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Advances in Mathematics 189 (2004) 413–438

ADVANCES IN Mathematics

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Universal $\mathcal{R}$-matrix for quantum affine algebras $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1\,|\,2))$ with Drinfeld comultiplication

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Received 22 April 2003; accepted 26 November 2003

Communicated by P. Etingof

Abstract

We derive an integral formula for the universal R-matrix for the twisted quantum affine algebra $U_q(A_2^{(2)})$ and quantum affine superalgebra $U_q(\widehat{osp}(1\,|\,2))$ with Drinfeld comultiplication.

MSC: 17B37; 81R50; 17B65

Keywords: Quantum affine algebra; Loop realization; Quantum Serre relation; Universal R-matrix

1. Introduction

In the works [DK,DKP] we described the universal R-matrix for quantum nontwisted affine algebras with the so-called Drinfeld comultiplication. It was presented up to a standard factor as a series of contour integrals of certain canonical tensor over the system of factorizable cycles in deformed configuration spaces. The geometric properties of the deformed configuration spaces, which first appeared in functional realization of Borel subalgebras of quantum affine algebra [FO,E], are crucial for this presentation.

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2003.11.013

Using these properties, we constructed first the system of factorizable cycles for the algebra $U_q(\widehat{sl}_2)$ and then extended the construction to other nontwisted quantum affine algebras with a help of 'current' braid group action constructed in [DK].

An analogous picture should take place for other types of quantum affine superalgebras and we develop here our approach for twisted quantum affine algebra $U_q(A_2^{(2)})$ and its superpartner $U_q(\widehat{osp}(1\,|\,2))$. The algebra $U_q(A_2^{(2)})$ plays the same fundamental role for twisted quantum affine algebras as the algebra $U_q(\widehat{sl}_2)$ for nontwisted algebras; its matrix elements and the deformed configuration space should be studied separately. Then the obtained results can be used as a foundation for further study of other twisted quantum affine algebras; such an extension of the theory requires the construction of 'current' group action for all twisted quantum affine algebras, which we plan to describe in a future publication.

The analytical properties of current operators of the algebra $U_q(\widehat{sl}_2)$ from one side and of the algebras $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1\,|\,2))$ from the other side, differ in a crucial way: the poles and zeroes of the current operators of the last two algebras together with vanishing 'Serre conditions' form a complicated structure of the deformed configuration space. Also the role of the long root current, which is generated by one of the two poles of the basic current operators, is quite delicate. All this makes the new cases far more complicated with respect to that of $U_q(\widehat{sl}_2)$.

As a first step towards our goals, we need a complete functional description of the Borel parts of the algebras $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1\,|\,2))$, or, equivalently, the description of the matrix elements of the products of the current operators in highest weight representations. Such a description is presented in Section 3. In particular, we derive a complete functional version of the Serre relations analogous to [E]. The proofs are given in the appendix using a new identity of delta functions unknown before. Our exposition goes in the unified ways for both algebras, and the two cases differ essentially by a sign of the parameter q_θ.

This gives us a possibility to introduce the corresponding configuration spaces and to construct the universal R-matrix as a series of integrals of canonical tensor over the systems of factorizable cycles according to the abstract theory from [DKP]. We find the desired systems of cycles explicitly, check directly the factorization properties and prove the formula. The final answer, given in Theorem 2, appears to be surprisingly simple at the first glance, at least the integration is taken over the product of unit circles. However, even the verification, that the integration form is nonsingular on the integration cycle is quite subtle and is based on the vanishing properties of the integration form, coming from the Serre relations.

At this moment, we still do not fully understand the new integral formula presented in this paper and its implications. For instance, the role played by the current operators for the long root is essentially unclear, and we still do not know how to derive a reasonable differential equation or recurrence relations for the integrals, as it was done in [DKP] for $U_q(\widehat{sl}_2)$.

2. Definitions

In this section we describe quantum affine algebras $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1\,|\,2))$ in current realization. It means, in particular, that we use their natural completions, acting on highest weight representations. See [DKP, Section 2] for details. We propose here the unified description these two algebras, analogous to [KLT], such that the difference between two algebras will be only in the parity sign $(-1)^\theta$.

The algebras $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1\,|\,2))$ are $\mathbb{Z}_2$-graded Hopf algebras, which means that in the tensor square of the algebra the multiplication rule is defined for the homogeneous elements a, b, c, d by

$$(a\otimes b)(c\otimes d)=(-1)^{\theta(b)\theta(c)}(ac\otimes bd),$$

where $\theta(x)\in\mathbb{Z}_2$ denotes the grading of the element x.

Define $\theta=1$ for $U_q(\widehat{osp}(1\,|\,2))$ and $\theta=0$ for $U_q(A_2^{(2)})$ and denote these two algebras as $\mathscr{A}_\theta$, so $\mathscr{A}_0$ is $U_q(A_2^{(2)})$ and $\mathscr{A}_1$ is $U_q(\widehat{osp}(1\,|\,2))$.

The algebra $\mathscr{A}_\theta$ is generated by the elements

$$x_n^\pm,\quad n\in\mathbb{Z},\quad a_n,\ n\neq 0,\quad d,\ k^{\pm1}\quad\text{and}\quad\text{central}\quad q^{\pm c}$$

with the parity

$$\theta(x_n^\pm)=\theta,\quad \theta(a_n)=\theta(k^{\pm1})=\theta(q^{\pm c})=\theta(d)=0.$$

These elements are gathered into generating functions

$$x^\pm(z)=\sum_{k\in\mathbb{Z}}x_k^\pm z^{-k},\quad K^\pm(z)=k^{\pm1}\exp\left(\pm(q-q^{-1})\sum_{n>0}a_{\pm n}z^{\mp n}\right),$$

which satisfy the following relations on the level of formal power series:

$$q^d a(z)q^{-d}=a(qz),\quad\text{for}\quad a=x^\pm,K^\pm,\tag{1}$$

$$(z-q^{\pm2}w)(z+q_\theta^{\mp1}w)x^\pm(z)x^\pm(w)=(q^{\pm2}z-w)(q_\theta^{\mp1}z+w)x^\pm(w)x^\pm(z),\tag{2}$$

$$K^+(z)x^\pm(w)K^+(z)^{-1}=g(q^{\mp\frac{1}{2}c}w/z)^{\mp1}x^\pm(w),\tag{3}$$

$$K^-(z)x^\pm(w)K^-(z)^{-1}=g(z/wq^{\mp\frac{1}{2}c})^{\pm1}x^\pm(w).\tag{4}$$

$$K^\pm(z)K^\pm(w)=K^\pm(w)K^\pm(z),\tag{5}$$

$$K^-(z)K^+(w)K^-(z)^{-1}K^+(w)^{-1}=g(q^{-c}z/w)g(q^cz/w)^1,\tag{6}$$

$$x^{+}(z)x^{-}(w)-(-1)^{\theta}x^{-}(w)x^{+}(z)$$
$$=\frac{1}{q-q^{-1}}\left(\delta\left(\frac{z}{w}q^{-c}\right)K^{+}(wq^{\frac{1}{2}c})-\delta\left(\frac{z}{w}q^{c}\right)K^{-}(zq^{\frac{1}{2}c})\right). \tag{7}$$

Here $q_\theta=(-1)^\theta q$ and

$$g_\theta(z)=(-1)^\theta\,\frac{(zq^2-1)(zq_\theta^{-1}+1)}{(z-q^2)(z+q_\theta^{-1})} \tag{8}$$

is treated as a formal power series over z.

The currents $x^{\pm}(z)$ satisfy also the following cubic Serre relations [CP,D]:

$$Sym_{z_1,z_2,z_3}(q_\theta^{-3}z_1^{\pm 1}-(q^{-2}+q_\theta^{-1})z_2^{\pm 1}+z_3^{\pm 1})x^{\pm}(z_1)x^{\pm}(z_2)x^{\pm}(z_3)=0, \tag{9}$$

$$Sym_{z_1,z_2,z_3}(q_\theta^{3}z_1^{\mp 1}-(q^{2}+q_\theta)z_2^{\mp 1}+z_3^{\mp 1})x^{\pm}(z_1)x^{\pm}(z_2)x^{\pm}(z_3)=0. \tag{10}$$

The coalgebra structure of $\mathscr{A}_\theta$, which we investigate here, is given by the relations:

$$\Delta(q^c)=q^c\otimes q^c, \tag{11}$$

$$\Delta(x^{+}(z))=x^{+}(z)\otimes 1+K^{-}(zq^{\frac{c_1}{2}})\otimes x^{+}(zq^{c_1}), \tag{12}$$

$$\Delta(x^{-}(z))=1\otimes x^{-}(z)+x^{-}(zq^{c_2})\otimes K^{+}(zq^{\frac{c_2}{2}}), \tag{13}$$

$$\Delta(K^{-}(z))=K^{-}(zq^{-\frac{c_2}{2}})\otimes K^{-}(zq^{\frac{c_1}{2}}), \tag{14}$$

$$\Delta(K^{+}(z))=K^{+}(zq^{\frac{c_2}{2}})\otimes K^{+}(zq^{-\frac{c_1}{2}}), \tag{15}$$

where $c_1=c\otimes 1$ and $c_2=1\otimes c$.

