

Diffusion in Poro-Elastic Media

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Existence, uniqueness, and regularity theory is developed for a general initial-boundary-value problem for a system of partial differential equations which describes the Biot consolidation model in poro-elasticity as well as a coupled quasi-static problem in thermoelasticity. Additional effects of secondary consolidation and pore fluid exposure on the boundary are included. This quasi-static system is resolved as an application of the theory of linear degenerate evolution equations in Hilbert space, and this leads to a precise description of the dynamics of the system. © 2000 Academic Press

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1. INTRODUCTION

We shall consider a system modeling diffusion in an elastic medium in the case for which the inertia effects are negligible. This *quasi-static* assumption arises naturally in the classical Biot model of consolidation for a linearly elastic and porous solid which is saturated by a slightly compressible viscous fluid. The fluid *pressure* is denoted by $p(x, t)$ and the *displacement* of the structure by $\mathbf{u}(x, t)$. In the special case of a homogeneous and

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isotropic medium, the *poro-elasticity system* takes the form

$$\begin{aligned} -(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}(x, t)) - \mu\Delta\mathbf{u}(x, t) + \alpha\nabla p(x, t) &= \mathbf{f}(x, t), \\ \frac{\partial}{\partial t}(c_0 p(x, t) + \alpha\nabla \cdot \mathbf{u}(x, t)) - \nabla \cdot k\nabla p(x, t) &= h(x, t), \end{aligned} \quad (1.1)$$

consisting of the *equilibrium equation* for momentum and the *diffusion equation* for Darcy flow. General anisotropic elastic structures will be included in the development below. The constant $c_0 \geq 0$ combines *compressibility* and *porosity*; it is a measure of the amount of fluid which can be forced into the medium by pressure increments with constant volume. Similarly, $k > 0$ involves the *permeability* of the medium and the *viscosity* of the fluid as a measure of the Darcy flow corresponding to a pressure gradient. The term $\alpha\nabla p(x, t)$ results from the additional stress of the fluid pressure within the structure, and $\alpha\nabla \cdot \mathbf{u}(x, t)$ represents the additional fluid content due to the local volume change.

Our objective here is to develop the existence-uniqueness-regularity theory for these systems as an application of the theory of evolution equations in Hilbert space [13, 32, 34]. Thereby we will not only obtain sharp results on the appropriate spaces and definitions of solution, but we will obtain corresponding estimates directly from the abstract theory. Additionally, we extend the development to include new terms arising from secondary consolidation effects and from the exposure of the pore fluid on the boundary. Of special interest is to determine when the evolution is *holomorphic* (i.e., *parabolic*) or merely *strongly continuous* (i.e., *hyperbolic*). The former case leads to sharp estimates of order $O(\frac{1}{t})$, additional *regularity* of the solution, and a larger class of data for which the initial-value problem is well-posed. An interesting point for us is the behaviour of $\lim_{t \rightarrow 0^+} \mathbf{u}(t)$ and $\lim_{t \rightarrow 0^+} p(t)$, especially when some of the coefficients are null. It is precisely such degenerate cases which are of most interest in the applications. The system (1.1) will be reduced below to an evolution equation in Hilbert space in three distinct formulations. We include these various approaches in order to illustrate not only the various ways to construct the dynamics and to compare their respective results, but also to illustrate the advantages of each in the obvious generalizations and to compare our results with those obtained elsewhere.

We briefly review the associated *conservation laws* from which the poro-elasticity system (1.1) arises. The elastic structure Ω forms a porous and permeable matrix of density ρ , and it is saturated by a slightly compressible and viscous fluid which diffuses through it. For each subdomain $B \subset \Omega$, the momentum of the corresponding portion of the matrix is given by $\int_B \rho \frac{\partial \mathbf{u}(x, t)}{\partial t} dx$. The forces acting on the body B consist of the

traction forces applied by the complement of B across its boundary ∂B with normal \mathbf{n} , given by $\int_{\partial B} \sigma_{ij}(x, t) n_j dS$ where the *stress* σ_{ij} is the symmetric tensor that represents the internal forces on surface elements. Thus we obtain the equation for *balance of momentum*

