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Dynamical analysis of fractional-order modified logistic model

Syed Abbas [a], Malay Banerjee [b], Shaher Momani [c,*]

[a] School of Basic Sciences, Indian Institute of Technology Mandi, Mandi, Himachal Pradesh - 175001, India
[b] Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur, Uttar Pradesh - 208016, India
[c] Department of Mathematics, The University of Jordan, Amman 11942, Jordan

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ABSTRACT

In this paper, we study a fractional differential equation model of the single species multiplicative Allee effect. First we study the stability of equilibrium points. Further we give some sufficient conditions ensuring the existence and uniqueness of integral solution. In the last section we perform several numerical simulations to validate our analytical findings.

1. Introduction

The growth equation of single species with multiplicative Allee effect is governed by the following nonlinear ordinary differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = rx(t)\left(1 - \frac{x(t)}{k}\right)(x(t) - m), \tag{1.1}$$

subjected to non-negative initial condition $x_0 \geq 0$. In the above model system 'r', 'm' and 'k' are positive constants and stand for per-capita growth rate, Allee effect threshold and carrying capacity, respectively. In ordinary logistic growth model it is assumed that the growth rate is positive below the threshold level 'k' and is negative above it. In reality, it is observed that a minimum population density is required for the growth of certain species and below which population goes to extinction. Hence growth rate of the population is positive only within the range $m < x < k$ and is negative outside this interval. Based upon this assumption the per capita logistic growth rate $r\left(1 - \frac{x}{k}\right)$ is modified to '$r\left(1 - \frac{x}{k}\right)(x-m)$' and results in the model as we have presented in (1.1). Model system (1.1) is said to have multiplicative Allee effect [1–4]. One can easily prove that the solutions of model (1) are always positive when starting from a point in $\mathbb{R}_+$. There are only three equilibria, $x_1 = 0$, $x_2 = m$ and $x_3 = k$. Among these three, x_2 is always locally unstable, x_1 is locally stable from right and all trajectories converge to x_3 starting from the initial points $x_0 > x_2$. We illustrate these results with help of numerical simulations in Section 4.

In this work we consider the fractional counterpart of Eq. (1.1)

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = rx(t)\left(1 - \frac{x(t)}{k}\right)(x(t) - m), \quad t > 0 \tag{1.2}$$

for $0 < \alpha < 1$ and $x(0) = x_0$. This work is motivated by work done in El-Sayed et al. [5].

Now we describe the definitions of fractional integral and derivative in the sense of Reimann–Liouville. For more details on the geometric and physical interpretation for the Reimann–Liouville fractional derivative and integral see [6].

* Corresponding author. Tel.: +962 777500326; fax: +962 6 5355522.
E-mail addresses: s.momani@ju.edu.jo, shahermm@yahoo.com (S. Momani).

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Definition 1.1. The Reimann–Liouville fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s,$$

provided the right side exists pointwise on $\mathbb{R}^+$. Γ is the gamma function. For instance, $I^\alpha f$ exists for all $\alpha > 0$, when $f \in C^0(\mathbb{R}^+) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+)$; note also that when $f \in C^0(\mathbb{R}_0^+)$ then $I^\alpha f \in C^0(\mathbb{R}_0^+)$ and moreover $I^\alpha f(0) = 0$.

Definition 1.2. The Reimann–Liouville fractional derivative of order $\alpha \in (0, 1)$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} f(s)\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} I_0^{1-\alpha} f(t).$$

Fractional differential equations are generalizations of ordinary differential equations to arbitrary non-integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventeenth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [7,8]. Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modeling encompassing different branches of physics, chemistry and biological sciences [9,6]. Magin [10] used fractional calculus to model some complex dynamics in biological tissues. There have been many excellent books and monographs available on this field [6,11–17]. In [14], the authors gave the most recent and up-to-date developments on fractional differential and fractional integro-differential equations with applications involving many different potentially useful operators of fractional calculus. Recently many mathematicians and scientists worked on the problem of existence and uniqueness of solutions of fractional differential equations, see [18–22]. In this work, we discuss the existence and uniqueness of solution for our model system (1.2).