3. Properties of correlation functions

Let V be a highest weight representation in the standard sense of the algebra $\mathscr{A}_\theta$, $v\in V$, $\xi\in V^*$. Analogously to the case of quantized nontwisted affine Lie algebra, we claim that the matrix coefficient

$$\langle \xi, x^{+}(z_1)\cdots x^{+}(z_m)v\rangle \tag{16}$$

and

$$\langle \xi, x^{-}(z_1)\cdots x^{-}(z_m)v\rangle \tag{17}$$

belong, as formal power series, to the space

$$\mathbb{C}[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}]\left[\left[\frac{z_2}{z_1}, \frac{z_3}{z_2}, \dots, \frac{z_m}{z_{m-1}}\right]\right], \tag{18}$$

that is, can be presented as Taylor series over the variables $z_2/z_1, \dots, z_{m-1}/z_m$ with coefficients being polynomials over $z_1, z_1^{-1}, \dots, z_m, z_m^{-1}$. These formal power series converge in the region $|z_1| \gg |z_2| \gg \cdots \gg |z_m|$ to analytical functions, which can be analytically continued to meromorphic functions having simple poles at hyperplanes

$$z_i = q^2 z_j, \quad \text{and} \quad z_i = -q_\theta^{-1} z_j, \quad i<j \tag{19}$$

for the matrix coefficients (16) and

$$z_i = q^{-2} z_j, \quad \text{and} \quad z_i = -q_\theta z_j, \quad i<j \tag{20}$$

for the matrix coefficients (17). Put

$$\begin{aligned} &P^{\pm}_{\xi,v}(z_1, \dots, z_m) \\ &\quad = \prod_{i<j} \left((z_i - q^{\pm 2} z_j)(z_i + q_\theta^{\mp 1} z_j)\right) \langle \xi, x^{\pm}(z_1) \cdots x^{\pm}(z_m) v \rangle . \end{aligned} \tag{21}$$

We claim, that the commutation relations (2) and Serre relations (9) and (10) imply the following properties of the correlation functions, generalizing analogous properties of the correlation functions of nontwisted quantum affine algebras [E].

Theorem 1. *The Laurent polynomials $P^{\pm}_{\xi,v}(z_1 \dots z_m)$ vanish on the diagonals $z_i = z_j$, $i \neq j$ and on all codimension two planes*

$$\{z_i = -q_\theta z_j\} \bigcap \{z_j = -q_\theta z_k\}, \quad i \neq j, \ j \neq k, \ i \neq k. \tag{22}$$

The proof of Theorem 1, based on certain delta-functions identities, is given in Appendix A.

The vanishing conditions on correlation functions, described in the Theorem 1 could be also derived from the nondegeneracy of the Hopf pairing between two Borel subalgebras, attached to the Hopf structure (11)–(15). This is shown in Appendix B.

In the next section we work with the currents

$$t(z) = (q^{-1} - q) x^-(z) \otimes x^+(z),$$

$$t^{(1)}(z) = x^-(z) \otimes K^+(z) \otimes x^+(z)$$

and

$$t^{(2)}(z) = 1 \otimes x^-(z) \otimes x^+(z).$$

Property (18) implies that for any two highest weight representations V and U and the vectors $v\in V\otimes U$, $\xi\in V^*\otimes U^*$ the matrix coefficients

$$\langle \xi, t(z_1)\cdots t(z_m)v\rangle \tag{23}$$

belong to space (18), as well as the matrix coefficients

$$\langle \xi, t^{(i_1)}(z_1)\cdots t^{(i_m)}(z_m)v\rangle, \tag{24}$$

where v and ξ are vectors and covectors in tensor product of three highest weight representations of $\mathscr{A}_\theta$.

Again, matrix coefficients (24) (and (23), as their particular case) converge in the region $|z_1|\gg|z_2|\gg\cdots\gg|z_m|$ to analytical functions. These functions admit meromorphic analytical continuations, which have simple poles at hyperplanes

$$z_i=q^2z_j,\quad \text{and}\quad z_i=-q_\theta^{-1}z_j,\quad i\neq j. \tag{25}$$

For any matrix coefficient (24) let us define Laurent polynomial $P_{\xi,v,i_1,\dots,i_m}(z_1,\dots,z_m)$ as follows:

$$\begin{aligned}&P_{\xi,v,i_1,\dots,i_m}(z_1,\dots,z_m)\\&\quad=\prod_{i\neq j}\left((z_i-q^2z_j)(z_i+q_\theta^{-1}z_j)\right)\langle \xi, t^{(i_1)}(z_1)\cdots t^{(i_m)}(z_m)\rangle.\end{aligned} \tag{26}$$

We have the following corollary of Theorem 1:

Proposition 1. *The Laurent polynomials $P^{\pm}_{\xi,v,i_1,\dots,i_m}(z_1\dots z_m)$ vanish on the diagonals $z_i=z_j$, $i\neq j$ and have zero of second order on all codimension two planes*

$$\{z_i=-q_\theta z_j\}\bigcap\{z_j=-q_\theta z_k\},\quad i\neq j,\ j\neq k,\ i\neq k.$$

The proof of Proposition 1 consists of application of Theorem 1 and relations (3) and (4) to the matrix coefficients (24).

4. The main result: the universal R-matrix

In this section we present an explicit expression for the universal R-matrix for the algebra $\mathscr{A}_\theta$. The answer will be given as a product of two factors: the first is certain canonical infinite product over generators a_n, while the second is given as a series of contour integrals analogous to [DK,DKP].

Let us remind that the universal R-matrix R for a quasitriangular Hopf algebra $\mathscr{A}$ is characterized by the properties

$$\Delta^{\mathrm{op}}(a)=R\Delta(a)R^{-1} \tag{27}$$

for any $a \in \mathscr{A}$ and

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \quad (\mathrm{id} \otimes \Delta)R = R_{13}R_{12}. \tag{28}$$

The universal R-matrix for quantum twisted affine algebra $U_q(A_2^{(2)})$ with standard comultiplication, defined as

$$\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_i^{-1} \otimes e_{\alpha_i}, \quad \Delta(e_{-\alpha_i}) = 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_i$$

on Chevalley generators, was presented in [KT1] in a form of infinite product over the roots of $A_2^{(2)}$. The limiting twisting procedure, developed in [KT2], gives the presentation of the universal R-matrix for coproduct (12)–(15) as

$$\mathscr{R} = \mathscr{K}\overline{\mathscr{R}}, \tag{29}$$

where

$$\begin{aligned}\mathscr{K} &= q^{-\frac{h \otimes h}{2}} q^{\frac{-c \otimes d - d \otimes c}{2}} \\ &\quad \times \exp\left(\sum_{n>0} \frac{-n(q-q^{-1})^2}{(q_\theta^n - q_\theta^{-n})(q_\theta^n + q_\theta^{-n} + (-1)^n)} a_n \otimes a_{-n}\right) q^{\frac{-c \otimes d - d \otimes c}{2}},\end{aligned} \tag{30}$$

and $\overline{\mathscr{R}}$ is an ordered product of q-exponents over all real roots of $A_2^{(2)}$.

The main result of this paper is a presentation of the factor $\overline{\mathscr{R}}$ as a series of contour integrals.

Let us introduce the new current $s(z)$ as

$$s(z) = \operatorname*{Res}_{z_1 = -q_\theta z} t(z_1) t(z) \frac{dz_1}{z_1}. \tag{31}$$

Theorem 2. *Let $|q| > 1$. Then for any $n > 0$ the contour integral*

$$\oint_{|z_1| = \cdots = |z_n| = 1} \cdots \oint (t(z_1) + s(z_1)) \cdots (t(z_n) + s(z_n)) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \tag{32}$$

is well defined and the factor $\overline{\mathscr{R}}$ of the universal R-matrix can be presented as a series

$$\begin{aligned}\overline{\mathscr{R}} &= 1 \\ &\quad + \sum_{n>0} \frac{1}{n!(2\pi i)^n} \oint_{|z_1| = \cdots = |z_n| = 1} \cdots \oint (t(z_1) + s(z_1)) \cdots (t(z_n) + s(z_n)) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}.\end{aligned} \tag{33}$$

Due to the delicate structure of the singularities of the integrand series (33) is very far from being an exponent. Also we do not have a direct way of comparing this series with its expression in Cartan–Weyl generators in [KT1].

The classical r-matrix, to which the R-matrix (33) degenerates when q tends to 1, can be described as follows. The Lie algebra $A_2^{(2)}$ can be identified with the central extension of the algebra of sl_3 valued functions $X(t)$ on $\mathbb{C}^*$, satisfying the condition $X^\tau(t) = X(-t)$, where τ is the automorphism of gl_3 defined as $E_{ij}^\tau = (-1)^{i+j}E_{\bar{j},\bar{i}}$ on matrix units E_{ij}. Here $\bar{k} = 4-k$.

In this description the Lie algebra $A_2^{(2)}$ is generated by central element c, grading element d, Cartan element $a_0 = E_{11} - E_{33}$, imaginary root vectors $a_{2n} = (E_{11} - E_{33}) \otimes t^{2n}$, $n \neq 0$ and $a_{2n+1} = (E_{11} - 2E_{22} + E_{33}) \otimes t^{2n+1}$; and real root vectors $x_n^+ = (E_{12} + (-1)^n E_{23}) \otimes t^n$, $x_n^- = (E_{21} + (-1)^n E_{32}) \otimes t^n$, $s_{2n+1}^+ = E_{13} \otimes t^{2n+1}$, $s_{2n+1}^- = E_{31} \otimes t^{2n+1}$.

We collect the real root vectors into generating functions $x^\pm(z) = \sum_{n\in\mathbb{Z}} x_n^\pm z^{-n}$ and $s^\pm(z) = \sum_{n \in ZZ} s_{2n+1}^\pm z^{-2n-1}$. In these notations the r matrix, corresponding to (33) looks as follows:

$$
\begin{aligned}
r = {} & \frac{1}{2}(a_0 \otimes a_0 + c \otimes d + d \otimes c) + \sum_{n>0}\left(a_{2n} \otimes a_{-2n} + \frac{1}{3} a_{2n-1} \otimes a_{-2n+1}\right) \\
& + \oint \frac{dz}{z}(x^-(z) \otimes x^+(z) + s^-(z) \otimes s^+(z)).
\end{aligned}
$$

5. Proof of the Theorem 2

5.1. A reformulation

We rewrite first the statement of Theorem 1 in terms of multiple integrals over the current $t(z)$ only. In order to do this, we introduce a family of integration cycles.

