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# On the Structure of Free Baxter Algebras

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A Baxter algebra is a commutative algebra $A$ together with a linear operator $P$ such that $P(ab) + P(a)\,P(b) = P(aP(b)) + P(bP(a))$ holds for any pair $a$, $b$ in $A$. We construct in an explicit way the *free* Baxter algebra $\mathfrak{B}(X)$ on a set $X$. A suitable completion $\hat{\mathfrak{B}}(X)$ of $\mathfrak{B}(X)$ is the algebra of formal power series in the indeterminates $D^n(x)$ for $n \geqslant 0$ and $x$ in $X$, where $D(a) = -\sum_{j=1}^{\infty} P^j(a)$, the series converging in $\hat{\mathfrak{B}}(X)$. Incidentally a new set of identities of combinatorial interest is derived.

## INTRODUCTION

In an important paper [1], Baxter deduced most of the known identities in the theory of fluctuations for random variables from a simple identity (formula (1)). Recently, Rota [2, 3] has undertaken an algebraic analysis of this identity and defined the category of the so-called Baxter algebras. From a combinatorial point of view the interest lies in the manifold of identities one can formally deduce from the Baxter identity. As is usual, the search for such identities may be pursued in the free algebras within the category of Baxter algebras. The existence of free Baxter algebras follows from well-known arguments in universal algebra but remains quite immaterial as long as the corresponding word problem is not solved in an explicit way as Rota was the first to do.

The purpose of this paper is to derive a new set of identities valid in any Baxter algebra (see Section 2) and to use it to give a direct explicit construction of free Baxter algebras. We then define the completion of a free Baxter algebra and show its isomorphism with a suitable formal power series algebra. This will give us a very powerful algorithm and enable us to make more explicit than in Rota [2] the connection with the symmetric polynomials. We conclude by a derivation of Spitzer's identity within the framework of Baxter algebras.

A final word about notations. For any integer $n \geqslant 1$, we let $[n]$ denote the set of integers between 1 and $n$. For any set $X$, we denote by $|X|$ the number of its elements.



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## 1. Definitions

Let $K$ be a commutative ring with unit.

By a *Baxter algebra* we mean a pair $(A, P)$ where $A$ is an algebra over $K$ (associative and commutative) and $P$ is a linear map from $A$ into $A$ satisfying the *Baxter identity*

$$P(ab) + P(a) \cdot P(b) = P(a \cdot P(b)) + P(b \cdot P(a)) \tag{1}$$

(for $a$, $b$ in $A$). We refer to [3] for various examples of Baxter algebras.

We don't assume that $A$ has a unit element. In any case, we embed $K$ and $A$ in a new algebra $A^+$ in such a way that any element in $A^+$ can be uniquely written as $\lambda + a$ with $\lambda$ in $K$ and $a$ in $A$, the multiplication being given in $A^+$ by

$$(\lambda + a) \cdot (\mu + b) = \lambda\mu + (\lambda \cdot b + \mu \cdot a + ab).$$

In particular, the unit element of $K$ acts as unit element of $A^+$.

By induction on $n \geqslant 1$, one defines as follows the $n$-bracket $[a_1, \ldots, a_n]$

$$[a] = P(a), \tag{2}$$

$$[a_1, \ldots, a_n] = P(a_1 \cdot [a_2, \ldots, a_n]) \qquad (n \geqslant 2) \tag{3}$$

for $a$ in $A$, $a_1, \ldots, a_{n-1}$ in $A^+$ and $a_n$ in $A$. Since $A$ is an ideal in $A^+$, the values of the $n$-bracket lie in $A$ for any $n \geqslant 1$. Since the $n$-bracket is obviously linear in each of its arguments, one can in principle consider only brackets, with arguments lying in $A \cup \{1\}$. One has, for example, $[1, 1, a] = P^3(a)$, $[a, 1, b] = P(a \cdot P^2(b))$ and our conventions dispense altogether of considering explicitly the iterated operators $P^2$, $P^3$ and so on.