$$\frac{\partial}{\partial t} \int_B \rho \frac{\partial \mathbf{u}(x, t)}{\partial t} dx = \int_{\partial B} \sigma(\cdot, t, \mathbf{n}) dS + \int_B \mathbf{f}(x, t) dx$$

for each subdomain B , where $\mathbf{f}(\cdot, t)$ denotes the volume-distributed external forces. The components of the *normal stress* $\sigma(\cdot, t, \mathbf{n})$ are given by $\sigma(\cdot, t, \mathbf{n})_i = \sigma_{ij}(\cdot, t) n_j$. With the divergence theorem this gives

$$\rho \frac{\partial^2 u_i(x, t)}{\partial t^2} - \partial_j \sigma_{ij}(x, t) = f_i(x, t), \quad 1 \leq i \leq 3.$$

The quantity of fluid in each such subdomain B is $\int_B \eta(x, t) dx$, and this defines the *fluid content* $\eta(x, t)$ of the medium. The *flux* is the mass flow rate $\mathbf{q}(x, t)$ of fluid relative to the matrix, so the rate at which fluid moves across the boundary ∂B is given by $\int_{\partial B} \mathbf{q}(x, t) \cdot \mathbf{n} dS$. Then the *conservation of mass* of fluid takes the integral form

$$\frac{\partial}{\partial t} \int_B \eta(x, t) dx + \int_{\partial B} \mathbf{q} \cdot \mathbf{n} dS = \int_B h(x, t) dx, \quad B \subset \Omega,$$

in which $h(\cdot, t)$ denotes any volume distributed *source density*. When the flux and content are differentiable, we obtain the differential form

$$\frac{\partial}{\partial t} \eta(x, t) + \nabla \cdot \mathbf{q}(x, t) = h(x, t), \quad x \in \Omega.$$

Any model of fluid flow through a deformable solid matrix must account for the coupling between the mechanical behavior of the matrix and the fluid dynamics. For example, compression of the medium leads to increased pore pressure, if the compression is fast relative to the fluid flow rate. Conversely, an increase in pore pressure induces a dilation of the matrix in response to the added stress. This coupled pressure-deformation interaction is the basis of the development of poro-elasticity starting with the work of Terzaghi [36] and Biot [7]. For the corresponding *constitutive equations*, we assume the total stress and fluid content are given respectively by

$$\begin{aligned} \sigma_{ij} &= \lambda \delta_{ij} \varepsilon_{kk}(\mathbf{u}) + 2\mu \varepsilon_{ij}(\mathbf{u}) - \alpha \delta_{ij} p, \\ \eta &= c_0 p + \alpha \nabla \cdot \mathbf{u}, \end{aligned}$$

where the small local *strain* of the solid is denoted by $\varepsilon_{kl}(\mathbf{u}) = \frac{1}{2}(\partial_k u_l + \partial_l u_k)$. The positive Lamé constants λ and μ are the *dilation* and *shear* moduli of elasticity, respectively. The coefficient $\alpha > 0$ is the *Biot–Willis constant* that accounts for the *pressure-deformation coupling*, and $c_0 \geq 0$ is the combined *porosity* of the medium and *compressibility* of the fluid. We also assume the *flux* \mathbf{q} is given by *Darcy’s law*

$$\mathbf{q} = -k\nabla p$$

for the diffusive flow through the medium. The constant k is the *hydraulic conductivity*, and it contains the permeability of the medium and the viscosity of the fluid. The momentum balance equations for the displacement of the medium and the mass balance equation for the pressure distribution are then given by the (fully dynamic) *classical Biot system*

$$\begin{aligned} \rho \ddot{\mathbf{u}} - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\Delta\mathbf{u} + \alpha\nabla p &= \mathbf{f}(x, t), \\ c_0 \dot{p} + \alpha\nabla \cdot \dot{\mathbf{u}} - \nabla \cdot k\nabla p &= h(x, t) \quad \text{in } \Omega. \end{aligned}$$