Fix an integer n and denote by U_n the complement in $\mathbb{C}^n$ to the union of hyperplanes

$$
\begin{aligned}
& H'_{k,l} = \{z_k = -q_\theta z_l\}, \quad H''_{k,l} = \{z_k = q^2 z_l\}, \quad && 1 \leqslant k \neq l \leqslant n, \\
& H_i = \{z_i = 0\}, && 1 \leqslant i \leqslant n.
\end{aligned}
$$

Let k be an integer such that $2k \leqslant n$ and $\{i_1, j_1, i_2, j_2, \ldots, i_k, j_k, l_1, \ldots, l_{n-2k}\}$ be a permutation of the set $\{1, 2, \ldots, n\}$. Denote by $T_{\{i_1,j_1\},\{i_2,j_2\},\ldots,\{i_k,j_k\},l_1,\ldots,l_{n-2k}}$ the following torus in $\mathbb{C}^n$:

$$
\begin{aligned}
T_{\{i_1,j_1\},\ldots,\{i_k,j_k\},l_1,\ldots,l_{n-2k}} = \{ & z_{i_1} \circlearrowleft -q_\theta z_{j_1}, z_{i_2} \circlearrowleft -q_\theta z_{j_2}, \ldots, \\
& z_{i_k} \circlearrowleft -q_\theta z_{j_k}, |z_{j_1}| = |z_{j_2}| = \cdots = |z_{j_k}| = |z_{l_1}| = \cdots = |z_{l_{n-2k}}| = 1\}, \qquad (34)
\end{aligned}
$$

that is,

$$T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}} = \{z_{i_1} = -q_\theta z_{j_1} + \varepsilon_1 e^{i\phi_{i_1}},\ z_{j_1} = e^{i\phi_{j_1}}, \dots,$$
$$z_{i_k} = -q_\theta z_{j_k} + \varepsilon_k e^{i\phi_{i_k}},\ z_{j_k} = e^{i\phi_{j_k}},$$
$$z_{l_1} = e^{i\phi_{l_1}}, \dots, z_{l_{n-2k}} = e^{i\phi_{l_{n-2k}}}\}, \tag{35}$$

where $\varepsilon_1, \dots, \varepsilon_k$ are small positive real numbers, $0 \leqslant \phi_j < 2\pi$ and the orientation of the torus $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ is given by natural order of the coordinates ϕ_j, that is, by the top form $d\phi_1 \wedge d\phi_2 \wedge \dots \wedge d\phi_n$. The notation for the cycles $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ is compatible with the action of the symmetric group S_n: if we keep for a permutation $\sigma \in S_n$ the same notation σ for the corresponding diffeomorphism of $\mathbb{C}^n$: $\sigma(z_i) = z_{\sigma(i)}$, then

$$T_{\{\sigma(i_1),\sigma(j_1)\},\dots,\{\sigma(i_k),\sigma(j_k)\},\sigma(l_1),\dots,\sigma(l_{n-2k})}$$
$$= (-1)^{\operatorname{sgn}\sigma} \sigma T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}} \tag{36}$$

in $H_n(U_n)$. Moreover, for any permutations $\sigma' \in S_k$ and $\sigma'' \in S_{n-2k}$ the homology class in $H_n(U_n)$ of the cycles

$$T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}} \text{ and } T_{\{i_{\sigma'(1)},j_{\sigma'(1)}\},\dots,\{i_{\sigma'(k)},j_{\sigma'(k)}\},l_{\sigma''(1)},\dots,l_{\sigma''(n-2k)}}$$

coincide. Thus we have $\frac{n!}{k!(n-2k)!}$ different cycles $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ in $H_n(U_n)$. Denote by $D_{k,n}$ their total normalized sum:

$$D_{k,n} = \frac{k!(n-2k)!}{n!} \sum_{\sigma \in S_n} T_{\{\sigma(1),\sigma(2)\},\dots,\{\sigma(2k-1),\sigma(2k)\},\sigma(2k+1),\dots,\sigma(n)}, \tag{37}$$

and D_n be the sum of all $D_{k,n}$ over k:

$$D_n = \sum_{k:0 \leqslant 2k \leqslant n} D_{k,n}. \tag{38}$$

The nontrivial statement which we would like to prove further becomes as following

Theorem 3. *Let $|q| > 1$. Then the factor $\overline{\mathscr{R}}$ of the universal R-matrix can be presented as a following series of correctly defined contour integrals*:

$$\overline{\mathscr{R}} = 1 + \sum_{n>0} \frac{1}{n!} \oint_{D_n} t(z_1)t(z_2)\cdots t(z_n) \frac{dz_1}{2\pi i z_1} \wedge \dots \wedge \frac{dz_n}{2\pi i z_n}. \tag{39}$$

The main Theorem 2 follows from Theorem 3 due to the description of the contours (37), (38) and the symmetry of the integrands in (39).

5.2. Factorizable cycles

The proof of Theorem 3 is strongly based on the results of [DKP]. In that paper we reformulated the properties of the universal R-matrix of the quantum affine algebra according to the properties of the integration cycles of the factor $\overline{\mathcal{R}}$. The reformulation is valid for quantum twisted affine algebras and superalgebras as well.

In the case of $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1\,|\,2))$ it looks as follows. Let $I=\{k_1,\dots,k_n\}$, $n=|I|$ be an ordered finite set of integers. Let X_I be the following stratified space. As a total space, X_I is isomorphic to $\mathbb{C}^n$ with coordinates $z_k, k\in I$. The closures of the strata are given by the intersections of hyperplanes

$$H'_{k,l}=\{z_k=-q_\theta z_l\},\quad H''_{k,l}=\{z_k=q^2z_l\},\quad k,l\in I,\ k\neq l,\quad H_i=\{z_i=0\}, i\in I.$$

By U_I we denote an open stratum: the complement to the union of hyperplanes. A holomorphic top form $\omega\in\Omega^{|I|}(U_I)$ is called admissible, $\omega\in\Omega_I$, if it has a form

$$\omega=\frac{P(z_{k_1},\dots,z_{k_n})}{\prod_{l\neq m}\big((z_{k_l}+q_\theta z_{k_m})(z_{k_l}-q^2z_{k_m})\big)}\frac{dz_{k_1}}{z_{k_1}}\wedge\frac{dz_{k_2}}{z_{k_2}}\wedge\dots\wedge\frac{dz_{k_n}}{z_{k_n}},\tag{40}$$

where $P(z_{k_1},\dots,z_{k_n})$ is a Laurent polynomial over z_{k_i}, satisfying the vanishing conditions (41), (42):

$$P(z_{k_1},\dots,z_{k_n})=0\quad\text{if}\quad z_{k_i}=z_{k_j},\tag{41}$$

and $P(z_{k_1},\dots,z_{k_n})$ has zero of the second order on any Serre stratum

$$\{z_{k_i}=-q_\theta z_{k_j}\}\bigcap\{z_{k_j}=-q_\theta z_{k_l}\},\quad i\neq j,\ j\neq l,\ i\neq l.\tag{42}$$

Introduce also the subspace

$$\Omega_{I^1,I^2}\subset\Omega_I$$

for any decomposition of $I=I^1\coprod I^2$ into disjoint union of its ordered subsets I^1 and I^2 (the order in I is as follows: first we count the elements of I^1 in their given order, then the elements of I^2 in their order): an admissible form ω belongs to Ω_{I^1,I^2}, if it has no singularities at hyperplanes $z_k=-q_\theta z_l$ and $z_k=q^{-2}z_l$ for any $k\in I^1$ and $l\in I^2$.

Suppose that for any I we have chosen an antisymmetric cycle $D_I\in H_n(U_I)$, where $n=|I|$. The symmetricity condition means that for any bijection $\sigma: I\to\sigma(I)$ of ordered sets (which induces the diffeomorphism σ: $X_I\to X_{\sigma(I)}$) and for any admissible form $\omega\in\Omega_{\sigma(I)}$ there is an equality

$$\oint_{D_{\sigma(I)}}\omega=\oint_{D_I}\sigma^*(\omega).\tag{43}$$

The system $\{D_I\}$ of antisymmetric cycles is called *factorizable* if for any ordered finite set I and for any its ordered decomposition $I = I^1 \coprod I^2$ the following equality holds for any admissible form $\omega \in \Omega_{I^1,I^2}$:

$$\oint_{D_I} \omega = \oint_{D_{I^1} \bowtie D_{I^2}} \omega, \tag{44}$$

where $D_{I^1} \bowtie D_{I^2}$ means the set-theoretical product $\eta D_{I^1} \times D_{I^2}$ of dilated cycle ηD_{I^1} and D_{I^2} with a sufficiently small $\eta > 0$.