2. Stability analysis

Consider the function

$$f(x(t)) = rx(t)\left(1 - \frac{x(t)}{k}\right)(x(t) - m).$$

To evaluate the equilibrium points, consider $\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = 0$, which implies that $f_1(x^*) = 0$. Now we first discuss the stability analysis of the model system (1.2). Let us perturb the equilibrium point by adding a positive term $\epsilon(t)$, that is

$$x(t) = x^* + \epsilon(t).$$

We get out system for any f

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(x^* + \epsilon) = f(x^* + \epsilon),$$

which gives

$$\frac{\mathrm{d}^\alpha \epsilon(t)}{\mathrm{d}t^\alpha} = f(x^* + \epsilon).$$

Using a Taylor series expansion, we get

$$f(x^* + \epsilon) = f(x^*) + f'(x^*)\epsilon + \cdots,$$

which implies

$$f(x^* + \epsilon) \simeq f'(x^*)\epsilon.$$

Thus we have

$$\frac{\mathrm{d}^\alpha \epsilon(t)}{\mathrm{d}t^\alpha} \simeq f'(x^*)\epsilon(t), \quad t > 0 \qquad \epsilon(0) = x_0 - x^*.$$

Thus we can easily deduce that if the solution exists for the above systems, then as $\epsilon(t)$ increases the equilibrium point x^* becomes unstable. The equilibrium point is locally asymptotically stable if $\epsilon(t)$ is decreasing.

It is also to note that if we replace x by $x + x^*$, in Eq. (1.2), then the linear part is given by

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = r\left(2\left(1 + \frac{m}{k}\right)x^* - \frac{3}{k}{x^*}^2 - m\right)x, \quad t > 0. \tag{2.3}$$

One can easily check that the above expression is the same as

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = f'(x^*)x, \quad t > 0. \tag{2.4}$$

Thus, in order to check the stability of the equilibrium points, we need to check the nature of f' at that point.

It is easy to see that the Eq. (1.2) has three equilibrium points $x_1 = 0$, $x_2 = m$ and $x_3 = k$. In order to study the stability, we calculate

$$f'(x(t)) = r\left(1 - \frac{x(t)}{k}\right)(x(t) - m) + rx(t)\left(1 - \frac{x(t)}{k}\right) - \frac{r}{k}x(t)(x(t) - m).$$

Now we get

$$f'(0) = -rm, \qquad f'(m) = rm\left(1 - \frac{m}{k}\right), \qquad f'(k) = -r(k - m).$$

For the first equilibrium point, consider the problem,

$$\frac{\mathrm{d}^\alpha \epsilon(t)}{\mathrm{d}t^\alpha} = f'(x^* = 0)\epsilon(t) = -rm\epsilon(t), \quad t > 0 \qquad \epsilon(0) = x_0.$$

The solution of the problem is given by

$$\epsilon(t) = \sum_0^\infty \frac{(-rm)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} x_0.$$

As r, m, k are positive constants and $m < k$, we have

$$f'(0) < 0, \qquad f'(m) > 0, \qquad f'(k) < 0.$$

Thus $x_1 = 0$ is asymptotically stable. For $x_2 = m$, we have

$$\frac{\mathrm{d}^\alpha \epsilon(t)}{\mathrm{d}t^\alpha} = f'(x^* = m)\epsilon(t) = rm\left(1 - \frac{m}{k}\right)\epsilon(t), \quad t > 0 \qquad \epsilon(0) = x_0 - m.$$

The solution of the problem is given by

$$\epsilon(t) = \sum_0^\infty \frac{\left(rm\left(1 - \frac{m}{k}\right)\right)^n t^{n\alpha}}{\Gamma(n\alpha + 1)}(x_0 - m).$$

Hence the equilibrium point $x_2 = m$ is unstable. Now for the third equilibrium point we have

$$\frac{\mathrm{d}^\alpha \epsilon(t)}{\mathrm{d}t^\alpha} = f'(x^* = k)\epsilon(t) = -r(k - m)\epsilon(t), \quad t > 0 \qquad \epsilon(0) = x_0 - k.$$

The solution of the problem is given by

$$\epsilon(t) = \sum_0^\infty \frac{(-r(k - m))^n t^{n\alpha}}{\Gamma(n\alpha + 1)}(x_0 - k).$$

Hence the equilibrium point $x_3 = k$ is asymptotically stable.

