REMARKS ON GEOMETRIC STRUCTURES ON COMPACT COMPLEX SURFACES

DIETER KOTSCHICK

(Received 34 September 1990)

In his beautiful paper [26] C. T. C. Wall made a detailed study of locally homogeneous geometric structures on compact complex surfaces and, more generally, on closed smooth 4-manifolds. It is the purpose of this note to correct an error in the treatment of hyperbolic manifolds in [26]. We take this opportunity to include a few remarks on other topics discussed in [26] and to update the references.

This paper is not intended to be understandable without reference to [26]. We generally refer to [26] as [W] and to reference [n] in [W] as [Wn].

1. HYPERBOLIC MANIFOLDS

We begin with an elementary observation.

**Lemma 1.** A closed hyperbolic 4-manifold M has positive Euler characteristic.

**Proof.** The Gauss-Bonnet-Chern-Weil formula for the Euler characteristic in dimension 4 is

$$\chi(M) = \frac{1}{8\pi^2} \int_M \frac{1}{24} s^2 + \|W\|^2 - \|Ric\_0\|^2,$$

with s the scalar curvature, W the Weyl tensor and Ric\_0 the traceless Ricci tensor.

A hyperbolic manifold is Einstein (Ric\_0 = 0) and conformally flat (W = 0) and we may normalize s = -1. Then

$$\chi(M) = \frac{1}{8\pi^2} \int_M \frac{1}{24} s^2 = \frac{1}{192\pi^2} vol(M) > 0.$$

**QED**

**Remark 1.** This argument can be generalized in various directions to yield the Hitchin-Thorpe inequality $\chi(M) \geq \frac{1}{2} |\sigma(M)|$ for Einstein 4-manifolds and the Chern-Milnor inequality $\chi(M) \geq 0$ for 4-manifolds all of whose sectional curvatures have the same sign. In any even dimension it gives $\chi(M) = c_k vol(M)$ with $sign(c_k) = (-1)^k$ for hyperbolic 2k-manifolds, a fact first noticed by Chern. Compare [20].

In view of Lemma 1, Theorem 6.1 in [W] should list the model space $X = \mathbb{H}^4$ in the class of geometries giving manifolds with $\sigma(M) = 0$, $\chi(M) > 0$ instead of $\chi(M) < 0$ as stated, together with its compact dual $S^4$ and the pair of mutually dual symmetric spaces $\mathbb{H}^2 \times \mathbb{H}^2$ and $S^2 \times S^2$. (The space $S^2 \times \mathbb{H}^2$ remains in a class by itself, being "self-dual").
The sign of the Euler characteristic of a hyperbolic manifold is already wrong in Wall's earlier paper [W49], p. 289. It is used there only to conclude that $\mathbb{H}^4$ is not Hermitian symmetric. This is indeed standard and can be proved by a Lie algebra calculation as mentioned in [W49] p. 275. Cf. [9].

Thus Theorem 1.1 of [W], quoted from [W49], is not affected. However, the third case considered in the proof of the main Theorems 10.1 and 10.2 is now incomplete. To prove Theorem 10.1 ("homotopy type determines geometry") one needs the following proposition.

**Proposition 1.** Geometric closed 4-manifolds modelled on $\mathbb{H}^2 \times \mathbb{H}^2$ and $\mathbb{H}^4$ respectively are never homotopy equivalent.

The missing part of Theorem 10.2 is the next proposition.

**Proposition 2.** There is no compact complex surface homotopy equivalent to a closed hyperbolic 4-manifold.

**Proof.** Let $M$ be complex, $N$ be hyperbolic and $f: M \to N$ be a homotopy equivalence. We have $c_1^2(f) = 3\chi(M) + 2\chi(N) = 3\chi(N) + 2\chi(N) > 0$ by Lemma 1. Now $c_1^2(M) > 0$ and $c_2(M) > 0$ imply, by the Enriques–Kodaira classification of complex surfaces [W2], [W3], that $M$ is either rational or a surface of general type. The first is impossible because rational surfaces are simply-connected.