## 2. Products of Brackets

Our subsequent work is based on the following general identity expressing a product of brackets

$$\prod_{j=1}^{k} [a^j_1, \ldots, a^j_{p_j}] = \sum_{n, P_1, \ldots, P_k} (-1)^{n+p_1+\cdots+p_k} \Phi_{n, P_1, \ldots, P_k}. \tag{4}$$

Summation runs over the integers $n$ with $1 \leqslant n \leqslant p_1 + \cdots + p_k$ and the sequences $P_1, \ldots, P_k$ of subsets of $[n]$ such that

$$P_1 \cup \cdots \cup P_k = [n], \qquad |P_1| = p_1, \ldots, |P_k| = p_k. \tag{5}$$

Moreover, for given $n$, $P_1, \ldots, P_k$, the $n$-bracket

$$\Phi_{n,P_1,\ldots,P_k} = [c_1, \ldots, c_n]$$

is defined by

$$c_j = a_{\sigma_1}^{t_1} \cdots a_{\sigma_q}^{t_q} \qquad (1 \leqslant j \leqslant n) \tag{6}$$

if $j$ belongs to $P_{t_1}, \ldots, P_{t_q}$ but to none other of the $P$'s and $\sigma_r$ is the rank of $j$ in $P_{t_r}$ arranged in increasing order (for $1 \leqslant r \leqslant q$).

We mention a few particular cases of (4). First of all, the product of two brackets is given as follows:

$$[a_1, \ldots, a_p] \cdot [b_1, \ldots, b_q] = \sum_{n,P,Q} (-1)^{n+p+q} \Phi_{n,P,Q} \tag{7}$$

with summation over the integers $n$ between 1 and $p + q$ and pairs of subsets $P$, $Q$ of $[n]$ such that

$$P \cup Q = [n], \qquad |P| = p, \qquad |Q| = q. \tag{8}$$

For given $n$, $P$, $Q$, the $n$-bracket $\Phi_{n,P,Q} = [c_1, \ldots, c_n]$ is defined by

$$c_j = \begin{cases} a_\alpha & \text{if} \quad j \text{ is the } \alpha\text{-th element in } P \text{ and } j \notin Q; \\ b_\beta & \text{if} \quad j \text{ is the } \beta\text{-th element in } Q \text{ and } j \notin P; \\ a_\alpha b_\beta & \text{if} \quad j \text{ is the } \alpha\text{-th element in } P \textit{ and} \text{ the } \beta\text{-th element in } Q \end{cases} \tag{9}$$

for $1 \leqslant j \leqslant n$.

For $p = 1$, formula (7) reduces to

$$\begin{aligned}[a] \cdot [b_1, \ldots, b_q] = &- \sum_{i=1}^{q} [b_1, \ldots, b_{i-1}, ab_i, b_{i+1}, \ldots, b_q] \\ &+ [a, b_1, \ldots, b_q] + \sum_{i=2}^{q} [b_1, \ldots, b_{i-1}, a, b_i, \ldots, b_q] + [b_1, \ldots, b_q, a]. \end{aligned} \tag{10}$$

If we let $p_1 = \cdots = p_k = 1$ in (4), we get

$$[a_1] \cdots [a_k] = \sum_{j=1}^{k} (-1)^{k-j} \sum_{H_1,\ldots,H_j} [a_{H_1}, \ldots, a_{H_j}], \tag{11}$$

where $(H_1, \ldots, H_j)$ runs over the set of all indexed partitions of $[k]$ and $a_H = \prod_{i \in H} a_i$ for any subset $H$ of $[k]$.

Letting finally $q = 1$ in (10) or $k = 2$ in (11) one gets the identity

$$[a] \cdot [b] = - [ab] + [a, b] + [b, a] \tag{12}$$

for $a$, $b$ in $A$. In a different notation, this is nothing else than the initial Baxter identity.

## 3. Proof of the Previous Identities

We prove first (7) by an induction on the pair of integers $p \geqslant 1$, $q \geqslant 1$. Indeed[1] letting $x = [a_2, ..., a_p]$ and $y = [b_2, ..., b_q]$, one gets by definition of brackets the relations

$$[a_1, ..., a_p] = P(a_1 \cdot x), \qquad [b_1, ..., b_q] = P(b_1 y).$$

By Baxter identity, one gets

$$P(a_1 x) \cdot P(b_1 y) = - P(a_1 b_1 xy) + P(a_1 x \cdot P(b_1 y)) + P(b_1 y \cdot P(a_1 x)),$$

that is,

$$\begin{aligned} &[a_1, ..., a_p] \cdot [b_1, ..., b_q] \\ &\quad = - P(a_1 b_1 [a_2, ..., a_p] \cdot [b_2, ..., b_q]) \\ &\qquad + P(a_1 [a_2, ..., a_p] \cdot [b_1, ..., b_q]) + P(b_1 [a_1, ..., a_p] \cdot [b_2, ..., b_q]). \end{aligned} \tag{13}$$