3. Existence and uniqueness of the solution

Define $C(I)$ be the class of continuous functions on $I = [0, T]$ for $T < \infty$, with norm

$$\|x\| = \sup_t |\mathrm{e}^{-Nt}x(t)|, \quad N > 0.$$

It is easy to see that the norm $\|\cdot\|$ is equivalent to the supremum norm $\sup_t |x(t)|$.

Now we prove the existence and uniqueness of the solution of problem (1.2). By a solution of (1.2), we mean that

- $(t, x(t)) \in D$, $t \in I$ where $D = I \times B = \{x \in \mathbb{R} : |x| \le b\}$,
- $x(t)$ satisfies (1.2).

Theorem. *The model (1.2) has a unique solution $x \in C(I)$ provided*

$$\frac{r}{N^{\alpha-1}}\left(-m + \left(1 + \frac{m}{k}\right)2b + \frac{b^2}{k}\right) < 1.$$

Proof. The model (1.2) can be written as

$$I^{1-\alpha}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = rx(t)\left(1 - \frac{x(t)}{k}\right)(x(t) - m).$$

By the operation I^α on both sides of the above equation, we get

$$x(t) - x_0 = I^\alpha \left(rx(t) \left(1 - \frac{x(t)}{k}\right)(x(t) - m) \right). \tag{3.5}$$

Define the operator $F : C(I) \to C(I)$ by

$$Fx(t) = x_0 + I^\alpha \left(rx(t) \left(1 - \frac{x(t)}{k}\right)(x(t) - m) \right).$$

We can easily see that

$$\begin{aligned} f(x(t)) - f(y(t)) &= rx(t) \left(1 - \frac{x(t)}{k}\right)(x(t) - m) - ry(t) \left(1 - \frac{y(t)}{k}\right)(y(t) - m) \\ &= -m(x(t) - y(t)) + \left(1 + \frac{m}{k}\right)(x^2(t) - y^2(t)) - \frac{1}{k}(x^3(t) - y^3(t)) \\ &= (x(t) - y(t)) \left(-m + \left(1 + \frac{m}{k}\right)(x(t) + y(t)) - \frac{1}{k}(x^2(t) + y^2(t) + x(t)y(t)) \right) \\ &= (x(t) - y(t)) \left(-m + \left(1 + \frac{m}{k}\right)(x(t) + y(t)) - \frac{1}{k}((x(t) - y(t))^2 - x(t)y(t)) \right). \end{aligned} \tag{3.6}$$

For $x, y \in B$, we have

$$\begin{aligned} \mathrm{e}^{-Nt}(Fx(t) - Fy(t)) &= r\mathrm{e}^{-Nt} I^\alpha (x(t) - y(t)) \left(-m + \left(1 + \frac{m}{k}\right)(x(t) + y(t)) - \frac{1}{k}((x(t) - y(t))^2 - x(t)y(t)) \right) \\ &\quad \times \frac{r}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \mathrm{e}^{-N(t-s)} (x(s) - y(s)) \\ &\quad \times \left(-m + \left(1 + \frac{m}{k}\right)(x(s) + y(s)) - \frac{1}{k}((x(s) - y(s))^2 - x(s)y(s)) \right) \mathrm{e}^{-Ns} \mathrm{d}s. \end{aligned} \tag{3.7}$$

Hence by the definition of defining $C(I)$, we have

$$\begin{aligned} \|Fx - Fy\| &\le \frac{r}{\Gamma(\alpha)} \left(-m + \left(1 + \frac{m}{k}\right) 2b + \frac{b^2}{k} \right) \|x - y\| \int_0^t (t - s)^{\alpha - 1} \mathrm{e}^{-N(t-s)} \mathrm{d}s \\ &\le \frac{r}{\Gamma(\alpha)} \left(-m + \left(1 + \frac{m}{k}\right) 2b + \frac{b^2}{k} \right) \|x - y\| \int_0^\infty (s)^{\alpha - 1} \mathrm{e}^{-Ns} \mathrm{d}s \\ &\le \frac{r}{N^{\alpha - 1}} \left(-m + \left(1 + \frac{m}{k}\right) 2b + \frac{b^2}{k} \right) \|x - y\|. \end{aligned} \tag{3.8}$$

