Note

Sphericity Exceeds Cubicity for Almost All Complete Bipartite Graphs

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We will prove that sphericity exceeds cubicity for complete bipartite graphs $K(m, n)$ with $\max(m, n) > n_0$, that $\text{sph} \ K(n, n) > n$, and that as a result of the latter there is a complete bipartite graph with sphericity exceeding cubicity for every value of cubicity at least 6. This answers the question of Fishburn [1].

1. INTRODUCTION

The sphericity of a graph $G$, $\text{sph} \ G$, is the smallest integer $n$ such that the vertices of $G$ can be embedded in $n$-space $\mathbb{R}^n$ in such a way that $|x - y| < 1$ if and only if $x$ and $y$ are adjacent in $G$, where $| |$ is the Euclidean norm. The cubicity of $G$, $\text{cub} \ G$, is defined similarly by using the $\sup$ norm instead of the Euclidean norm. Concerning the cubicity of a complete multipartite graph, there is a nice formula by Roberts [8]:

$$\text{cub} \ K(n_1, \ldots, n_p) = \lceil \log_2 n_1 \rceil + \cdots + \lceil \log_2 n_p \rceil.$$  

Havel observed in [2] that cubicity can exceed sphericity. For example, $\text{cub} \ K(1, 5) = 3 > \text{sph} \ K(1, 5) = 2$; $\text{cub} \ K(2, 2, \ldots, 2) = \lceil \text{the number of 2's} \rceil$ by the above formula, but $\text{sph} \ K(2, 2, \ldots, 2) = 2$ (see [2, 4, 5]). And there arose the question whether sphericity can exceed cubicity.

Fishburn [1] constructed graphs $G$ that satisfy

$$\text{sph} \ G > \text{cub} \ G = n$$  

for $n = 2$ and 3. But he left open the question whether there are graphs $G$ satisfying ($\ast$) for large $n$.

Now we state our results.
THEOREM 1. \( \text{sph } K(n,n) \geq n \).

As a corollary we have the following which answers the question of Fishburn.

COROLLARY 1. For any \( n \geq 6 \), there exists a complete bipartite graph that satisfies (*).

Remark 1. It can be shown that \( \text{sph } K(n,n) > n \) for \( n \geq 3 \) by a further geometric consideration [6]. The values \( \text{sph } K(4,4) = 5 \) and \( \text{sph } K(3,8) = 6 \) are also derived in [6]. These cover the cases \( n = 4 \) and 5, since \( \text{cub } K(4,4) = 4 \) and \( \text{cub } K(3,8) = 5 \).

THEOREM 2. \( \text{sph } K(1,n) > (2.49) \log_2 n + o(1) \).

As a corollary we have

COROLLARY 2. If \( \max(m,n) > n_0 \) then \( \text{sph } K(m,n) > \text{cub } K(m,n) \).

Remark 2. From [4, Table 11], it would be expected that \( \text{sph } K(m,n) \geq \text{cub } K(m,n) \) for \( m, n \geq 2 \).

Finally, we comment on complete multipartite graphs. Let \( K_p(n) \) denote the complete \( p \)-partite graph \( K(n,n,...,n) \). It is known that \( \text{sph } K_p(n) \leq 2(n-1) \) by [5] and \( \text{cub } K_p(n) = p \lceil \log_2 n \rceil \) by Roberts' cubicity formula. Since \( K_p(n) \) contains \( K(n,n) \) as an induced subgraph, it follows that

\[
\text{sph } K_p(n) > \text{cub } K_p(n) \quad \text{if} \quad p < n/\lceil \log_2 n \rceil
\]

whereas

\[
\text{sph } K_p(n) < \text{cub } K_p(n) \quad \text{if} \quad p > 2(n-1)/\lceil \log_2 n \rceil.
\]

2. PROOFS OF THEOREMS AND COROLLARIES

Proof of Theorem 1. Let \( \text{sph } K(n,n) = k \). Then in \( k \)-space we can take two point sets \( X \) and \( Y \), each having \( n \) points, such that for any \( x, x' \) of \( X \) and \( y, y' \) of \( Y \), \( |x-x'| \geq 1, |y-y'| \geq 1 \), and \( |x-y| < 1 \). Let \( r \) and \( s \) be the circumradii of \( X \) and \( Y \), respectively. We first show that \( r^2 + s^2 < 1 \). Assume the circumcenter of \( X \) is at the origin \( O \), and let \( \{z_1,...,z_m\} = \{z \in X; |z| = r\} \). Then it is easily seen that the origin \( O \) belongs to the convex hull of \( z_1,...,z_m \), whence \( O = \sum a_i z_i \) with \( \sum a_i = 1 \) and always \( a_i \geq 0 \). For each \( y \) of \( Y \) and each \( 1 \leq i \leq m \),

\[
1 > |y-z_i|^2 = |y|^2 + r^2 - 2(y,z_i),
\]
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where \((, )\) denotes the inner product. Multiplying both sides by \(a_i\) and summing on \(i\),

\[ 1 > |y|^2 + r^2 - 2\left(y, \sum a_i z_i \right) = |y|^2 + r^2. \]

Thus \(|y|^2 < 1 - r^2\), and hence \(r^2 + s^2 < 1\).