On the other hand, let $n$ be any integer between 1 and $p + q$. The pairs $(P, Q)$ of subsets of $[n]$ satisfying conditions (8) fall into three classes according to the following scheme:

$$\text{(I)} \quad 1 \in P, 1 \in Q; \qquad \text{(II)} \quad 1 \in P, 1 \notin Q; \qquad \text{(III)} \quad 1 \notin P, 1 \in Q.$$

Consider, for instance, any pair $(P, Q)$ of class I and let $P'$ be the set of all integers $i$ in $[n - 1]$ such that $i + 1 \in P$; define similarly $Q'$ to consist of the elements $j$ of $[n - 1]$ such that $j + 1 \in Q$. From (9), one gets $\Phi_{n,P,Q} = [a_1 b_1, c_2, ..., c_n]$ where the $(n - 1)$-bracket

$$\Phi'_{n-1,P',Q'} = [c_2, ..., c_n]$$

is defined by the sequences $a_2, ..., a_p$ and $b_2, ..., b_q$ and the subsets $P'$ and

[1] By convention, an empty bracket is the unit element 1 of $A^+$.

$Q'$ of $[n-1]$ using the same process as in the definition of $\Phi_{n,P,Q}$. Otherwise stated, one has

$$\Phi_{n,P,Q} = P(a_1 b_1 \Phi'_{n-1,P',Q'}).$$

If we use as an inductive assumption that the product of a $(p-1)$-bracket by a $(q-1)$-bracket is given according to (7), we conclude therefore that the sum $\sum_{n,P,Q} (-1)^{n+p+q} \Phi_{n,P,Q}$ extended over all pairs $(P, Q)$ of class I, equal to $-P(a_1 b_1 [a_2, ..., a_p] \cdot [b_2, ..., b_q])$. The pairs of class II or III are treated in a similar way and the corresponding sums are, respectively, equal to

$$P(a_1 [a_2, ..., a_p] \cdot [b_1, ..., b_q]) \qquad \text{and} \qquad P(b_1 [a_1, ..., a_p] \cdot [b_2, ..., b_q])$$

under the corresponding inductive assumptions. Using (13) concludes the proof of (7).

One goes from (7) to (4) by a straightforward induction on $k$, the details of which are left to the reader[2].

To conclude this section, let us mention that (10) can be proved directly by an easy induction on $q$, and that (11) can be derived from (10) by another inductive proof.

## 4. Free Baxter Algebras

Let $X$ be any set. We let $M$ be the free commutative semigroup with unit on $X$, that is, the set of all monomials in the "variables" taken from $X$. By $\mathfrak{B}(X)$ we denote the free $K$-module with a basis consisting of the symbols $u \cdot [\,]$ with $u$ in $M$, $u \neq 1$ and $u \cdot [u_1, ..., u_p]$ with $p \geqslant 1$, $u, u_1, ..., u_p$ in $M$ and $u_p \neq 1$. One defines in $\mathfrak{B}(X)$ a bilinear multiplication whose effect on the basic elements is given by

$$\{u \cdot [\,]\}\{v \cdot [\,]\} = uv \cdot [\,],$$

$$\{u \cdot [\,]\}\{v \cdot [v_1, ..., v_q]\} = \{v \cdot [v_1, ..., v_q]\}\{u \cdot [\,]\} = uv \cdot [v_1, ..., v_q],$$

$$\{u \cdot [u_1, ..., u_p]\}\{v \cdot [v_1, ..., v_q]\} = \sum_{n,P,Q} (-1)^{n+p+q} uv \cdot \Psi_{n,P,Q}. \tag{14}$$

[2] See also the proof of $(A)$ in Section 4 for a similar argument.

The conventions over $n$, $P$, $Q$ are as in (7) and $\Psi_{n,P,Q}$ is the $n$-bracket $[c_1, \ldots, c_n]$ where $c_j$ is defined for $1 \leqslant j \leqslant n$ by

$$c_j = \begin{cases} u_\alpha & \text{if} \quad j \text{ is the } \alpha\text{-th element in } P \text{ and } j \notin Q; \\ v_\beta & \text{if} \quad j \text{ is the } \beta\text{-th element in } Q \text{ and } j \notin P; \\ u_\alpha v_\beta & \text{if} \quad j \text{ is the } \alpha\text{-th element in } P \textit{ and } \text{the } \beta\text{-th element in } Q. \end{cases} \tag{15}$$

Finally one defines a linear map $\Pi$ from $\mathfrak{B}(X)$ into $\mathfrak{B}(X)$ by

$$\Pi(u \cdot [\,]) = [u], \qquad \Pi(u \cdot [u_1, \ldots, u_p]) = [u, u_1, \ldots, u_p]. \tag{16}$$

We identify any element $x$ in $X$ with the basic element $x \cdot [\,]$ of $\mathfrak{B}(X)$. Therefore $X$ is a subset of $\mathfrak{B}(X)$.