Thus for N large enough, we have

$$\frac{r}{N^{\alpha - 1}} \left(-m + \left(1 + \frac{m}{k}\right) 2b + \frac{b^2}{k} \right) < 1.$$

Thus the operator F has a fixed point and hence the integral equation has a unique solution $x \in C(I)$.

$$\begin{aligned} x(t) &= x_0 + r \left[\frac{t^{\alpha - 1}}{\Gamma(\alpha)} \left(x_0 \left(1 - \frac{x_0}{k}\right)(x_0 - m) \right) + I^{\alpha + 1} \frac{\mathrm{d}}{\mathrm{d}t} x(t) \left(\left(1 - \frac{x(t)}{k}\right)(x(t) - m) \right) \right] \\ &= x_0 + r \left[\frac{t^{\alpha - 1}}{\Gamma(\alpha)} \left(x_0 \left(1 - \frac{x_0}{k}\right)(x_0 - m) \right) + I^{\alpha + 1} \left(-mx'(t) + 2\left(1 + \frac{m}{k}\right) x(t)x'(t) - \frac{3}{k} x^2(t)x'(t) \right) \right]. \end{aligned} \tag{3.9}$$

Thus

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r \left[\frac{t^{\alpha - 1}}{\Gamma(\alpha)} \left(x_0 \left(1 - \frac{x_0}{k}\right)(x_0 - m) \right) + I^\alpha \left(-mx'(t) + 2\left(1 + \frac{m}{k}\right) x(t)x'(t) - \frac{3}{k} x^2(t)x'(t) \right) \right].$$

Applying e^{-Nt} on both side of the above equation we get

$$\mathrm{e}^{-Nt} x'(t) = r\mathrm{e}^{-Nt} \left[\frac{t^{\alpha - 1}}{\Gamma(\alpha)} \left(x_0 \left(1 - \frac{x_0}{k}\right)(x_0 - m) \right) + I^\alpha \left(-mx'(t) + 2\left(1 + \frac{m}{k}\right) x(t)x'(t) - \frac{3}{k} x^2(t)x'(t) \right) \right]$$

which implies that $x' \in C(I_\sigma)$ for some $\sigma > 0$.

Now we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha} f(x(t))$$
$$I^{1-\alpha}\frac{\mathrm{d}x}{\mathrm{d}t} = I^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha} f(x(t))$$
$$I^{1-\alpha}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} I^{\alpha} f(x(t))$$
$$\frac{\mathrm{d}^{\alpha}x}{\mathrm{d}t^{\alpha}} = \frac{\mathrm{d}}{\mathrm{d}t} I(f(x(t)))$$
$$\frac{\mathrm{d}^{\alpha}x}{\mathrm{d}t^{\alpha}} = (f(x(t))) \tag{3.10}$$

and

$$x(0) = x_0 + I^{\alpha}(f(x(t)))|_{t=0}$$

which implies $x(0) = x_0$. Thus the problem (1.2) is equivalent to corresponding integral equation (3.5). □

4. Numerical methods and results

In this section we use the Adams-type predictor–corrector method for the numerical simulations of nonlinear problems (1.1) and (1.2). This is a very effective tool to give numerical solutions of fractional order differential equations [5,23,24]. It may be used both for linear and for nonlinear problems. We first consider a nonlinear ordinary differential equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = rx(t)(x(t) - m)\left(1 - \frac{x(t)}{K}\right), \tag{4.11}$$

subjected to non-negative initial condition $x_0 \geq 0$. In equation the parameters 'r', 'm' and 'K' are positive constants and satisfy the restriction $m < K$. One can easily prove that the solution of model (4.11) is always positive starting from a point in $\mathbb{R}_+$ based upon the following result

$$x(t) = x(0)\exp\left[\int_0^t (x(s) - m)\left(1 - \frac{x(s)}{K}\right)\mathrm{d}s\right].$$