The translation of the Theorem 1 from [DKP] to the case of $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1\mid 2))$ says

Proposition 2. *For any factorizable system of antisymmetric cycles $\{D_I\}$ with initial condition $D_k = \{|z_k| = 1\}$ the tensor*

$$\mathscr{K}\overline{\mathscr{R}},$$

where

$$\overline{\mathscr{R}} = 1 + \sum_{n>0} \frac{1}{n!} \oint_{D_{\{1,2,\dots,n\}}} t(z_1)t(z_2)\cdots t(z_n)\frac{dz_1}{2\pi i z_1}\cdots\frac{dz_n}{2\pi i z_n} \tag{45}$$

satisfy properties (27) *and* (28) *of the universal R-matrix.*

Indeed, repeating the arguments of [DKP], we can prove, that if the factorization property (44) holds for any form $\omega = g(z)dz_1 \wedge \cdots \wedge dz_n$, where $g(z)$ is an arbitrary matrix coefficient (24), then the factor $\overline{\mathscr{R}}$ of the universal R-matrix coincides with the canonical integral (45) over the corresponding system $\mathscr{D}$ of symmetric cycles. Due to Proposition 1 any such form is admissible and it is enough to check the factorization property for all admissible forms.

5.3. *The cycles $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ and their deformations*

Before the proof of Theorem 3 we would like to verify that the integrals of admissible forms over the cycles $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ are correctly defined and do not depend of certain perturbations of the cycles. In particular, this implies the first statement of the Theorem 2: the integrand in (32) has no singularity at the integration contour $|z_1| = \cdots = |z_n| = 1$.

We attach to any cycle $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ the following diagram to illustrate our arguments:

In Fig. 1 the vertical position represents the absolute value of the variables, so the points on the same horizontal line have the same absolute value, and the higher the vertical position is, the bigger absolute value the point has.

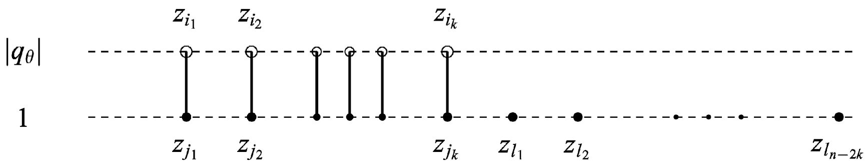

Fig. 1. The diagram of $|z_k|$ for the cycle $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$.

The cycle $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ is a product of one- and two-dimensional tori. Each thick point on the lower horizontal line represents the circle $|z_j| = 1$. Fig. 2 represents the two-dimensional torus $|z_{j_k}| = 1$, $|z_{i_k} + q_\theta z_{j_k}| = \varepsilon$, where ε is a small positive number. The vertical line, connecting the point z_{i_k} and z_{j_k} on the diagrams shows that an admissible form ω has a singularity along the hyperplane $z_{i_k} = -q_\theta z_{j_k}$ and this singularity is enclosed in the integration contour.

This implies that the only singularities of the admissible form, which can cross the contour $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$, are $z_{i_a} = -q_\theta z_{l_b}$, $a = 1,\dots,k$, $b = 1,\dots,n-2k$, or $z_{i_a} = -q_\theta z_{j_b}$, $a = 1,\dots,k$, $b = 1,\dots,k$, $a \neq b$. By dimensional reasons we can deform the torus $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ such that it does not intersect the hyperplane $h(z) = 0$ of singularity of the form, but the resulting integral will not depend on the deformation, if the corresponding residue at the hyperplane $h = 0$ vanishes.

Consider first the case, when the contour crosses the singularity $z_{i_a} = -q_\theta z_{l_b}$. Since the torus $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$ decomposes into a product of one- and two-dimensional tori, it is sufficient to prove the vanishing of the double residue

$$\operatorname*{Res}_{z_{i_a}=-q_\theta z_{l_b}} \ \operatorname*{Res}_{z_{i_a}=-q_\theta z_{j_a}} \omega = 0. \tag{46}$$

The double residue (46) is equal to zero since any admissible form ω vanishes on the diagonal $z_{l_b} = z_{j_a}$, that is, the form ω can be written as

$$\omega = \frac{z_{l_b} - z_{j_a}}{(z_{i_a} + q_\theta z_{j_a})(z_{i_a} + q_\theta z_{l_b})}\,\omega_0,$$

where ω_0 does not have the singularities on the hyperplanes $z_{i_a} = -q_\theta z_{j_a}$ and $z_{i_a} = -q_\theta z_{l_b}$.

This case is depicted in the diagram of Fig. 3. Again, the horizontal position denotes the absolute values of coordinates, thick lines denote the possible simple poles, which are $z_{i_a} = -q_\theta z_{l_b}$ and $z_{i_a} = -q_\theta z_{j_a}$, and the crossed line denotes a zero $z_{l_b} = z_{j_a}$. The diagram demonstrates that finally we have only a first order pole, thus the second order residue is zero.

In the second case, there are two hyperplanes, $z_{i_a} = -q_\theta z_{j_b}$ and $z_{i_b} = -q_\theta z_{j_a}$, where the integrand could have poles. So, we have to show that the integral over the torus, which can be obtained from torus (34) by a replacing of the condition $|z_{j_b}| = 1$ with $z_{j_b} \circlearrowleft -q_\theta^{-1} z_{j_a}$, that is $|z_{j_b} + q_\theta^{-1} z_{j_a}| = \varepsilon$, vanishes. Again, for this it is sufficient to

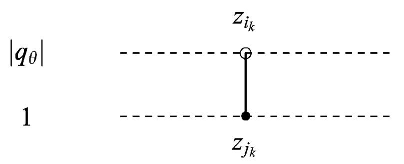

Fig. 2. The diagram of two-dimensional cycle $|z_{j_k}| = 1$, $|z_{i_k} + q_\theta z_{j_k}| = \varepsilon$.

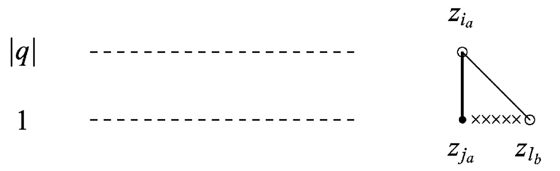

Fig. 3. The vanishing of the double residue $\operatorname*{Res}_{z_{i_a}=-q_\theta z_{l_b}} \operatorname*{Res}_{z_{i_a}=-q_\theta z_{j_a}} \omega$.

show that the triple residue

$$\operatorname*{Res}_{z_{i_a}=-q_\theta z_{j_a}} \operatorname*{Res}_{z_{i_b}=-q_\theta z_{j_b}} \operatorname*{Res}_{z_{i_a}=-q_\theta z_{j_b}} \omega = 0 \tag{47}$$

vanishes. This is true, since ω has now zero of second order on the intersection of hyperplanes, defining the triple residue, while the order of the pole on this intersection is four, that is,

$$\omega = \frac{(z_{i_a} - z_{i_b})(z_{j_a} - z_{j_b})}{(z_{i_a} + q_\theta z_{j_a})(z_{i_a} + q_\theta z_{j_b})(z_{i_b} + q_\theta z_{j_a})(z_{i_b} + q_\theta z_{j_b})}\,\omega_0,$$

where ω_0 has no singularities at the hyperplanes $z_{i_a} + q_\theta z_{j_a} = z_{i_a} + q_\theta z_{j_b} = z_{i_b} + q_\theta z_{j_a} = z_{i_b} + q_\theta z_{j_b} = 0$ and thus the triple residue of ω vanishes. This is depicted at Fig. 4.

We have checked that the integral of admissible form over cycle (34) is well defined. We would like now to go further, namely we show below that there are some nontrivial deformations of torus (34) which do not change the value of the integral of admissible form.

Remind that torus (34) decomposes naturally into direct product of circles $|z_{l_a}| = 1$, or two-dimensional tori $z_{i_a} \circlearrowleft -q_\theta z_{j_a},\ |z_{j_a}| = 1$. For a one-dimensional torus $|z_a| = \lambda$ or for a two-dimensional torus $z_a \circlearrowleft -q_\theta z_a,\ |z_{j_a}| = \lambda$ we call λ to be the basic radius of the torus. So, all basic radia of the one-dimensional and two-dimensional tori, composing cycle (34) are close to 1. We can dilate independently all these tori, changing their basic radii. Here we would like to make the following useful remark about such deformations of torus (34).

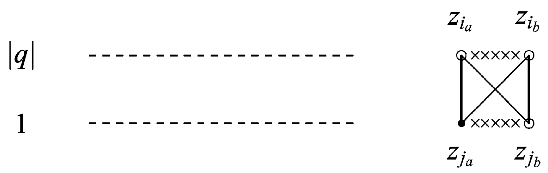

Fig. 4. The vanishing of the triple residue $\underset{z_{i_a}=-q_0z_{j_a}}{\mathrm{Res}}\ \underset{z_{i_b}=-q_0z_{j_b}}{\mathrm{Res}}\ \underset{z_{i_a}=-q_0z_{j_b}}{\mathrm{Res}}\ \omega.$

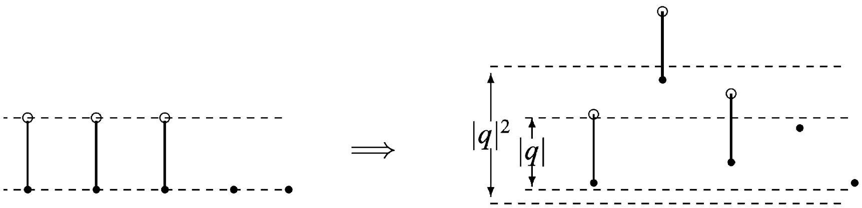

Fig. 5. Admissible deformations of $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$.

Lemma 1. *The independent dilations of one- and two- dimensional tori of torus* (34) *do not change the value of the integral of admissible form, if the dilated tori satisfy the following conditions*: (i) *the ratio of basic radii of any two of elementary one-dimensional tori* $|z_{l_a}| = r_a$ *is less then* $|q|$; (ii) *the ratio of basic radii of two-dimensional tori* $z_{i_a} \circlearrowleft -q_\theta z_{j_a}$, $|z_{j_a}| = r_a$, *as well as the ratio between the radii of one and two-dimensional tori* (*in any order*) *is less then* $|q|^2$.