Theorem 1. *The pair* $(\mathfrak{B}(X), \Pi)$ *is a free Baxter algebra on* $X$.

The contention of the theorem is the following list of properties:

(A) The multiplication in $\mathfrak{B}(X)$ is *associative.*

(B) The multiplication in $\mathfrak{B}(X)$ is *commutative.*

(C) *Baxter identity:*

$$\Pi(ab) + \Pi(a) \cdot \Pi(b) = \Pi(a \cdot \Pi(b)) + \Pi(b \cdot \Pi(a))$$

for $a$, $b$ in $\mathfrak{B}(X)$.

(D) *Universal property:* Let $(A, P)$ be any Baxter algebra and $\varphi$ a map from $X$ into $A$. There exists a unique homomorphism of $K$-algebras $f : \mathfrak{B}(X) \to A$ extending $\varphi$ and such that $Pf = f\Pi$.

*Proof of* (A). In order to make the proof easier to follow, we shall change the notations slightly. An infinite sequence $\mathbf{u} = (u_1, u_2, \ldots)$ of elements in $M$ is called *admissible* if there are only finitely many terms different from 1 among the $u_n$'s; the *rank* of $\mathbf{u}$ is the largest among the integers $n$ with $u_n \neq 1$ or 0 if there exists no such integer. We denote by $[\,]$ the unit element in $\mathfrak{B}(X)^+$ (see footnote 3) and identify each monomial $u$ in $M$ with $u \cdot [\,]$. The submodule of $\mathfrak{B}(X)$ generated by these monomials is therefore identified to the polynomial algebra $K[X]$ (free associative and commutative algebra with unit on $X$). For any admissible sequence $\mathbf{u}$ of rank $n \geqslant 1$ we let $\epsilon(\mathbf{u}) = (-1)^n [u_1, \ldots, u_n]$ and let $\epsilon(\mathbf{u}) = [\,]$ if $\mathbf{u}$ is the (unique) admissible sequence of rank 0. We can

[3] The definition of $\mathfrak{B}(X)^+$ is similar to the definition of $A^+$ in Section 1. Note that the multiplication in $\mathfrak{B}(X)$ is associative if and only if it is in $\mathfrak{B}(X)^+$.

in an obvious way consider $\mathfrak{B}(X)^+$ as a module over the ring $K[X]$; then the elements $\epsilon(\mathbf{u})$, where $\mathbf{u}$ runs over the set of admissible sequences, form a basis of $\mathfrak{B}(X)^+$ and the multiplication in $\mathfrak{B}(X)^+$ is bilinear over $K[X]$. Consequently it suffices to check the associativity in the form

$$(\epsilon(\mathbf{u})\,\epsilon(\mathbf{v}))\,\epsilon(\mathbf{w}) = \epsilon(\mathbf{u})\,(\epsilon(\mathbf{v})\,\epsilon(\mathbf{w})), \tag{17}$$

whenever $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are three admissible sequences.

We have first to compute $\epsilon(\mathbf{u})\,\epsilon(\mathbf{v})$ for any pair of admissible sequences $\mathbf{u}$ and $\mathbf{v}$. Let $p$ be the rank of $\mathbf{u}$ and $q$ the rank of $\mathbf{v}$. Let $I$ be the set of integers 1, 2,... and $\alpha$ be any strictly increasing map from $[p]$ to $I$. Define a new admissible sequence $\alpha\mathbf{u}$ by the following recipe

$$(\alpha\mathbf{u})_n = \begin{cases} u_i & \text{if} \quad \text{there is an integer } i \text{ in } [p] \text{ with } n = \alpha(i) \\ 1 & \qquad \text{otherwise.} \end{cases} \tag{18}$$

Furthermore, two strictly increasing maps $\alpha : [p] \to I$ and $\beta : [q] \to I$ are called *compatible* if the union of their ranges is an interval of the form $[m]$ in $I$. With these notations, formula (14) takes the equivalent form

$$\epsilon(\mathbf{u})\,\epsilon(\mathbf{v}) = \sum_{\alpha,\beta} \epsilon(\alpha\mathbf{u} * \beta\mathbf{v}), \tag{19}$$

where the summation is over the pairs of compatible maps $\alpha : [p] \to I$ and $\beta : [q] \to I$ and where $*$ denotes the term-wise multiplication of sequences of elements in $M$.