This is a system of mixed wave-parabolic type. The small deformations of the matrix are described by the Navier equations of linear elasticity, and the diffusive fluid flow is described by Duhamel’s equation. We note also that the system is formally equivalent to the classical coupled *thermo-elasticity system* which describes the flow of heat through an elastic structure. In that context, $p(x, t)$ denotes the *temperature*, $c_0 > 0$ is the *specific heat* of the medium, and $k > 0$ is the *conductivity*. Then $\alpha\nabla p(x, t)$ arises from the *thermal stress* in the structure, and the term $\alpha\nabla \cdot \frac{\partial \mathbf{u}(x, t)}{\partial t}$ corresponds to the *internal heating* due to the *dilation rate*. We have *not* made the uncoupling assumption in which this term is deleted from the diffusion equation.

Finally, we remark that in certain models of secondary consolidation in clays [23] there arises an additional term in the total stress of the form

$$\lambda^* \delta_{ij} \varepsilon_{kk} \left(\frac{\partial \mathbf{u}}{\partial t} \right).$$

This leads to the *fully dynamic coupled system* of mixed hyperbolic-parabolic type

$$\begin{aligned} \rho \frac{\partial^2 \mathbf{u}(t)}{\partial t^2} - \lambda^* \nabla \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}(t)) + \mathcal{E}(\mathbf{u}(t)) + \alpha \nabla p(t) &= \mathbf{f}(t), \\ \frac{\partial}{\partial t} (c_0 p(t) + \alpha \nabla \cdot \mathbf{u}(t)) + A(p(t)) &= h(t). \end{aligned} \tag{1.2}$$

Here we have denoted by \mathcal{E} the elasticity operator, and A is the diffusion operator. For the consolidation problem, we shall study various initial-boundary-value problems for (1.2) with $\lambda^* \geq 0$, $c_0 \geq 0$ in the quasi-static case with $\rho = 0$.

Earlier works on the system (1.1) have been concerned with boundary conditions of either the very simplest (Dirichlet) type on both displacement and pressure, i.e., portions of the boundary which are *clamped* or *drained*, respectively, or a combination of these with the complementary boundary conditions involving *traction* forces and *flux*. These complementary boundary conditions are contained in the variational forms associated with the elasticity operator and the diffusion operator, respectively. They can be mixed naturally by prescribing two independent partitions of the boundary, one which determines those portions of the boundary on which displacement and tractions are prescribed, and the other on which pressure and flux are prescribed. Here we have included a general class of boundary conditions which obtain all of these. Moreover, we introduce another feature in order to designate the fraction of the matrix pores which are *sealed* along the boundary. The complement consists of those for which the diffusive flow paths are directly *exposed* by the boundary. On the sealed portion one prescribes the classical total traction on the combined elastic stress and internal fluid pressure, while on the exposed portion of the boundary only the matrix structure is directly supported. Thereby we obtain a new boundary condition in which only the elastic component of traction or *effective stress* is prescribed on one part, and on the complement there is a corresponding contribution to the flux. The complementary portions will be determined through Stokes' formula by the gradient and divergence operators in the coupling of the system. These general boundary conditions are displayed explicitly below in (3.2) and described in more detail immediately thereafter.

In this work we restrict our considerations to the linear quasi-static case ($\rho = 0$), and we develop the simplest and most direct connections with the classical theory of semigroups [21], i.e., the case of those generated by linear *single-valued* operators in Hilbert space. In dealing with functional equations with operators which may be degenerate, it frequently occurs, e.g., from the use of inverse operators, that one is required to consider *multi-valued* operators. This is not only a natural consideration from the aspect of degenerate operators, but it is also well known to be the natural framework for the nonlinear *monotone* operators; these are most effectively regarded as *multi-valued*. We shall develop this approach in a following work, in which the operators will be permitted to be both degenerate and nonlinear. Then the *multi-valued* case is used effectively to handle these generalizations and the *fully dynamic* hyperbolic-parabolic case (1.2) simultaneously [34]. Although one can develop an intermediate

theory for *linear multi-valued* operators, or *generalized inverses*, and this could be of some interest for the regularity theory of these problems, we prefer for purposes of exposition to describe the simplest cases here and then go to the most general case of *multi-valued nonlinear* operators in the following work. In order to remain within the classical framework below, it will be necessary to verify a *kernel condition* in order to force a natural composition of operators to be *single-valued*. It is rather remarkable that this artificial condition is fulfilled in our applications here.

