H, \mathcal{F}) = \langle x \in G \mid x \in H \langle H, H^x \rangle^{\mathcal{F}} \rangle$. Because of Theorem 2.1 and since $\mathcal{F}$ is subgroup-closed, if $H \triangleleft L \leq G$ and $L$ is $\mathcal{F}$-subnormal in $G$, then $S \leq L$. Thus the first statement holds, if we prove that $S$ is $\mathcal{F}$-subnormal in $G$.

$G^\mathcal{F} \leq G$, because $\langle H \rangle \leq \mathcal{F}$ and $G \leq \mathcal{F} \mathcal{F}$, and moreover we can assume that $G^\mathcal{F} \neq 1$. If we proceed by induction on $|G|$, we can also assume, using Lemma 3.1, that $SN$ is $\mathcal{F}$-subnormal in $G$ for a minimal normal subgroup $N$ of $G$ contained in $G^\mathcal{F}$. If $SN = L < G$, then $S = T_0(H, \mathcal{F})$ is $\mathcal{F}$-subnormal in $L$, and the result follows easily. Therefore we must have $SN = G$, and thus $S$ is maximal in $G$. Arguing as above, we also obtain $\text{core}_G(S) = 1$, and $G$ is a primitive group of type $I$.

Choose $H$ of minimal order among those subgroups of $G$ such that $T_0(H, \mathcal{F})$ is not $\mathcal{F}$-subnormal in $G$. Thus, if $M$ is a maximal subgroup of $H$ satisfying $H = T_H(M, \mathcal{F})$, we have that $H = T_H(M, \mathcal{F}) \leq T_0(M, \mathcal{F})$, and $T_0(M, \mathcal{F})$ is $\mathcal{F}$-subnormal in $G$. Consequently, $S = T_0(H, \mathcal{F}) \leq T_0(M, \mathcal{F}) \leq T_0(H, \mathcal{F})$, and $S$ is $\mathcal{F}$-subnormal in $G$, a contradiction. Therefore each maximal subgroup of $H$ is $\mathcal{F}$-subnormal in $H$. This implies that $H \leq \mathcal{F}$. Let $N_0$ be a minimal $H$-invariant subgroup of $N$. Clearly, we can assume $A = HN_0 < G$. If $HN_0$ is not an $\mathcal{F}$-group, then $N_0 = A^\mathcal{F}$. Therefore, $A = T_0(H, \mathcal{F}) = T_0(H, \mathcal{F}) \leq S$, which is not possible. Consequently, $HN_0$ is an $\mathcal{F}$-group. Moreover, if $\text{soc}_H(N)$ denotes the product of all minimal $H$-invariant subgroups of $N$, we deduce that $\text{Hsoc}_H(N) \leq \mathcal{F}$. We claim that $HN \leq \mathcal{F}$. Let $L$ be an $\mathcal{F}$-maximal subgroup of $HN$ containing $H \text{soc}_H(N)$. Because of Lemmas 1.2 and 1.3, it follows that $HN = L(HN)^\mathcal{F}$ and $(HN)^\mathcal{F} \cap L = 1$, since $(HN)^\mathcal{F} < N$. But then, if $(HN)^\mathcal{F} \neq 1$, we have that $\{f(q)\}_q \leq (HN)^\mathcal{F} \cap \text{soc}_H(N) \leq (HN)^\mathcal{F} \cap L$, which is a contradiction. Therefore, if $1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r = N$ is an $H$-composition series of $N$, and $\{f(q)\}_q$ is the local integrated and full definition of $\mathcal{F}$, then $H/C_H(N_i/N_{i-1}) \in f(p)$, for $i = 1, \ldots, r$, and $p$ the prime divisor of $|N|$. This implies that $H^{(p)} \leq \{C_H(N_i/N_{i-1})\}_i = f(p)$, and so that $H^{(p)} = C_H(N_i/N_{i-1}) = H^{(p)}$, which is a $p$-group, because $H^{(p)}$ stabilizes a series of $N$. Thus, we have $H \in \mathcal{F}_{p'} f(p) = f(p)$.