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be admissible sequences of respective ranks $p$, $q$ and $r$. From (19) one deduces

$$(\epsilon(\mathbf{u})\,\epsilon(\mathbf{v}))\,\epsilon(\mathbf{w}) = \sum_{\alpha,\beta,\gamma,\delta} \epsilon(\gamma(\alpha\mathbf{u} * \beta\mathbf{v}) * \delta\mathbf{w}). \tag{20}$$

The previous summation is restricted to the systems of strictly increasing maps

$$\alpha : [p] \to I, \qquad \beta : [q] \to I, \qquad \gamma : [n] \to I, \qquad \delta : [r] \to I,$$

where $[n]$ is the union of the ranges of $\alpha$ and $\beta$ and the union of the ranges of $\gamma$ and $\delta$ is some interval $[m]$. It is clear that $\gamma(\alpha\mathbf{u} * \beta\mathbf{v})$ is equal to $\gamma\alpha\mathbf{u} * \gamma\beta\mathbf{v}$ and that any pair of strictly increasing maps $\rho : [p] \to I$ and $\sigma : [q] \to I$ whose ranges comprise together $n$ elements can be uniquely written as $\rho = \gamma\alpha$, $\sigma = \gamma\beta$ with $\alpha$, $\beta$, and $\gamma$ as before. From (20) follows the identity

$$(\epsilon(\mathbf{u})\,\epsilon(\mathbf{v}))\,\epsilon(\mathbf{w}) = \sum_{\rho,\sigma,\tau} \epsilon((\rho\mathbf{u} * \sigma\mathbf{v}) * \tau\mathbf{w}), \tag{21}$$

the summation being over the triples of strictly increasing maps

$$\rho : [p] \to I, \qquad \sigma : [q] \to I, \qquad \tau : [r] \to I$$

such that the union of their ranges be some interval $[m]$.

A completely similar reasoning gives the identity

$$\epsilon(\mathbf{u})\,(\epsilon(\mathbf{v})\,\epsilon(\mathbf{w})) = \sum_{\rho,\sigma,\tau} \epsilon(\rho\mathbf{u} * (\sigma\mathbf{v} * \tau\mathbf{w})) \tag{22}$$

and associativity of the multiplication in $\mathfrak{B}(X)^+$ follows therefore from the associativity in the semigroup $M$.

*Proof of* (B). Commutativity of multiplication in $\mathfrak{B}(X)$ follows immediately from the corresponding property in $M$.

*Proof of* (C). By linearity it suffices to prove the Baxter formula for basic elements $a = u_1 \cdot [u_2, ..., u_p]$ and $b = v_1 \cdot [v_2, ..., v_q]$ in $\mathfrak{B}(X)$[4]. By definition of multiplication and $\Pi$ in $\mathfrak{B}(X)$ we get

$$\Pi(a) \cdot \Pi(b) = [u_1, ..., u_p] \cdot [v_1, ..., v_q] = \sum_{n,P,Q} (-1)^{n+p+q}\, \Psi_{n,P,Q}\,. \tag{23}$$

Divide the pairs $(P, Q)$ in three classes as in Section 3 and split accordingly the last sum in (23) as $\Sigma_{\mathrm{I}} + \Sigma_{\mathrm{II}} + \Sigma_{\mathrm{III}}$. A calculation similar to the one in Section 3 gives

$$\Pi(ab) = -\Sigma_{\mathrm{I}}\,, \qquad \Pi(a \cdot \Pi(b)) = \Sigma_{\mathrm{II}}\,, \qquad \Pi(b \cdot \Pi(a)) = \Sigma_{\mathrm{III}}\,,$$

whence the sought-for relation.

*Proof of* (D). We first extend $\varphi$ to a multiplicative map $\varphi'$ from $M$ into $A^+$ by $\varphi'(x_1 \cdots x_n) = \varphi(x_1) \cdots \varphi(x_n)$ for $x_1, ..., x_n$ in $X$. Let then the linear map $f^+ : \mathfrak{B}(X)^+ \to A^+$ be given by its action on the basic elements

$$f^+(u \cdot [u_1, ..., u_p]) = \varphi'(u)\,[\varphi'(u_1), ..., \varphi'(u_p)].$$

It is immediate that $f^+$ induces a map $f$ from $\mathfrak{B}(X)$ into $A$, that $f$ extends $\varphi$ and that $Pf = f\Pi$.