Our plane is as follows. Section 2 contains a description of appropriate Lebesgue and Sobolev spaces and the construction of the differential operators that represent the elasticity system, the diffusion equation, and the divergence and gradient operators that occur in the coupling [1, 15, 19, 35]. Section 3 begins with an explicit description of the initial-boundary-value problem, and there we prove that the quasi-static case is a well-posed parabolic problem. The *strong solution* obtained there is sufficiently regular that both the momentum and the diffusion equations can be decoupled into a system of partial differential equations together with boundary conditions. In Section 4 we prove existence and uniqueness of a *weak solution* for which only the equilibrium equation is so regular, and in the parabolic case the problem is resolved with a more general class of data. These results are obtained again in Section 5 by an entirely independent method which also permits the inclusion of the case of secondary consolidation $\lambda^* \geq 0$. There we find that $\lambda^* > 0$ has a *deregularizing effect* for the momentum equation similar to that of $c_0 > 0$ for the diffusion equation. Section 6 is a self-contained presentation of the theory of implicit degenerate evolution equations in Hilbert space based solely on the classical semigroup theory as presented in [21], for example. All of our results are obtained from the theorems of this section, so for logical reasons it should be read before Section 3. However, it has been placed at the end as an appendix in order to direct attention to the primary objective, the systems (1.1) and (1.2). (Theorem 6.3 and Theorem 6.4 are not essential, since they are used only to obtain alternate proofs of Theorem 4.1 and Theorem 4.2.) The development in Section 6 is a brief alternative to earlier works on implicit evolution equations, e.g., [13, 32]. One could also consult the latter reference for an expanded discussion of the abstract Green's theorem [33] that is used to make precise the decoupling of a functional equation into a partial differential operator and the complementary boundary operator. This would avoid the necessity of using the regularity theory for strongly elliptic systems.

Remark on the Literature. For the theory of the system (1.2) with $\rho > 0$ and $\lambda^* = 0$, and especially its relation to thermo-elasticity, see the fundamental work of Dafermos [16], the exhaustive and complementary accounts of Carlson [12] and Kupradze [22], and the development in the

context of strongly elliptic systems by Fichera [20]. By contrast, very few references were to be found in the thermoelasticity literature for well-posedness of even the simplest linear problem for the *coupled quasi-static* case of (1.1) in which the system degenerates to a mixed elliptic-parabolic type. Such a system in one spatial dimension is developed by classical methods in the book of Day [17]. There is a collection of papers on the nonlinear contact problem for coupled quasi-static thermoelasticity in one dimension [2], and there is one paper [31] for the problem on the 2D disk in which the displacement is decoupled to get a single integral-differential equation. Recently the existence of a solution for the general N dimensional contact problem was given in Shi and Shillor [30] under the assumption that the coupling coefficient is sufficiently small. This “smallness” condition was removed and more realistic boundary conditions were included by Xu [39]. According to a scaling argument in Boley and Wiener [10], it appears that the reasons for taking $\rho = 0$ apply as well to simultaneously delete the term $\alpha \nabla \cdot \dot{\mathbf{u}}(t)$ and thereby uncouple the system, so these two assumptions are frequently taken together. This may explain in part the limited attention given to this case until recently in the thermoelasticity literature; also see Esham and Weinacht [18] for the behaviour as $\rho \rightarrow 0$.