The possible deformations of cycle (34) are depicted at Fig. 5. During such moves the cycle can cross the following singularities:

$$\begin{aligned}
&\text{(a)}\ \ z_{j_a} = -q_\theta z_{l_b}\ \text{ or }\ z_{i_a} = q^2 z_{l_b};\\
&\text{(b)}\ \ z_{l_b} = -q_\theta z_{i_a}\ \text{ or }\ z_{l_b} = q^2 z_{j_a};\\
&\text{(c)}\ \ z_{i_a} = -q_\theta z_{i_b}\quad \text{or}\quad z_{j_a} = -q_\theta z_{j_b}\ \text{ or }\quad z_{i_a} = q^2 z_{j_b},
\end{aligned} \tag{48}$$

see Fig. 6.

For example, case (a) occurs for the dilation from the product

$$\{|z_{j_a}| = 1,\ |z_{i_a} + q_\theta z_{j_a}| = \varepsilon\} \times \{|z_{l_b}| = 1\}$$

to

$$\{|z_{j_a}| = \lambda_1,\ |z_{i_a} + q_\theta z_{j_a}| = \varepsilon\} \times \{|z_{l_b}| = \lambda_2\},$$

where $\lambda_1/\lambda_2 > |q_\theta|$.

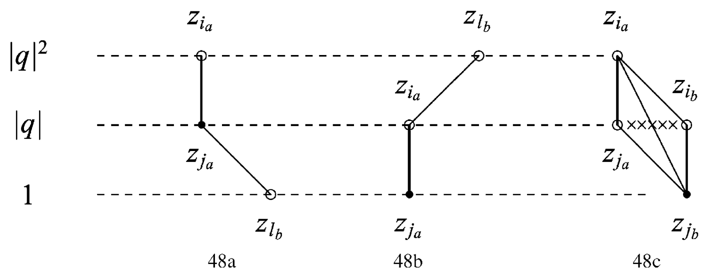

Fig. 6. Vanishing residues in admissible moves of $T_{\{i_1,j_1\},\dots,\{i_k,j_k\},l_1,\dots,l_{n-2k}}$.

In cases (a) and (b) the double residues vanish, namely

$$\operatorname*{Res}_{z_{l_b}=-q_\theta^{-1}z_{j_a}} \operatorname*{Res}_{z_{i_a}=-q_\theta z_{j_a}} \omega = 0 \quad \text{and} \quad \operatorname*{Res}_{z_{l_b}=-q_\theta z_{i_a}} \operatorname*{Res}_{z_{i_a}=-q_\theta z_{j_a}} \omega = 0 \tag{49}$$

because of the Serre condition on codimension two planes

$$\{z_{l_b} = -q_\theta^{-1} z_{j_a}\} \bigcap \{z_{i_a} = -q_\theta z_{j_a}\} \quad \text{and} \quad \{z_{l_b} = -q_\theta z_{i_a}\} \bigcap \{z_{i_a} = -q_\theta z_{j_a}\}.$$

The Serre condition gives additional second order zero (see Proposition 1) on the above intersections which is more than enough for the vanishing properties (49). In case (c), the third order residue vanishes, namely

$$\operatorname*{Res}_{z_{j_a}=-q_\theta z_{j_b}} \operatorname*{Res}_{z_{i_b}=-q_\theta z_{j_b}} \operatorname*{Res}_{z_{i_a}=-q_\theta z_{j_a}} \omega = 0 \tag{50}$$

because of the following. Any admissible form ω could have the pole of order at most 5 at the codimension three plane

$$\{z_{j_a} = -q_\theta z_{j_b}\} \bigcap \{z_{i_a} = -q_\theta z_{j_a}\} \bigcap \{z_{i_b} = -q_\theta z_{j_b}\}.$$

These poles come from the factors $z_{i_a} + q_\theta z_{j_a}$, $z_{i_a} + q_\theta z_{i_b}$, $z_{i_b} + q_\theta z_{j_b}$, $z_{j_a} + q_\theta z_{j_b}$ and $z_{i_a} - q^2 z_{j_a}$ in the denominator of ω, but the numerator of ω is divisible by $z_{j_a} - z_{i_b}$ and has additional second order zero due to the Serre condition on codimension two plane $\{z_{j_a} = -q_\theta z_{j_b}\} \bigcap \{z_{i_a} = -q_\theta z_{j_a}\}$. Thus the total order of the pole is at most two and the triple residue vanishes.

5.4. The factorizability of D_n

Due to Proposition 2 it is sufficient to prove the factorization property of the cycles D_n. The cycles D_n are clearly antisymmetric with respect to the action of symmetric group on the configuration space; this imply the antisymmetricity condition of the Section 5.2. Thus we have to prove an equality

$$\oint_{D_{n+m}} \omega = \oint_{D_n \bowtie D_m} \omega \tag{51}$$

for any admissible $\omega \in \Omega_{I',I''}$, where $I' = \{1, \dots, n\}$ and $I'' = \{n+1, \dots, n+m\}$, so ω has no poles at hyperplanes

$$z_{k'} = -q_\theta z_{k''} \quad \text{and} \quad z_{k'} = q^{-2} z_{k''} \tag{52}$$

for any $k' \in I'$, that is $1 \leqslant k' \leqslant n$ and $k'' \in I''$, that is $n+1 \leqslant k'' \leqslant n+m$.

Let us fix such a form ω. Consider the right-hand side of relation (51). This is a sum of the integrals over the tori, in which the absolute values of the first n coordinates are much smaller then of the last m. Let us dilate the first n coordinates simultaneously, making them bigger with a final goal to make their absolute value being equal 1. The relative move of one specific torus from D_n with respect to a torus from D_m is depicted at Fig. 7. The indices of the first n variables are equipped with one prime', the indices of the last m variables are equiped with two primes''.

Consider such a move of the torus $T_{\{i'_1,j'_1\},\dots,\{i'_{k'},j'_{k'}\},l'_1,\dots,l'_{n-2k'}}$ relative to the torus $T_{\{i''_1,j''_1\},\dots,\{i''_{k''},j''_{k''}\},l''_1,\dots,l''_{n-2k''}}$. During the move , we can meet only the singular planes $z_{a'} = -q_\theta^{-1} z_{b''}$, where $1 \leqslant a' \leqslant n$ and $n+1 \leqslant b'' \leqslant n+m$ (note that $|q|>1$). First we meet the hyperplanes $z_{i'_a} = q_\theta^{-1} z_{j''_b}$, crossing two-dimensional tori $z_{i'_a} \circlearrowleft q_\theta z_{j'_a}$ and $z_{i''_b} \circlearrowleft q_\theta z_{j''_b}$, see Fig. 8. In this case the ratio of basic radii is equal to $|q|^2$ (that is, the 'level' η is equal to $|q|^{-2}$).

Let us prove that the triple residue

$$\operatorname*{Res}_{z_{i'_a} = -q_\theta^{-1} z_{j''_b}} \operatorname*{Res}_{z_{i''_b} = -q_\theta z_{j''_b}} \operatorname*{Res}_{z_{i'_a} = -q_\theta z_{j'_a}} \omega = 0 \tag{53}$$

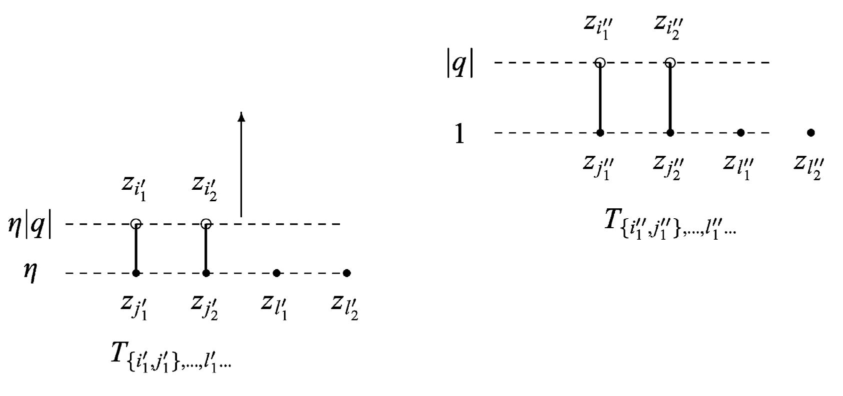

Fig. 7. Relative move of the cycles.

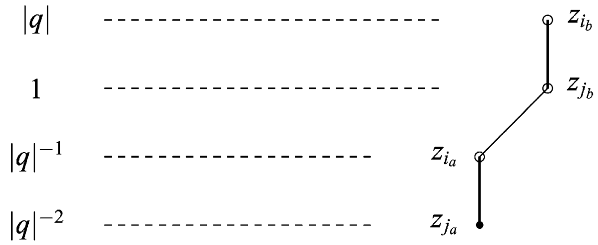

Fig. 8. Vanishing residue at the level $\eta = |q|^{-2}$.

at the codimension three plane

$$\{z_{i'_a} = -q_\theta^{-1} z_{j''_b}\} \bigcap \{z_{i'_a} = -q_\theta z_{j'_a}\} \bigcap \{z_{i''_b} = -q_\theta z_{j''_b}\} \tag{54}$$

vanishes for any admissible form ω satisfying zero conditions (52). From (52) we conclude, that the form ω can have the pole of order three at the plane (54) , arising from the factors $z_{i'_a} + q_\theta^{-1} z_{j''_b}$, $z_{i'_a} + q_\theta z_{j'_a}$ and $z_{i''_b} + q_\theta z_{j''_b}$ in its denominator. But the Serre condition on the codimension two plane

$$\{z_{i'_a} = -q_\theta^{-1} z_{j''_b}\} \bigcap \{z_{i'_a} = -q_\theta z_{j'_a}\} \tag{55}$$

says that the form ω should have one additional zero on the plane (55) in addition to the first order zero, prescribed by (52), which cancels original factor $z_{j''_b} - q^2 z_{j'_a}$ in the denominator of ω. So, ω has the pole of order two at codimension three plane (54) and thus any triple residue at this plane vanishes.