[4] We allow $p = 1$ or $q = 1$, that is, $a = u_1 \cdot [\ ]$ or $b = v_1 \cdot [\ ]$.

It remains to check the relation $f(ab) = f(a)f(b)$ for $a$, $b$ in $\mathfrak{B}(X)$. By linearity we need only consider the case of basic elements $a$ and $b$. But it checks because the *definition* (14) of the product of two basic elements in $\mathfrak{B}(X)$ matches with identity (7) valid in any Baxter algebra.

Any element in $\mathfrak{B}(X)$ can be obtained from the elements of $X$ by applying finitely many times the basic operations in $\mathfrak{B}(X)$: addition, multiplication, scalar multiplication by an element of $K$, action of $\Pi$. This property entails uniqueness of $f$.

## 5. Completion of a Free Baxter Algebra

Notations are as in Section 4. We let $\hat{\mathfrak{B}}(X)$ be the set of unrestricted linear combinations of the basic elements $u \cdot [u_1, \ldots, u_p]$ with coefficients in $K$. We allow therefore for infinitely many terms in such a linear combination. It is easily seen that a given basic element enters with a nonzero coefficient in the product of finitely many pairs of basic elements only and that it is the image under $\Pi$ of at most one basic element. We can therefore extend to $\hat{\mathfrak{B}}(X)$ the multiplication of $\mathfrak{B}(X)$ and extend $\Pi$ to an operator $\hat{\Pi}$ in $\hat{\mathfrak{B}}(X)$. Obviously the pair $(\hat{\mathfrak{B}}(X), \hat{\Pi})$ is a Baxter algebra, to be called the *completion of the free Baxter algebra on X*.

A given basic element can be written finitely many times only in the form $\Pi^n(a)$, where $a$ is another basic element and $n \geqslant 0$ an integer. It follows immediately that the series $\sum_{n=0}^{\infty} \hat{\Pi}^n(a)$ converges for any element $a$ in $\hat{\mathfrak{B}}(X)$. The linear operator $I - \hat{\Pi}$ in $\hat{\mathfrak{B}}(X)$ is therefore invertible with the operator $\sum_{n=0}^{\infty} \hat{\Pi}^n$ as inverse. Define the operator

$$\Delta = -\hat{\Pi} \cdot (I - \hat{\Pi})^{-1}.$$

We have then

$$\Delta = -\hat{\Pi} - \hat{\Pi}^2 - \hat{\Pi}^3 - \cdots .$$

$$\hat{\Pi} = -\Delta - \Delta^2 - \Delta^3 - \cdots .$$

The following lemma by Rota [3] implies that $\Delta$ is a ring endomorphism of $\hat{\mathfrak{B}}(X)$.

Lemma 1. *Let $A$ be a commutative ring and $P$ an additive operator in $A$. Assume that $I - P$ is invertible and set $D = -P \cdot (I - P)^{-1}$. Then $P$ satisfies the Baxter identity if and only if $D$ is a ring endomorphism of $A$.*

The graph $\Gamma$ of $D$ is the set of pairs $(a - P(a), -P(a))$ in $A \times A$ for $a$ running over $A$. A pair $(x, y)$ is in $\Gamma$ if and only if $y - P(y) = -P(x)$ holds. Moreover $D$ is a ring endomorphism of $A$ if and only if $\Gamma$ is closed under multiplication in $A \times A$, that is, if and only if the following identity holds in $A$:

$$P(a) \cdot P(b) - P(P(a) \cdot P(b)) = -P((a - P(a)) \cdot (b - P(b))). \tag{24}$$

After simplifying, this goes over to

$$P(a) \cdot P(b) = -P(ab) + P(a \cdot P(b)) + P(b \cdot P(a)),$$

that is Baxter identity. Q.E.D.

We are now in a position to prove our main theorem. Let $\mathfrak{S}$ be the ring of formal power series without constant term with coefficients in $K$ in a family of indeterminates $T_{x,n}$ for $x$ in $X$ and $n \geqslant 0$ an integer.

THEOREM 2. *There exists an isomorphism of algebras from* $\mathfrak{S}$ *onto* $\hat{\mathfrak{B}}(X)$ *mapping* $T_{x,n}$ *into* $\Delta^n(x)$ *for* $x$ *in* $X$ *and* $n \geqslant 0$.