The poro-elasticity consolidation model of Biot [7] requires the quasi-static case, $\rho = 0$. An additional degeneracy occurs in the *incompressible* case in which we have also $c_0 = 0$, and then the system is formally of elliptic type. The mathematical issues of well-posedness for the quasi-static case were first studied in the fundamental work of Auriault and Sanchez-Palencia (1977) [3]. They derived a non-isotropic form of the Biot system (1.1) by homogenization and then obtained a *strong* solution. In the later paper of Zenisek [41] the *weak* solution was obtained in the first order Sobolev space $H^1(\Omega)$, so the equations hold in the dual space, $H^{-1}(\Omega)$ (see below). Additional issues of analysis and approximation of this case were developed in [11, 24–26, 28, 29, 40, 42]. Problems with secondary consolidation ($\lambda^* > 0$) were introduced and developed by Cushman and Murad [23]. There are a number of works on similar processes in composite media in which one has parallel systems of fluid which interact with each other as well as with the structure. These *multiporosity* or *multipermeability* systems are of the form

$$\begin{aligned} \mathcal{E}(\mathbf{u}(t)) + \alpha_1 \nabla p_1(t) + \alpha_2 \nabla p_2(t) &= \mathbf{0}, \\ c_1 \dot{p}_1(t) + A_1 \nabla p_1(t) + \alpha_1 \nabla \cdot \dot{\mathbf{u}}(t) + \gamma(p_1(t) - p_2(t)) &= h_1(t), \\ c_2 \dot{p}_2(t) + A_2 \nabla p_2(t) + \alpha_2 \nabla \cdot \dot{\mathbf{u}}(t) + \gamma(p_2(t) - p_1(t)) &= h_2(t), \end{aligned} \quad (1.3)$$

and they can be used to model *fissured media* [4–6, 37, 38].

Finally, we mention in passing the *uncoupled quasi-static case* which is obtained from (1.1) in the context of thermoelasticity by also deleting the dilation rate $\alpha \nabla \cdot \frac{\partial \mathbf{u}(x,t)}{\partial t}$ from the energy equation. Then one can solve for the independent temperature field and use this as data in the elasticity equation with time as a parameter for which most results of the equilibrium theory carry over [12, 14, 27]. Although this decoupling assumption is appropriate in many thermoelasticity applications, it is *never* permissible for the consolidation problems of poro-elasticity [5, 8, 9].

2. PRELIMINARIES

Sobolev Spaces

We describe the spaces which will be used to develop the variational formulation of the system. Let Ω be a smoothly bounded region in \mathbb{R}^3 and denote its boundary by $\Gamma = \partial\Omega$. Denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with support contained in Ω and by $L^2(\Omega)$ the Lebesgue space of (equivalence classes of) functions whose modulus squared is integrable on Ω . For any $w(\cdot) \in L^2(\Omega)$ and $j, 1 \leq j \leq 3$, we denote by $\partial_j w$ its distributional derivative,

$$\langle \partial_j w, \varphi \rangle = - \int_{\Omega} w(x) \overline{\partial_j \varphi(x)} dx, \quad \varphi \in C_0^\infty(\Omega).$$

Let $H^k(\Omega)$ be the *Sobolev space* consisting of those functions in $L^2(\Omega)$ having each of their partial derivatives through order k also in $L^2(\Omega)$. The *trace map* $\gamma : H^1(\Omega) \rightarrow L^2(\Gamma)$ is the restriction to the boundary Γ denoted by $\gamma(w) = w|_{\Gamma}$; we shall denote the range of this map by $\gamma(V) = \text{Rg}(\gamma) = H^{\frac{1}{2}}(\Gamma)$. The space $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$, and it is characterized as the subspace of $H^1(\Omega)$ consisting of those functions whose trace is zero. The dual of $H_0^1(\Omega)$ is the space $H^{-1}(\Omega)$ of distributions on Ω which are first order derivatives of functions in $L^2(\Omega)$. We shall also use the quotient space $L^2(\Omega)/\mathbb{R}$ with the norm $\inf_{c \in \mathbb{R}} \|p + c\|_{L^2}$. Corresponding spaces of vector-valued functions will be denoted by bold-face symbols. For example, we denote the product space $L^2(\Omega)^3$ by $\mathbf{L}^2(\Omega)$ and the corresponding triple of Sobolev spaces by $\mathbf{H}^1(\Omega) \equiv H^1(\Omega)^3$. Additional information on these spaces will be recalled from [1, 35] as needed.