So we cross the level $\eta = |q|^{-2}$ without any change. Next, we meet the singularities at the level $\eta = |q|^{-1}$. The hyperplane of singularity can cross:

(a) two two-dimensional tori $\{z_{i'_a} \circlearrowleft - q_\theta z_{j'_a},\ |z_{j'_a}| = q_\theta\}$ and $\{z_{i''_b} \circlearrowleft - q_\theta z_{j''_b},\ |z_{j''_b}| = 1\}$;
(b) one two-dimensional torus from D_n and a circle from D_m: $\{z_{i'_a} \circlearrowleft - q_\theta z_{j'_a},\ |z_{j'_a}| = q_\theta\}$ and $|z_{l''_b}| = 1$;
(c) one two-dimensional torus from D_m and a circle from D_n: $|z_{l'_a}| = 1$ and $\{z_{i''_b} \circlearrowleft - q_\theta z_{j''_b}, |z_{j'_a}| = q_\theta\}$;
(d) two circles $|z_{l'_a}| = q$ and $|z_{l''_b}| = 1$.

We have shown already in Lemma 1 that the multiple residues appearing in the first three cases vanish for any admissible form ω even without vanishing conditions (52). These are respectively cases (a)–(c) of (48). The nonzero residues appear only in case (d). This shows that we can make under the integral of ω the basic radii of all two-dimensional tori $\{z_{i'_a} \circlearrowleft -q_\theta z_{j'_a},\ |z_{j'_a}| = q_\theta\}$ to be equal one. In other words, we can replace the integral over $T = \eta T_{\{i'_1,j'_1\},\dots,\{i'_{k'},j'_{k'}\},l'_1,\dots,l'_{n-2k'}} \times T_{\{i''_1,j''_1\},\dots,\{i''_{k''},j''_{k''}\},l''_1,\dots,l''_{n-2k''}}$ by the integral over $T = T' \times T''$, where $T' = \eta T_{l'_1,\dots,l'_{n-2k'}}$, $T'' = T_{\{i'_1,j'_1\},\dots,\{i'_{k'},j'_{k'}\},\dots,\{i''_{k''},j''_{k''}\},l''_1,\dots,l''_{n-2k''}}$, and $0<\eta<|q|^{-1}$. What is left, is to move the circles $|z_{l'_a}| = v$ to the positions $|z_{l'_a}| = 1$. Perturb first slightly all the radii of one-dimensional circles. Then during the move we meet one by one the planes of type (d)

$$z_{l'_a} = -q_\theta^{-1} z_{l''_b},$$

which adds the residue at $z_{l'_a} = -q_\theta^{-1} z_{l''_b}$ with a negative sign, see Fig. 9.

More precisely, we get instead of $T = T' \times T''$ the cycle $\tilde{T} = \tilde{T}' \times T''$, where the coordinate $z_{l'_a}$ in $\tilde{T}'$ runs in a unit circle, that is,

$$\tilde{T} = \eta T_{l'_1,\dots,l'_a,\dots,l'_{n-2k'}} \times T_{\{i'_1,j'_1\},\dots,\{i'_{k'},j'_{k'}\},\dots,\{i''_{k''},j''_{k''}\},l'_a,l''_1,\dots,l''_{n-2k''}},$$

minus the cycle $\tilde{\tilde{T}}$, which differs from T by

$$z_{l'_a} = -q_\theta^{-1} z_{l''_b} + \varepsilon_{l''_b} e^{i\phi_{l'_a}}, \quad z_{l''_b} = e^{i\phi_{l''_b}} \tag{56}$$

instead of $z_{l'_a} = \eta e^{i\phi_{l'_a}}$ and $z_{l''_b} = e^{i\phi_{l''_b}}$. This is depicted in Fig. 10.

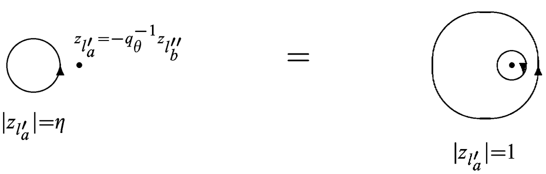

Fig. 9. The sign of the residue $\mathrm{Res}_{z_{l'_a}=-q_\theta^{-1}z_{l''_b}}\,\omega$.

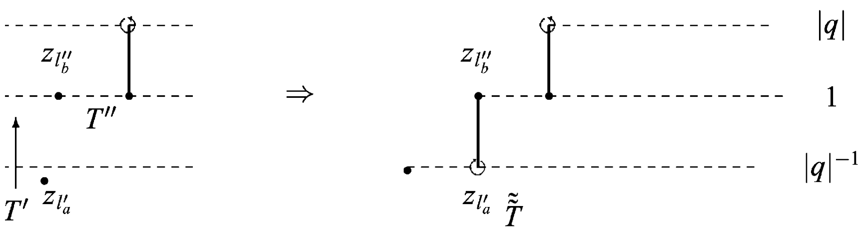

Fig. 10. The appearance of a new cycle.

The two-dimensional torus (56) is homotopic to

$$z_{l'_a} = |q|^{-1} e^{i\phi_{l''_b}}, \quad z_{l''_b} = -q_\theta z_{l'_a} + \varepsilon_{l''_b} e^{i\phi_{l'_a}}. \tag{57}$$

Again, by Lemma 1, we can change the basic radius of this torus from $|q|^{-1}$ to 1 and finally we get, taking in mind the change of the order of local parameters in (57), that

$$\check{T} \approx -\eta T_{l'_1,\ldots,l'_a,\ldots,l'_{n-2k'}} \times T_{\{l''_b,l'_a\},\{i'_1,j'_1\},\ldots,\{i'_{k'},j'_{k'}\},\ldots,\{i''_{k''},j''_{k''}\},l''_1,\ldots,l''_b,\ldots,l''_{n-2k''}}$$

and $T \approx \tilde{T} + \check{T}$. In other words, we get one more cycle from D_{n+m} which contains one two-dimensional torus

$$z_{l''_b} = -q_\theta z_{l'_a}, \quad |z_{l'_a}| = 1, \tag{58}$$

where the coordinate $z_{l'_a}$ is from the first group in (44), while $z_{l''_b}$ is from the second.

Continuing the move, we get all the tori $T_{\{i_1,j_1\},\ldots,\{i_k,j_k\},l_1,\ldots,l_{n+m-2k}}$ where for any two-dimensional torus $z_{i_a} = -q_\theta z_{j_a}$, $|z_{j_a}| = 1$ either both coordinates are from the same group I' or I'', or the coordinate z_{j_a} is from the first group I' in (44), while z_{i_a} is from I''. But we can also freely add under the integral of ω , satisfying (52) any torus $T_{\{i_1,j_1\},\ldots,\{i_k,j_k\},l_1,\ldots,l_{n+m-2k}}$, which contains a two-dimensional torus $z_{i_a} = -q_\theta z_{j_a}$, $|z_{j_a}| = 1$, for which the coordinate z_{i_a} is from the first group I' in (44), and z_{j_a} is from the second, I'', since the integral of ω over such cycle is zero due to (52), and so get all possible tori $T_{\{i_1,j_1\},\ldots,\{i_k,j_k\},l_1,\ldots,l_{n+m-2k}}$.

This proves equality (51) and the Theorem 3.

Acknowledgments

This work was partially supported by the Taft Memorial Fund of the University of Cincinnati.

The work of J.D. was also supported by University Research Council of the University of Cincinnati and NSA and the work of S.Kh. was supported by grants RFBR 02-01-00668, INTAS OPEN 00-00055 and by the grant for the support of scientific schools, n. 00-15-96557. The authors also thank the referee for useful remarks and comments.

Appendix A. Serre relations and the properties of the correlation functions for $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1\mid 2))$

This appendix contains the proof of Theorem 1, that is a deduction of the properties of the correlation functions listed in the Theorem 1 from relations (2), (9) and (10). The technique is the same as in Enriquez paper [E].