Let $D$ be the ring endomorphism of $\mathfrak{S}$ which maps a power series into the power series obtained by the substitutions $T_{x,n} \to T_{x,n+1}$. It is easy to see that the series $\sum_{n=0}^{\infty} D^n(f)$ converges for any $f$ in $\mathfrak{S}$, hence the additive operator $I - D$ in $\mathfrak{S}$ has an inverse equal to $\sum_{n=0}^{\infty} D^n$. Set $P = -D \cdot (I - D)^{-1}$. Then $I - P$ is invertible and $D = -P \cdot (I - P)^{-1}$. By Lemma 1 one concludes that $(\mathfrak{S}, P)$ is a Baxter algebra.

We identify any $x$ in $X$ with the indeterminate $T_{x,0}$ in $\mathfrak{S}$; hence $T_{x,n} = D^n(x)$. As before, let $M$ be the set of monomials on the elements of $X$. Then a monomial in $\mathfrak{S}$ can be uniquely written in one of the following forms : $u \neq 1$ in $M$; or $u \cdot D(u_1) \cdot D^2(u_2) \cdots D^p(u_p)$ with $u, u_1, \ldots, u_p$ in $M$ and $u_p \neq 1$.

Let us denote by $\{a_1, \ldots, a_n\}$ the $n$-bracket in the Baxter algebra $(\mathfrak{S}, P)$. By definition of $P$, one gets $P(a) = -\sum_{j=1}^{\infty} D^j(a)$ for any $a$ in $A$. By induction on $n$, one gets

$$\{a_1, \ldots, a_n\} = (-1)^n \sum_{1 \leqslant j_1 < \cdots < j_n} D^{j_1}(a_1) \cdots D^{j_n}(a_n) \tag{25}$$

for $a_1, \ldots, a_n$ in $\mathfrak{S}$.

Since $(\mathfrak{S}, P)$ is a Baxter algebra, there exists a unique homomorphism $\Phi : \mathfrak{B}(X) \to \mathfrak{S}$ such that $P\Phi = \Phi\Pi$ and $\Phi(x) = x$ for any $x$ in $X$. Therefore $\Phi$ maps $u \cdot [\ ]$ into $u$ for any monomial $u \neq 1$ in $M$. Define the degree of $T_{x,n}$ to be $n$ and the degree of a monomial in these indeterminates to be the sum of the degrees of its factors. Using (25) one shows that $\Phi$ maps the basic element $u \cdot [u_1, ..., u_p]$ into $(-1)^p u \cdot D(u_1) \cdots D^p(u_p) + R$ where $R$ is a sum of monomials each having a degree strictly greater than the degree of $u \cdot D(u_1) \cdots D^p(u_p)$. By an easy induction one concludes that any element in $\mathfrak{S}$ can be uniquely written as a linear combination (possibly infinite) of the images under $\Phi$ of the basic elements of $\mathfrak{B}(X)$.

Therefore $\Phi$ extends to an isomorphism $\hat{\Phi}$ of the algebra $\hat{\mathfrak{B}}(X)$ onto the algebra $\mathfrak{S}$. From $P\hat{\Phi} = \hat{\Phi}\hat{\Pi}$ one deduces $D\hat{\Phi} = \hat{\Phi}\Delta$ hence $D^n\hat{\Phi} = \hat{\Phi}\Delta^n$ for any integer $n \geqslant 0$ and $\hat{\Phi}$ maps therefore $\Delta^n(x)$ into $D^n(x) = T_{x,n}$. Q.E.D.

The map $\Phi$ of the previous proof induces an isomorphism of $\mathfrak{B}(X)$ onto a subalgebra of $\mathfrak{S}$ compatible with the corresponding Baxter operators. We have therefore the following general principle, first stated and proved by Rota [2, p. 328].

*Suppose we want to prove an identity of the form $F(a_1, ..., a_n) = 0$ to be valid for any choice of elements $a_1, ..., a_n$ in any Baxter algebra, where $F$ is formed using also the Baxter operator. It suffices to prove the following particular case. Let $\mathfrak{S}_n$ be the ring of power series without constant term in a sequence of indeterminates $X_{i,j}$ with $1 \leqslant i \leqslant n$ and $j \geqslant 0$. Let $D$ be the shift endomorphism in $\mathfrak{S}_n$ arising from the substitutions $X_{i,j} \to X_{i,j+1}$ and let $P = -D \cdot (I - D)^{-1}$. Then it suffices to prove the identity $F(X_{1,0}, ..., X_{n,0}) = 0$ in $\mathfrak{S}_n$.*

Formula (25) proves very helpful to evaluate the brackets in the Baxter algebra $(\mathfrak{S}_n, P)$. We leave as an exercise to the reader to rederive the identities in Section 2 along these lines.