As $M$ and $N$ are compact and $N$ has negative sectional curvature, we may assume $f$ to be harmonic [8]. Now $M$ is of general type and therefore Kählerian [W8], [W29], so that by Sampson's theorem [19] any harmonic map $f: M \to N$ has rank at most 2. This contradicts the assumption that $f$ is a homotopy equivalence. QED

J. A. Carlson and D. Toledo [2] have proved the most general results of this type. One of their theorems is the following.

**Theorem 1 [2].** Let $\Gamma$ be the fundamental group of a compact hyperbolic manifold of dimension $n > 2$. Then $\Gamma$ is not the fundamental group of any compact Kähler manifold.

We will return to a discussion of this later.

**Proof of Proposition 1.** As $\mathbb{H}^2 \times \mathbb{H}^2$ is complex Kähler this is a consequence of Proposition 2. Alternatively, it follows from Mostow's strong rigidity theorem [17] because one of the model spaces has no $\mathbb{H}^2$-factor. QED

2. SURFACES WITH NON-NEGATIVE SIGNATURE

The following is Theorem 6.2 of [W].

**Theorem 2.** For all complex surfaces of general type $c_1^2(S) \leq 3c_2(S)$. The equality $c_1^2(S) = 3c_2(S)$ holds if and only if $S$ has a geometric structure of type $\mathbb{H}^2(C)$.

In [W] this was ascribed to Yau [W50] and Miyaoka [W33]. However, the characterization of the case of equality does not follow from those references. Clearly, if $S$ has a geometric structure of type $\mathbb{H}^2(C)$ then $c_1^2(S) = 3c_2(S)$. The converse follows from Yau's theorem [W50] if the canonical bundle $K_S$ is ample, in other words $S$ does not contain any $(-2)$-curves. That this assumption is always satisfied only follows from Miyaoka's later work in [13].
For an independent proof of Theorem 2 due to C. T. Simpson see [21], Proposition 9.9.

Complex surfaces with $c_2^2(S) > 2c_2(S)$ have positive signature, or positive index in pre-Atiyah–Singer terminology. Except for $P^1(C)$ they are all of general type. These surfaces attracted a lot of attention for some years, mainly due to a dearth of examples. A note added in proof to [W] (p. 136) mentioned the first construction of simply-connected surfaces in this class due to B. Moishezon and M. Teicher. Their construction has now appeared in [14], [15]. The surfaces in question are smoothings of unions of projective planes. Another construction of simply-connected surfaces with positive signature has been given by Xiao Gang and Zhijie Chen [3]. This leads to hyperelliptic fibrations.

Although there are now many different kinds of surfaces with positive signature, cf. [1], the earlier feeling that they are special may yet be vindicated. In this direction Shin-Yi Lu and S.-T. Yau [12] have proved that surfaces in this class are $C$-hyperbolic, meaning that there is a proper subvariety $V \subset S$ such that any non-constant holomorphic map $\phi: C \to S$ has image contained in $V$.

As noted in [W] p. 136, the zero signature example of Moishezon and Teicher [14], [15] shows that surfaces of general type with $c_2^2(S) = 2c_2(S)$, i.e. with zero signature, need not have a geometric structure of type $\mathbb{H}^2 \times \mathbb{H}^2$. Indeed a complete characterization of surfaces with this geometric structure parallel to the second part of Theorem 2 is still missing. However, there is the following partial result.

**Theorem 3 [21].** Let $S$ be a compact Kähler surface. If $TS = L_1 \oplus L_2$ is a direct sum of line bundles of degrees $c_1(L_1)[\alpha] < 0$ and if $c_2^2(S) = 2c_2(S)$, then $S$ has a geometric structure of type $\mathbb{H}^2 \times \mathbb{H}^2$.

For surfaces with a geometric structure of type $\mathbb{H}^2 \times \mathbb{H}^2$ there is the following strengthening of Theorem 10.2 in [W].

**Theorem 4 [10].** Let $M$ be a compact complex surface homotopy equivalent to a compact complex surface $N$ with a geometric structure of type $\mathbb{H}^2 \times \mathbb{H}^2$. Then $M$ also has a geometric structure of this type.

Moreover, if $N$ is not covered by a product of compact Riemann surfaces, then, after possibly changing orientations suitably, $M$ and $N$ are biholomorphic.