Consider relation (9) for the current $x^+(z)$:

$$\mathrm{Sym}_{z_1,z_2,z_3}(q_\theta^{-3}z_1 - (q_\theta^{-2}+q_\theta^{-1})z_2 + z_3)x^+(z_1)x^+(z_2)x^+(z_3) = 0. \tag{A.1}$$

Relations (2) imply that the correlation function of the product

$$\prod_{1\leqslant i<j\leqslant 3}(z_i - q_\theta^2 z_j)(z_i + q_\theta^{-1}z_j)x^+(z_1)x^+(z_2)x^+(z_3)$$

is an antisymmetric Laurent polynomial and for any $v\in V$, $\xi\in V^*$, where V is a highest weight representation of $U_q(A_2^{(2)})$ or $U_q(\widehat{osp}(1\,|\,2))$ the formal power series

$$\begin{aligned}&E_{\xi,v}(z_1,z_2,z_3)\\ &\quad = <\xi, \prod_{1\leqslant i<j\leqslant 3}\frac{(z_i - q_\theta^2 z_j)(z_i + q_\theta^{-1}z_j)}{z_i - z_j}x^+(z_1)x^+(z_2)x^+(z_3)v>,\end{aligned} \tag{A.2}$$

defined originally in a region $|z_1|\gg|z_2|\gg|z_3|$ is a symmetric Laurent polynomial. Denote by $F(z_1,z_2,z_3)$ the following power series in the region $|z_1|\gg|z_2|\gg|z_3|$

$$\begin{aligned}&F(z_1,z_2,z_3)\\ &\quad = (q_\theta^{-3}z_1 - (q_\theta^{-2}+q_\theta^{-1})z_2 + z_3)\prod_{1\leqslant i<j\leqslant 3}\frac{z_i - z_j}{(z_i - q_\theta^2 z_j)(z_i + q_\theta^{-1}z_j)}.\end{aligned} \tag{A.3}$$

Then the Serre relation (A.1) can be written as

$$\mathrm{Sym}_{z_1,z_2,z_2}F(z_1,z_2,z_3)E_{\xi,v}(z_1,z_2,z_3) = 0,$$

or, due to the symmetricity of $E_{\xi,v}(z_1,z_2,z_3)$,

$$\mathrm{Sym}_{z_1,z_2,z_2}F(z_1,z_2,z_3)\cdot \mathrm{Sym}_{z_1,z_2,z_2}E_{\xi,v}(z_1,z_2,z_3) = 0. \tag{A.4}$$

Then the Theorem 1 follows from the following

Lemma A.1. *The following equality of the formal power series takes place*:

$$\begin{aligned}&\mathrm{Sym}_{z_1,z_2,z_2}F(z_1,z_2,z_3)\\ &\quad = \frac{q_\theta^{-1}}{1-q_\theta+q_\theta^2}\mathrm{Sym}_{z_1,z_2,z_2}\delta(z_3 - q_\theta^2 z_1)\delta(z_2 + q_\theta^{-1}z_3).\end{aligned} \tag{A.5}$$

Here $\delta(x-y) = \sum_{n\in\mathbf{Z}}x^n y^{-n-1}$.

The longest technical step of the proof of the lemma is the decomposition of the rational function $F(z_1,z_2,z_3)$ into a sum of rational functions having only two poles. In order to perform such a decomposition, we consider first $F(z_1,z_2,z_3)$ as a function over z_2 depending on the parameters z_1 and z_3 and decompose it into a sum of elementary (over z_2) fractions:

$$F(z_1, z_2, z_3) = \frac{A_1(z_1, z_3)}{z_1 - q_\theta^2 z_2} + \frac{B_1(z_1, z_3)}{z_1 + q_\theta^{-1} z_2} + \frac{A_3(z_1, z_3)}{z_2 - q_\theta^2 z_3} + \frac{B_3(z_1, z_3)}{z_2 + q_\theta^{-1} z_3}. \tag{A.6}$$

The answer is

$$\begin{aligned} F(z_1, z_2, z_3) = & - \frac{q_\theta^{-2} - 1}{(q_\theta^3 + 1)(z_1 - q_\theta^2 z_2)} \cdot \frac{z_1 - z_3}{(q_\theta^{-1} z_1 + z_3)(z_1 + q_\theta^{-1} z_3)} \\ & + \frac{q_\theta^{-3}(q_\theta^2 - 1)}{(q_\theta^3 + 1)(z_2 - q_\theta^2 z_3)} \cdot \frac{z_1 - z_3}{(q_\theta^{-1} z_1 + z_3)(z_1 + q_\theta^{-1} z_3)} \\ & + \frac{1 + q_\theta}{(1 + q_\theta^3)(z_1 + q_\theta^{-1} z_2)} \cdot \frac{(z_3 + (1 + q_\theta^{-1} + q_\theta^{-3}) z_1)(z_1 - z_3)}{(-q_\theta^2 z_1 + z_3)(q_\theta^{-1} z_1 + z_3)(z_1 - q_\theta^2 z_3)} \\ & - \frac{q_\theta^{-2}(1 + q_\theta)}{(1 + q_\theta^3)(z_2 + q_\theta^{-1} z_3)} \cdot \frac{(z_1 + (1 + q_\theta + q_\theta^3) z_3)(z_1 - z_3)}{(-q_\theta^2 z_1 + z_3)(q_\theta^{-1} z_1 + z_3)(z_1 - q_\theta^2 z_3)}. \end{aligned} \tag{A.7}$$

The crucial point here that in this decomposition no new poles appear. Now, to get the desired decomposition, it is sufficient to decompose the coefficients $A_2(z_1, z_3)$, $B_2(z_1, z_3)$, $A_3(z_1, z_3)$ and $B_3(z_1, z_3)$ into the sum of elementary (over z_1, e.g.) fractions. We get finally

$$\begin{aligned} F(z_1, z_2, z_3) = & \frac{q_\theta^{-1}}{1 - q_\theta + q_\theta^2} \left(\frac{q_\theta^{-1}}{(z_1 + q_\theta^{-1} z_2)(-q_\theta^2 z_1 + z_3)} \right. \\ & - \frac{q_\theta^{-2}}{(z_1 + q_\theta^{-1} z_2)(q_\theta^{-1} z_1 + z_3)} - \frac{1}{(z_1 + q_\theta^{-1} z_2)(z_1 - q_\theta^2 z_3)} - \frac{1}{(z_2 + q_\theta^{-1} z_3)(-q_\theta^2 z_1 + z_3)} \\ & - \frac{q_\theta^{-2}}{(z_2 + q_\theta^{-1} z_3)(q_\theta^{-1} z_1 + z_3)} + \frac{q_\theta^{-1}}{(z_2 + q_\theta^{-1} z_3)(z_1 - q_\theta^2 z_3)} - \frac{1}{(z_1 - q_\theta^2 z_2)(q_\theta^{-1} z_1 + z_3)} \\ & + \frac{1}{(z_1 - q_\theta^2 z_2)(z_1 + q_\theta^{-1} z_3)} + \frac{q_\theta^{-1}}{(z_2 - q_\theta^2 z_3)(q_\theta^{-1} z_1 + z_3)} \\ & \left. - \frac{q_\theta^{-1}}{(z_2 - q_\theta^2 z_3)(z_1 + q_\theta^{-1} z_3)} \right). \end{aligned} \tag{A.8}$$

Here we treat the r.h.s. of (A.8) as a formal power series in the region $|z_1| \gg |z_2| \gg |z_3|$. We also follow the convention

$$\frac{1}{x - y} = \sum_{n \geqslant 0} y^n x^{-n-1}, \tag{A.9}$$

where the order of the arguments in the denominator indicate the region of the decomposition into power series.

The role of the 10 summands in the r.h.s. of (A.8) is different. The 6 of them (numbers 1,2,4,5,7,9) have singularities on the Serre strata while the singularities of the rest (numbers 3,6,8,10) are not of the Serre type.

It is not difficult to observe why after the symmetrization the last terms vanish. For instance, the third term in r.h.s. of (A.8)

$$-\frac{1}{(z_1+q_\theta^{-1}z_2)(z_1-q_\theta^2 z_3)},$$

which is defined in the region $|z_1|\gg|z_2|$, $|z_1|\gg|z_3|$, appears also with a different sign as the 8th term of $F(z_1,z_3,z_2)$ where it is defined in the same region, so their sum in the symmetrization is zero. We can slightly simplify the six terms, contributing into the symmetrization of $F(z_1,z_2,z_3)$, reducing their number to 4. This is done by means of identities like

$$\frac{1}{(z_1+q_\theta^{-1}z_2)(q_\theta^{-1}z_1+z_3)}=\frac{q_\theta}{(q_\theta^{-1}z_1+z_3)(z_2-q_\theta^2 z_3)}-\frac{q_\theta^2}{(z_1+q_\theta^{-1}z_2)(z_2-q_\theta^2 z_3)}.$$

We have finally

$$\mathrm{Sym}_{z_1,z_2,z_2}F(z_1,z_2,z_3)=\frac{q_\theta^{-1}}{1-q_\theta+q_\theta^2}\cdot\mathrm{Sym}_{z_1,z_2,z_2}G(z_1,z_2,z_3), \tag{A.10}$$

where

$$\begin{aligned}G(z_1,z_2,z_3)=&\frac{q_\theta^{-1}}{(z_1+q_\theta^{-1}z_2)(-q_\theta^2 z_1+z_3)}-\frac{1}{(z_2+q_\theta^{-1}z_3)(-q_\theta^2 z_1+z_3)}\\&+\frac{1}{(z_1+q_\theta^{-1}z_2)(z_2-q_\theta^2 z_3)}-\frac{q_\theta^{-1}}{(z_2+q_\theta^{-1}z_3)(z_1-q_\theta^2 z_2)}.\end{aligned} \tag{A.11}$$

Now the deduction of (A.5) is just an application of the identity

$$\delta(x-y)=\frac{1}{x-y}-\frac{1}{-y+x}$$

to (A.10) under convention (A.9). For instance, the term $\frac{q_\theta^{-1}}{1-q_\theta+q_\theta^2}\delta(z_3-q_\theta^2z_1)\delta(z_2+q_\theta^{-1}z_3)$ is the result of the summation of the four terms up to overall factor $\dfrac{q_\theta^{-1}}{1-q_\theta+q_\theta^2}$:

$$\begin{aligned}\delta(z_3-q_\theta^2z_1)\delta(z_2+q_\theta^{-1}z_3) = &\frac{q_\theta^{-1}}{(z_1+q_\theta^{-1}z_2)(-q_\theta^2z_1+z_3)}\\ &-\frac{1}{(z_2+q_\theta^{-1}z_3)(-q_\theta^2z_1+z_3)}\\ &+\frac{1}{(z_2+q_\theta^{-1}z_3)(z_3-q_\theta^2z_1)}\\ &-\frac{q_\theta^{-1}}{(z_1+q_\theta^{-1}z_2)(z_3-q_\theta^2z_1)},\end{aligned}$$

where the first two terms are taken from $G(z_1,z_2,z_3)$, the third is from $G(z_2,z_3,z_1)$ and the fourth is from $G(z_3,z_1,z_2)$.