## 6. Baxter Algebras and Symmetric Polynomials

We consider in more detail the free Baxter algebra $\mathfrak{B}_1$ in one generator $x$ and its completion $\mathfrak{S}_1$; we denote by $P$ the Baxter operator in $\mathfrak{S}_1$ unlike the notation $\hat{\Pi}$ in previous sections. We define the shift operator $D = -\sum_{n=1}^{\infty} P^n$ in $\mathfrak{S}_1$. We know that $\mathfrak{S}_1$ can be considered as the algebra of power series without constant term in the infinitely many indeterminates $T_n = D^n(x)$ (for $n \geqslant 0$).

Any element in $\mathfrak{S}_1$ can be uniquely written as an unrestricted linear combination of the symbols $x^j \cdot [\ ]$ for $j > 0$ and $x^j \cdot [x^{k_1},..., x^{k_p}]$ for $j \geqslant 0$, $p \geqslant 1$, $k_1 \geqslant 0,..., k_{p-1} \geqslant 0$, $k_p > 0$. Furthermore, $\mathfrak{B}_1$ consists of the finite linear combinations of the same symbols and one has

$$[x^{k_1},..., x^{k_p}] = (-1)^p \sum_{1 \leqslant i_1 < \cdots < i_p} T_{i_1}^{k_1} \cdots T_{i_p}^{k_p}. \tag{26}$$

according to (25).

Let $p \geqslant 1$ be an integer and let $U_p$ be the submodule of $\mathfrak{S}_1$ with basis the set of $p$-brackets $[x^{k_1},..., x^{k_p}]$ with $k_1 > 0,..., k_p > 0$. The symmetric group $S_p$ of order $p$ acts in an obvious way on the set of such symbols whence results a linear action of $S_p$ in $U_p$. Let $V_p$ be the submodule of $U_p$ consisting of all invariants of $S_p$ in $U_p$. A basis of $V_p$ as a $K$-module consists of the elements

$$M(d_1,..., d_p) = (-1)^p \sum [x^{k_1},..., x^{k_p}] \tag{27}$$

for $d_1 \geqslant \cdots \geqslant d_p > 0$, where the summation extends over all distinct rearrangements $k_1,..., k_p$ of $d_1,..., d_p$. The unrestricted direct sum $\mathfrak{B}$ of the submodules $V_p$ for $p \geqslant 1$ is called the *symmetric part* of $\mathfrak{S}_1$.

According to (26) and (27), $M(d_1,..., d_p)$ is the sum of all distinct monomials in the indeterminates $T_1$, $T_2$ and so on, containing $p$ distinct variables with exponents $d_1,..., d_p$. By construction $\mathfrak{B}$ is the set of all unrestricted linear combinations of the elements $M(d_1,..., d_p)$ for variable $p$. By known results $\mathfrak{B}$ *is nothing else than the set of symmetric power series in the infinitely many indeterminates* $T_1, T_2,..., T_n,...$ .

As an example,

$$M(\overbrace{1,..., 1}^{p}) = (-1)^p [\overbrace{x,..., x}^{p}]$$

is equal to the $p$-th elementary symmetric function $a_p$ of the $T_n$'s. Moreover, $M(p) = -P(x^p)$ is equal to $s_p = \sum_{n=1}^{\infty} T_n{}^p$, the $p$-th power sum.

Suppose that $K$ is a field of characteristic 0. Using the classical results, one sees that $\mathfrak{B}$ is the ring of power series in the indeterminates $a_1, a_2,...$ and also in the indeterminates $s_1, s_2,...$ . The two sets of indeterminates are related by the well-known *Waring's formula*

$$1 + \sum_{p=1}^{\infty} a_p U^p = \exp \sum_{n=1}^{\infty} (-1)^{n-1} s_n U^n/n, \tag{28}$$

where $U$ is another indeterminate.

If we replace $a_p$ and $s_p$ by their values in the free Baxter algebra $\mathfrak{B}_1$, we get from Waring's formula the following general version of *Spitzer's formula*

$$1 + P(x) \cdot U + P(x \cdot P(x)) \cdot U^2 + \cdots = \exp \sum_{n=1}^{\infty} P(x^n) \cdot U^n/n \qquad (29)$$

valid for any element $x$ in any Baxter algebra $(A, P)$. For another combinatorial proof of this identity, see [3].

## References


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