Now we may assume without loss of generality that \(r \leq s\) (hence \(r < (1/2)^{1/2}\)). In this case the \(n\) points \(x_1, \ldots, x_n\) of \(X\) must be affinely independent. This is seen as follows. Suppose \(X\) is affinely dependent. It follows then by Caratheodory’s theorem that the circumcenter of \(X\) (i.e., the origin \(O\)) is expressed as the convex combination of at most \(n - 1\) points of \(X\), say, of \(x_2, \ldots, x_n\): \(O = a_2 x_2 + \cdots + a_n x_n\) \((a_2 + \cdots + a_n = 1, a_i \geq 0)\). Now for each \(i \geq 2\),

\[ 1 \leq |x_1 - x_i|^2 = |x_1|^2 + |x_i|^2 - 2(x_1, x_i) \leq 2r^2 - 2(x_1, x_i). \]

Multiplying both sides by \(a_i\) and summing on \(i\) (from 2 to \(n\)), we have \(1 \leq 2r^2\), a contradiction. Therefore \(X\) must be affinely independent, and \(k \geq n - 1\). It also follows that for all \(x\) of \(X\), \(|x| = r\). To see this, suppose \(|x_1| < r\). Then the circumcenter \(O\) is contained in the convex hull of \(X - \{x_1\}\), which similarly leads to a contradiction.

Finally, to see \(k \geq n\), suppose \(k = n - 1\). We will obtain the contradiction \(s < r\) by showing that \(|y| < r\) for every \(y\) of \(Y\). Let \(A(X)\) be the simplex spanned by \(X\). If \(y \in Y\) is contained in \(A(X)\) then clearly \(|y| < r\). Suppose now \(y \in Y\) lies outside \(A(X)\). Then for some \(c, 0 < c < 1\), the point \(c y\) lies on a face of \(A(X)\), say, on the face opposite to \(x_1\). So \(c y\) is expressed as \(a_2 x_2 + \cdots + a_n x_n\) with \(a_2 + \cdots + a_n = 1, a_i \geq 0\). Since \(1 \leq |x_1 - x_i|^2 = 2r^2 - 2(x_1, x_i), i \geq 2\), we have \(2(x_1, x_i) \leq 2r^2 - 1\), and

\[ 2c(x_1, y) = 2(x_1, cy) = 2(x_1, a_2 x_2 + \cdots + a_n x_n) \]

\[ \leq (a_2 + \cdots + a_n)(2r^2 - 1) = 2r^2 - 1 < 0. \]

Hence \(2(x_1, y) < 2r^2 - 1\), and

\[ 1 > |x_1 - y|^2 = r^2 + |y|^2 - 2(x_1, y) > r^2 + |y|^2 - 2r^2 + 1. \]

Therefore \(|y| < r\). This completes the contradiction, forcing \(k \geq n\).

Proof of Corollary 1. For a given \(n \geq 6\), let \(j = \lfloor n/2 \rfloor, k = \lceil n/2 \rceil\), and let \(G = K(2^j, 2^k)\). Then since \(j \geq 3\) and \(G\) contains \(K(2^j, 2^j)\) as an induced subgraph, \(\text{sph } G \geq 2^j > j + k\). On the other hand, by Roberts’ cubicity formula, we have \(\text{cub } G = j + k = n\). Thus \(\text{sph } G > \text{cub } G = n\). This verifies (*) for \(n \geq 6\).
3. PROOFS OF THEOREM 2 AND COROLLARY 2

Let \( \tau_n \) be the maximum number of equal nonoverlapping spheres that touch another of the same size in \( n \) dimensions. Odlyzko and Sloane [7] present certain bounds on \( \tau_n \) for \( n \leq 24 \), including the exact values \( \tau_8 = 240 \) and \( \tau_{24} = 196560 \). Kabatiansky and Levenshtein [3, p. 13] obtained the following asymptotic result:

\[
\left( \frac{1}{n} \right) \log_2 \tau_n < 0.401 + o(1).
\]

Using this we prove Theorem 2.

**Proof of Theorem 2.** First we show that \( \text{sph } K(1, n) \leq m \) implies \( \tau_m \geq n \). Suppose \( \text{sph } K(1, n) \leq m \). Then in the \( m \)-dimensional Euclidean space, \( n \) points can be placed inside a unit sphere \( S \) so that no pair of these \( n \) points may be closer than unit distance. Project these \( n \) points on the surface of \( S \) by the projection from the center of \( S \). Then clearly no pair of the resulting \( n \) points are closer than unit distance. Hence the unit diameter spheres centered at these \( n \) points on the surface of \( S \) are mutually nonoverlapping, and all touch the unit diameter sphere concentric with \( S \). Hence \( \tau_m \geq n \).

Now let \( m \geq \left( \frac{1}{0.401} \right) \log_2 n \). Then for large \( n \)

\[
\log_2 \tau_m < (0.401)m \leq \log_2 n,
\]

that is, \( \tau_m < n \), which implies \( \text{sph } K(1, n) > (2.49) \log_2 n \).

**Proof of Corollary 2.** By Theorem 2, there is an \( n_0 \) such that if \( n > n_0 \) then \( \text{sph } K(1, n) > (2.49) \log_2 n \). Suppose \( \text{max}(m, n) > n_0 \). Since \( K(m, n) \) contains \( K(1, m) \) and \( K(1, n) \) as induced subgraphs, we have

\[
\text{sph } K(m, n) > (2.49) \max(\log_2 m, \log_2 n) \\
> \lceil \log_2 m \rceil + \lceil \log_2 n \rceil = \text{cub } K(m, n).
\]

Thus, if \( \text{max}(m, n) > n_0 \) then \( \text{sph } K(m, n) > \text{cub } K(m, n) \).

**References**


6. H. Maehara, Dispersed points and geometric embeddings of complete bipartite graphs. submitted for publication.