Consider now $g \in \langle H, H^x \rangle \mathcal{F}$, but $g \notin H$. It is clear that $H < \langle H, H^x \rangle = T \notin \mathcal{F}$. Obvious, $T = HT^x \triangleleft H$. Denote $T^x = R$. Let $1 = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_s = N$ be a $T$-composition series of $N$. We claim that there exists a $T$-chief factor $K_i/K_{i-1}$, with $i \in \{1, \ldots, s\}$, which is not centralized by $R$. Otherwise, arguing as before, we could deduce that $R$ is a $p$-group, and since $H \in f(p)$, it would follow that $H^{(p)} \in \mathcal{F}_{p'} f(p)$ and hence that $T \in f(p) \leq \mathcal{F}$, a contradiction.

Let $L = K_1$, denote with bars the images in $\bar{L} = L/K_{i-1}$. Now we have that $\bar{K}_i \leq \bar{L}^\mathcal{F}$, because otherwise $\bar{K}_i \cap \bar{L}^\mathcal{F} = 1$, and then $\bar{R} \leq C_{\bar{F}}(\bar{K}_i)$, a contradiction. Therefore $\bar{L}^\mathcal{F} = \bar{R}$. Assume that $|\bar{L}| < |G|$. Then $S_{\bar{F}}(\bar{H}, \mathcal{F}) = T_{\bar{F}}(\bar{H}, \mathcal{F}) = \bar{L}$. Because of Lemma 3.1, we have that
$T_f(\tilde{H}, \tilde{F}) = T_f(H, \tilde{F}) K_i^{-1}/K_i^1$, and so $T_f(H, \tilde{F}) K_i^{-1} = K_i RH$. If $T_f(H, \tilde{F}) \cap K_i = 1$, then $T_f(H, \tilde{F}) = RH$ and $K_i = K_i^{-1}$, a contradiction. But then $1 \neq T_f(H, \tilde{F}) \cap K_i \leq S \cap N$, which is also impossible. Therefore we must have $|\tilde{L}| = |G|$, that is $N = K_i$, $G = NT$ and $S = T = \langle H, g \rangle$.

Take $n \in N$ such that $[H, n] \neq 1$, and consider $M = \langle H, H^n \rangle$. Since it must be that $G = HG^\varphi$, we have that $M < G$, because otherwise $n \in S$, and so $n \in S$, a contradiction. Let $L$ be a maximal subgroup of $G$ containing $M$. If $L - S$, then $H^n < S$, but this is impossible because then we would deduce $1 \neq [h, n] = h^{-1}h^n \in S \cap N$, for some $h \in H$. If core$_G(L) = 1$, then $L = H(L \cap G^\varphi) = HL^\varphi$, and by our choice of $G$, this implies that $L = T_f(H, \tilde{F}) \leq S$, a contradiction. Thus $N \leq L$. But then $H^n \leq L$, and so $S = \langle H, H^n \rangle \leq L$, which provides the final contradiction.

On the other hand, it is clear that $S_G(H, \tilde{F}) \leq L_G(H, \tilde{F}) = \langle T \leq G \mid T \leq H, T \rangle \varphi$. Now, if $K$ is an $\varphi$-subnormal subgroup of $G$ containing $H$ and $T$ is a generator of $L_G(H, \tilde{F})$, we have that $T \leq K \langle K, T \rangle \varphi$. Thus, if $t \in T$, then $t = k_i x_i$ with $k_i \in K$, $x_i \in \langle K, T \rangle \varphi$. Denote by $R = \langle x_i \mid t \in T \rangle$. It is clear that $\langle K, T \rangle = \langle K, R \rangle$ and $R \leq \langle K, R \rangle \varphi$. Since $K$ is $\tilde{F}$-subnormal in $G$, $R \leq K$. Consequently, $T \leq K$ and $L_G(H, \tilde{F}) \leq K$. Since $S_G(H, \tilde{F})$ is an $\tilde{F}$-subnormal subgroup of $G$ containing $H$, we have that $S_G(H, \tilde{F}) = L_G(H, \tilde{F})$ and the theorem is proved.