Thus, in this case the complex analytic structure is determined by the homotopy type, with the obvious exceptions. No such statement is true for surfaces without geometric structures, as shown by the existence of homeomorphic surfaces of different Kodaira dimension [W14]. It is conjectured that such pairs always give rise to exotic smooth structures detected by Donaldson theory [7]. In all cases where invariants have been successfully computed this conjecture has been verified.

### 3. Surfaces of Class $\text{VII}_0$

A compact complex analytic surface is of class $\text{VII}_0$ if it is minimal and has $b_1(S) = 1$. Combining the results of M. Inoue [W23] and F. A. Bogomolov [W5], [W6] one obtains the following theorem. (Cf. also [18].)

**Theorem 5.** A surface is of class $\text{VII}_0$ with $b_1(S) = 0$ and contains no curves if and only if it is one of the surfaces $S_M$, $S_N^\times$, $S_N^-$ described in [W23]. (Cf. [W] p. 145.)

Wall showed that the Inoue surfaces $S_M$, $S_N^\times$ and $S_N^-$ are precisely those surfaces
admitting a geometric structure of type \( Sol^*_1 \), \( Sol^*_1 \) or \( Sol^*_1 \) ([W], Prop. 9.1). This gives the following equivalent formulation of Theorem 5.

**Theorem 5 (bis).** A surface is of class VII\(_0\) with \( b_2(S) = 0 \) and contains no curves if and only if it has a geometric structure of type \( Sol^*_1 \), \( Sol^*_1 \) or \( Sol^*_1 \).

This statement reveals the similarity with Theorems 2 and 3. It is then not surprising that Bogomolov's arguments [W5], [W6] can be replaced by an argument using Yang–Mills theory, as observed by J. Li, Yau and F. Zheng [11], in the same spirit as Simpson's proof [21] of Theorems 2 and 3. In both cases one proceeds by constructing a Yang–Mills connexion on a bundle satisfying a suitable stability condition. The assumptions on characteristic classes then imply that this connexion is flat. In the situation considered in [21] this always works and the flat connexion provides the uniformization. In [11] the required stability condition holds only if one assumes that the surface in question is not an Inoue surface. The existence of the flat Yang–Mills connexion leads to a contradiction.

These arguments are generalizations of S. K. Donaldson's result in [6], which is itself a generalization of the theorem of M. Narasimhan and C. Seshadri [W36].

4. **Strong Rigidity**

Theorem 10.1 in [W] asserts that if geometric closed 4-manifolds are homotopy equivalent, then their geometries are of the same type. Sometimes more is true: up to normalization the manifolds are even isometric. This type of statement is known as **strong rigidity**.

G. D. Mostow [17] proved that two homotopy equivalent closed locally symmetric spaces of non-positive curvature are isometric (up to normalization), unless they both have closed one or two dimensional geodesic subspaces which are locally direct factors. As we remarked earlier this implies Proposition 1.

S.-T. Yau suggested that a similar result might be true for Kähler manifolds instead of locally symmetric spaces: If a compact Kähler manifold is homotopy equivalent to a suitably negatively curved compact Kähler manifold, then they are biholomorphic. Y.-T. Siu [24] proved the first such result, and since then many more have been proved, see e.g. Siu [25], J. Jost and Yau [10] and N. Mok [16]. Theorem 4 is an instance of this phenomenon.

Proposition 2 that we need here to complete the proof of Theorem 10.2 in [W] is of a different kind. It mixes the two points of view above to compare a locally symmetric space with a Kähler manifold. This situation was first studied by J. H. Sampson [19], whose main result we used in the proof. His method has been applied more systematically by Carlson and Toledo [2], proving general theorems like Theorem 1.

Finally, K. Corlette [4], [5] and Simpson [22], [23] develop the connections between this rigidity theory and Yang–Mills theory in the style of [6], [21].

All these methods work in arbitrary dimensions. In applications the results are a bit more powerful in low dimensions because they can be combined with the classification of structures (geometric, analytic, algebraic), which is more manageable in low dimensions. Conceptually, the 4-dimensional case is not distinguished.

**Acknowledgement**—This note was conceived during an enjoyable stay at MSRI, supported by the NSF.
REFERENCES


Queens' College,
Cambridge, CB3 9ET,
U.K.

Present address:
Mathematisches Institut,
Universität Basel,
Rheinsprung 21,
4051 Basel,
Switzerland