If we want to deduce the properties of the correlation functions from another Serre relation:

$$\mathrm{Sym}_{z_1,z_2,z_3}(q_\theta^3z_1^{-1}-(q_\theta^2+q_\theta)z_2^{-1}+z_3^{-1})x^+(z_1)(z_2)x^+(z_3)=0, \tag{A.12}$$

we use the symmetric Laurent polynomial

$$\begin{aligned}&\tilde{E}_{\xi,v}(z_1,z_2,z_3)\\ &\quad=\prod_{1\leqslant i<j\leqslant 3}\frac{(z_i^{-1}-q_\theta^{-2}z_j^{-1})(z_i^{-1}+q_\theta z_j^{-1})}{z_i^{-1}-z_j^{-1}}x^+(z_1)(z_2)(z_3)\end{aligned} \tag{A.13}$$

and prove the delta function decomposition for the symmetrization of the rational function

$$\begin{aligned}&\tilde{F}(z_1,z_2,z_3)\\ &\quad=(q_\theta^3z_1^{-1}-(q_\theta^2+q_\theta)z_2^{-1}+z_3^{-1})\prod_{1\leqslant i<j\leqslant 3}\frac{z_i^{-1}-z_j^{-1}}{(z_i^{-1}-q_\theta^{-2}z_j^{-1})(z_i^{-1}+q_\theta z_j^{-1})}\end{aligned} \tag{A.14}$$

which can be done by a formal change $z_i\to z_i^{-1}$ and $q_\theta\to q_\theta^{-1}$ in the proof of lemma.

We can reverse the arguments. Then, by Lemma A.1, the vanishing conditions listed in Theorem 1 imply equality (A.4), that is, that any matrix coefficient of the left hand side of the first Serre relation (9) vanishes,

$$\langle\xi,\mathrm{Sym}_{z_1,z_2,z_3}(q_\theta^{-3}z_1-(q_\theta^{-2}+q_\theta^{-1})z_2+z_3)x^+(z_1)x^+(z_2)x^+(z_3)v\rangle=0.$$

The delta functions identity for (A.14) show that the vanishing conditions of Theorem 1 also imply the vanishing of the matrix coefficients of the second Serre relation (10),

$$\langle \xi, \mathrm{Sym}_{z_1,z_2,z_3}(q_\theta^3 z_1^{-1} - (q_\theta^2 + q_\theta)z_2^{-1} + z_3^{-1})x^+(z_1)x^+(z_2)x^+(z_3)v \rangle = 0.$$

We see that in the highest weight representations any one of the Serre relations (9), (10) implies another. This indicates that one can probably find a direct algebraic proof of the equivalence of relations (9) and (10).

Appendix B. Vanishing properties of the correlation functions from the pairing

In this appendix we demonstrate the vanishing properties of correlation functions by means of the Hopf pairing between two opposite Borel subalgebras related to Drinfeld comultiplication. We show that the pairing

$$\langle x^-(w_1)x^-(w_2)x^-(w_3), x^+(z_1)x^+(z_2)x^+(z_3) \rangle$$

between the products of three opposite currents, multiplied by the polynomial

$$f(z_1,z_2,z_3) = \prod_{i<j} (z_i - q_\theta^2 z_j)(z_i + q_\theta^{-1} z_j) \tag{B.1}$$

can be restricted to 'Serre planes' (22) and this restriction is zero, that is, Serre relations in this form lie in the kernel of the Hopf pairing.

Denote by $\bar{g}_\theta(z)$ the Taylor expansion at a point $z = 0$ of the following rational function:

$$\bar{g}_\theta(z) = \frac{(q^2 - z)(q_\theta^{-1} + z)}{(1 - q^2 z)(1 + q_\theta^{-1} z)}. \tag{B.2}$$

The Hopf pairing between the products of the currents looks like:

$$\langle x^-(w_1)\cdots x^-(w_n), x^+(z_1), \cdots x^+(z_n) \rangle$$
$$= \frac{1}{(q_\theta^{-1} - q_\theta)^n} \sum_{\sigma \in S_n} \prod_{k=1}^{n} \delta\left(\frac{z_k}{w_{\sigma(k)}}\right) \prod_{\substack{k<l \\ \sigma(k)>\sigma(l)}} \bar{g}_\theta\left(\frac{z_l}{z_k}\right). \tag{B.3}$$

Note that there are no overall factors $(-1)^\theta$ in the r.h.s. of (B.3). During the proof of (B.3) they appear twice: once from translating the pairing of tensor products to the products of the pairing, and the second time during the permutation of $K_-(z_i)$ via

$x^+(z_k)$ and cancel each other. Put

$$\bar{E}_{w_1,w_2,w_3}(z_1,z_2,z_3) \\ = (q_\theta^{-1}-q_\theta)^3 f(z_1,z_2,z_3)\langle x^-(w_1)x^-(w_2)x^-(w_3),\ x^+(z_1)x^+(z_2)x^+(z_3)\rangle.$$

Presentation (B.3) implies that $\bar{E}_{w_1,w_2,w_3}(z_1,z_2,z_3)$ can be considered as a Laurent series $\bar{E}_{w_1,w_2,w_3}(z_1,z_2,z_3) = \sum_{n_1,n_2,n_3\in\mathbb{Z}} E_{n_1,n_2,n_3}(z_1,z_2,z_3)w_1^{n_1}w_2^{n_2}w_3^{n_3}$ over w_1,w_2,w_3 with coefficients $E_{n_1,n_2,n_3}(z_1,z_2,z_3)$ in $\mathbb{C}[z_1^{\pm 1},z_2^{\pm 1},z_3^{\pm 1}][[\frac{z_2}{z_1},\frac{z_3}{z_2}]]$.

These coefficients correspond to matrix coefficients of the product of the currents $\prod_{1\leqslant i<j\leqslant 3}(z_i-q_\theta^2 z_j)(z_i+q_\theta^{-1}z_j)x^+(z_1)x^+(z_2)x^+(z_3)$; they are antisymmetric and converge in a region $|z_1|\gg|z_2|\gg|z_3|$ to rational functions. The arguments here repeat [E,DKP]. In this sense the restriction of $\bar{E}_{w_1,w_2,w_3}(z_1,z_2,z_3)$ to a subvariety of $\mathbb{C}^3$ is well defined. Due to antisymmetricity for the deduction of the properties of correlation functions it is sufficient to prove that the series $\bar{E}_{w_1,w_2,w_3}(z_1,z_2,z_3)$ vanishes on the line $z_1=-q_\theta z_2, z_2=-q_\theta z_3$. We compute from (B.3):

$$\bar{E}_{w_1,w_2,w_3}(z_1,z_2,z_3) \\ = \sum_{\sigma\in S_3}(-1)^{l(\sigma)}\prod_{\sigma(i)<\sigma(j)}(z_i-q_\theta^2 z_j)(z_i+q_\theta^{-1}z_j)\prod_{k=1,2,3}\delta\left(\frac{z_k}{w_{\sigma(k)}}\right). \tag{B.4}$$

The equality shows that the coefficients at the delta functions in $\bar{E}_{w_1,w_2,w_3}(z_1,z_2,z_3)$ are proportional to $f(z_{\sigma(1)},z_{\sigma(2)},z_{\sigma(3)})$, so we have only to observe that $f(z_1,z_2,z_3)$ vanishes on all six lines

$$\{z_{\sigma(1)}=-q_\theta z_{\sigma(2)}\}\bigcap\{z_{\sigma(2)}=-q_\theta z_{\sigma(3)}\},\quad \sigma\in S_3,$$

which is an elementary check.

References

[CP] V. Chari, A. Pressley, Twisted quantum affine algebras, Comm. Math. Phys. 198 (2) (1998) 461–476.

[D] V.G. Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 36 (1988) 212–216.

[DK] J. Ding, S. Khoroshkin, Weyl group extension of quantized current algebras, Transform. Groups 5 (1) (2000) 35–59.

[DKP] J. Ding, S. Khoroshkin, S. Pakuliak, Integral presentations for the universal *R*-matrix, Lett. Math. Phys. 53 (2) (2000) 121–141.

[E] B. Enriquez, On correlation functions of Drinfeld currents and shuffle algebras, Transform. Groups 5 (2) (2000) 111–120.

[FO] B.L. Feigin, A.V. Odesskii, Vector bundles on elliptic curve and Sklyanin algebras, topics in Quantum Groups and Finite-type Invariants, AMS Translation Series 2, AMS, Providence, RI, Vol. 185, pp. 65–84.

[KT1] S.M. Khoroshkin, V.N. Tolstoy, The uniqueness theorem for the universal R-matrix, Lett. Math. Phys. 24 (3) (1992) 231–244.

[KLT] S.M. Khoroshkin, J. Lukiersky, V.N. Tolstoy, Quantum affine superalgebras $U_q(A_2^{(2)})$ and $U_q(C_2^{(2)})$, Comm. Math. Phys. 220 (3) (2001) 537–560.

[KT2] S.M. Khoroshkin, V.N. Tolstoy, The Cartan–Weyl basis and the universal R-matrix for quantum Kac–Moody algebras and superalgebras, in: H.-D. Doebner, V.K. Dobrev (Eds.), Quantum Symmetries, World Scientific Publishing Co, Singapore, New Jersey, London, Hong Kong, 1993, pp. 336–351.