This shows that $\mathbb{R}_+$ is invariant manifold for model (4.11). There are three equilibria for model (4.11) lying on $\mathbb{R}_+$ are $x_1 = 0, x_2 = m$ and $x_3 = K$. Among these three, x_2 is always unstable, x_1 is stable from right and all trajectories converge to x_3 starting from the initial points $x_0 > x_2$. Depending upon initial conditions the asymptotic behavior of trajectories are as follows

$$\lim_{t\to+\infty} x(t) = 0 \quad \text{for } 0 < x_0 < m,$$
$$\lim_{t\to+\infty} x(t) = K \quad \text{for } x_0 > m, \text{ and } x_0 \neq K.$$

Before proceeding further we are interested in looking at the numerical simulation results of Eq. (4.11) for the chosen parameter values $r = 0.5, m = 1$ and $K = 10$. For a numerical simulation we have chosen $\Delta t = 0.01$ and checked the limiting behavior of solution trajectories for different choices of initial conditions with (0, 15]. From Fig. 1 it is clear that the rate of convergence for the trajectories converging at $x_3 = K$ are faster compared to the trajectories converging at x_1.

Now we consider the numerical solution of following fractional differential equation for different values of α and starting from different initial conditions. We consider the specific equation

$$\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} = rx(t)(x(t) - 1)\left(1 - \frac{x(t)}{10}\right), \quad t > 0,$$

for positive α and subjected to the initial condition $x_0 > 0$. We start our numerical simulation with $\alpha = 0.5$. Solution trajectories starting from an initial point x_0 converge to '0' asymptotically for $0 < x_0 < 1$. On the other hand all trajectories converge rapidly to the equilibrium level $x_3 = 10$ starting from different initial points satisfying the restriction $x_0 > 1$. Hence the domain of attraction for $x_1 = 0$ is $x < 1$ and that of $x_3 = 10$ is $x > 1$. Further, time required to reach the vanishing equilibrium point $x_1 = 0$ is more than that required to reach $x_3 = 10$. Fig. 1 clearly depicts the fact that all trajectories reached $x_3 = 10$ within the time interval [0, 2] but the trajectory starting from $x(0) = 0.5$ is unable to reach the locally stable equilibrium $x_1 = 0$ even within the time interval [0, 3]. Thus we can conclude that the rate of convergence of solution trajectories to different steady states are not same for a fractional order differential equation rather it depends upon the initial point. A similar type of simulation with $\alpha = 0.8$ is presented in Fig. 2. Now it is clear that the magnitude of α is also a crucial parameter behind the convergence of solution trajectories to various equilibrium points.

Next we consider how the rate of convergence of solution trajectories towards their steady state depend upon the magnitude of α. We fix the initial condition at $x_0 = 8$ and perform the numerical simulation for $\alpha = 0.5, \alpha = 0.75$ and $\alpha = 1$. Simulation results are presented in Fig. 3 and it is clear that the time required for the convergence of solution

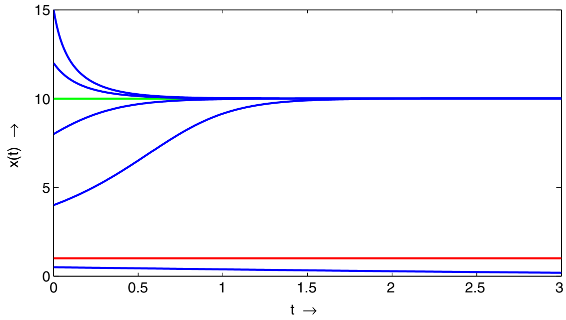

Fig. 1. Numerical simulation results of ordinary differential equation model. Green line and red line correspond to stable steady state $x_3 = 10$ and unstable steady state $x_2 = 1$ respectively. Trajectories starting from $x_0 > 1$ all converge to x_3. The trajectory starting from $x_0 = 0.5$ converges to $x_1 = 0$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

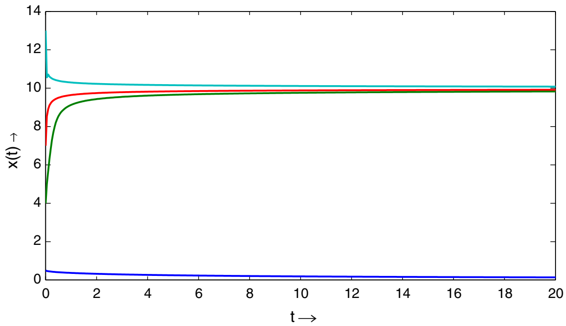

Fig. 2. Time evolution of solution trajectories starting from different initial conditions for $\alpha = 0.8$.