Note that 2.1 and 3.2 are not true in general. It is enough to take for example $\varphi = \tilde{F}$, the formation of all nilpotent groups and $G = \text{Alt}(5)$, the alternating group of degree 5. The trivial subgroup $H = \{1\}$ satisfies the property: “if $g \in \langle H, H^\varphi \rangle \varphi$, then $g \in H$,” but $H$ is not $\tilde{F}$-subnormal in $G$. Even if $G^\varphi < G$ and $H \neq \{1\}$, they do not hold. We could take $\tilde{F} = \hat{1}$ as above, $G = GL(2, 5)$ and $H = Z(GL(2, 5))$.

4. A CRITERION FOR $\tilde{F}$-PRONORMALITY

Müller in [7], generalized the property of pronormality of a subgroup of a soluble group to $\tilde{F}$-pronormality, where $\tilde{F}$ is a saturated formation with a subgroup-closed integrated local definition $\{f(q)\}_{q \in \tilde{F}}$, and he obtained the following characterization:

“A subgroup $H$ of a soluble group $G$ is $\tilde{F}$-pronormal in $G$ if and only if for every $g \in G$ there exists an element $x \in \langle H, H^\varphi \rangle \varphi$ such that $H^x = H^\varphi$.”

This motivates the following definition for an arbitrary group and an arbitrary saturated formation $\tilde{F}$:

**Definition 1.** If $H \leq G$, then $H$ is called $\tilde{F}$-pronormal in $G$, denoted by $H\tilde{F}$-pr $G$, if for every $g \in G$ there exists an element $x \in \langle H, H^\varphi \rangle \varphi$ such that $H^x = H^\varphi$. 
Now it is easy to prove:

**Lemma 2.** (i) If $H \mathcal{F}\text{-pr} G$ and $N$ is a normal subgroup of $G$, then $HN/N \mathcal{F}\text{-pr} G/N$.

(ii) (Gashütz's pronormality criterion). Let $H \trianglelefteq G$ and $N \trianglelefteq G$. If $HN/N \mathcal{F}\text{-pr} G/N$ and $H \mathcal{F}\text{-pr} N_G(HN)$, then $H \mathcal{F}\text{-pr} G$.

In the following theorem, $\mathcal{F}$ denotes again a subgroup-closed saturated formation containing $\mathcal{F}$, the class of all nilpotent groups.

**Theorem 3.** For a subgroup $H$ of a group $G \in \mathcal{F}$, the following conditions are equivalent:

(i) $H \mathcal{F}\text{-pr} G$;

(ii) If $H \leq L \leq G$, then $L = S_L(H, \mathcal{F}) N_L(H)$.

**Proof.** Assume first that $H \mathcal{F}\text{-pr} G$ and $H \trianglelefteq L \leq G$. If $g \in L$, then there exists an element $x \in \langle H, H^x \rangle^\mathcal{F}$ such that $H^x = H^x$. Thus $x \in S_L(H, \mathcal{F})$ and $gx^{-1} \in N_L(H)$. Therefore $g \in S_L(H, \mathcal{F}) N_L(H)$.

Conversely, suppose that (ii) does not imply (i), and let $G$ be a group of minimal order with a subgroup $H$ satisfying (ii) but not $\mathcal{F}$-pronormal in $G$. Then it is clear that $G$ is not an $\mathcal{F}$-group. Now let $N$ be a minimal normal subgroup of $G$ contained in $GF$. Then, because of our choice of $G$, it is $HN/N \mathcal{F}\text{-pr} G/N$. Now let $T = N_G(HN)$. If $T < G$, it follows again that $H \mathcal{F}\text{-pr} T$. By Lemma 4.2, we have that $H$ is $\mathcal{F}$-pr $G$, a contradiction. Then, we can assume that $HN$ is a normal subgroup of $G$. Consequently, $S_G(H, \mathcal{F}) \leq HN$ and $G = N_G(H) N$.

If $HN < G$, then $H$ is $\mathcal{F}$-pr $HN$. Take $g \in G$. Then $H^x = H^n$, with $n \in N$. Since $H \mathcal{F}\text{-pr} HN$, there exists $x \in \langle H, H^x \rangle^\mathcal{F} = \langle H, H^x \rangle^\mathcal{F}$ such that $H^x = H^n$. Thus, $H \mathcal{F}\text{-pr} G$, a contradiction. Therefore we must have $HN = G$. So $H = G$ or $H$ is a maximal subgroup of $G$. In both cases $H \mathcal{F}\text{-pr} G$, which provides the final contradiction.

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