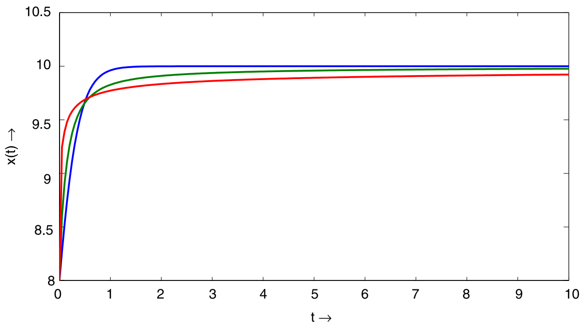

Fig. 3. Solution trajectories converging to $x_3 = 10$ starting from $x_0 = 8$ and different values of α, $\alpha = 0.5$ (red curve); $\alpha = 0.75$ (green curve); and $\alpha = 1$ (blue curve). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

trajectories to the stable steady-state $x_3 = 10$ increases with the decreasing magnitude of α. The time required for convergence to the steady-state x_3 not only varies with α it also depends upon the initial conditions. To illustrate this issue in Fig. 4 we have plotted the time required to reach x_3 by the solution trajectories starting from $x_0 = 8$ for different values of $\alpha \in [0.8, 1]$. As there are no transient oscillations for the solution trajectories before reaching the steady-state $x_3 = 10$, we have calculated the time point t_f such that $|x(t_f) - x_3| < 10^{-8}$. In Fig. 4 we have plotted t_f (correct up to 8th places of decimals) for a range of values of α appearing as the range of α along the horizontal axis. It is clear that the time taken to reach the steady-state $x_3 = 10$ deceases gradually with the increasing magnitude of α.

Finally we consider the time required to converge to the steady-state $x_3 = 10$ starting from a range of initial conditions ($x_0 \in [7, 9]$) and for a fixed value of $\alpha = 0.9$. A plot of t_f against x_0 is presented in Fig. 5. Here we like to remark that the distance of initial point from equilibrium level as well as the fractional order of the differential equation have significant impacts on the time required to converge to the equilibrium level.

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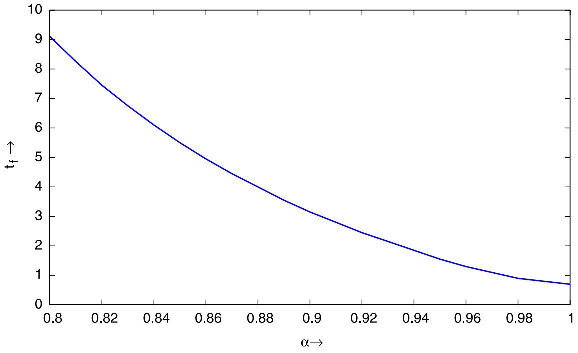

Fig. 4. Plot of t_f against a range of values of α.

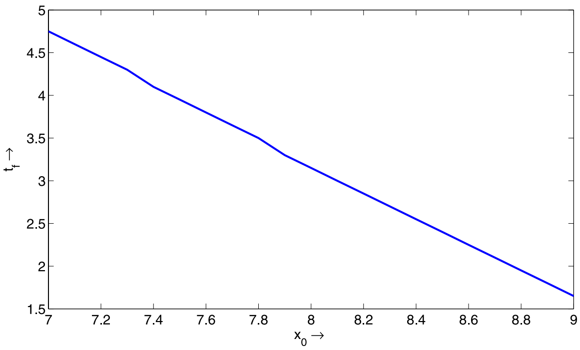

Fig. 5. Plot of t_f for a range of initial conditions x_0.

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