The Elasticity Operator. The (small) *displacement* $\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x))$ from the position $x \in \Omega$ gives the (linearized) *strain tensor* $\varepsilon_{kl}(\mathbf{u}) \equiv \frac{1}{2}(\partial_k u_l + \partial_l u_k)$ which provides a measure of the local deformation of the body. We shall assume that this determines the *stress tensor* σ_{ij}

by means of the *generalized Hooke's law*

$$\sigma_{ij} = a_{ijkl} \varepsilon_{kl}(\mathbf{u}).$$

The positive definite symmetric *elasticity* tensor a_{ijkl} provides a model for general anisotropic materials. Let Γ_0 and Γ_t be specified complementary parts of the boundary $\Gamma = \partial\Omega$. The general stationary elasticity system is given by the equations of equilibrium

$$\begin{aligned} -\partial_j \sigma_{ij} &= f_i & \text{in } \Omega, \\ u_i &= 0 \text{ on } \Gamma_0, & \sigma_{ij} n_j = g_i \text{ on } \Gamma_t, \end{aligned} \quad (2.1)$$

for each $1 \leq i \leq 3$. The boundary conditions correspond to a *clamped* portion Γ_0 and a constraint on the complement Γ_t which involves the surface density of forces or *traction* $\sigma_{ij} n_j$ determined by the unit outward normal vector $\mathbf{n} = (n_1, n_2, n_3)$ on Γ .

We shall seek a weak solution \mathbf{u} in the *complex* Sobolev product space

$$\mathbf{V} \equiv \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = 0 \text{ on } \Gamma_0\}.$$

(This space could be further delimited by additional linear constraints.) Multiply the system equations by the complex conjugates of the respective components of a $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and use Stokes' theorem to integrate by parts and obtain

$$\int_{\Omega} a_{ijkl} \varepsilon_{kl}(\mathbf{u}) \overline{\partial_j v_i} dx = \int_{\Omega} f_i \overline{v_i} dx + \int_{\Gamma} a_{ijkl} \varepsilon_{kl}(\mathbf{u}) n_j \overline{v_i} ds.$$

From the symmetry $a_{ijkl} = a_{jikl}$ and boundary condition $v_i = 0$ on Γ_0 we obtain the weak formulation of the problem,

$$\mathbf{u} \in \mathbf{V} : \int_{\Omega} a_{ijkl} \varepsilon_{kl}(\mathbf{u}) \overline{\varepsilon_{ij}(\mathbf{v})} dx = \int_{\Omega} f_i \overline{v_i} dx + \int_{\Gamma_t} g_i \overline{v_i} ds,$$

for all $\mathbf{v} \in \mathbf{V}$. This is of the form

$$\mathbf{u} \in \mathbf{V} : e(\mathbf{u}, \mathbf{v}) = \mathbf{f}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}$$

with the appropriate definition of the sesquilinear form $e(\cdot, \cdot)$ and the conjugate-linear functional $\mathbf{f}(\cdot)$ on the Hilbert space \mathbf{V} . Hereafter we denote the resulting *elasticity operator* above by $\mathcal{E}(\mathbf{u}) = \mathbf{f}$. That is, $\mathcal{E} : \mathbf{V} \rightarrow \mathbf{V}'$ is the linear operator determined by the sesquilinear form $e(\cdot, \cdot)$ on \mathbf{V} . Assume hereafter that $\text{meas}(\Gamma_0) > 0$; it follows from *Korn's inequality* that this form is coercive, and then \mathcal{E} is an isomorphism. We can recover the

boundary-value problem (2.1) from \mathcal{E} as follows. For $\mathbf{u} \in \mathbf{V}$ we define the restriction to $\mathbf{C}_0^\infty(\Omega)$ of $\mathcal{E}(\mathbf{u}) \in \mathbf{V}'$ by $\mathcal{E}_0(\mathbf{u})$. Then we find that

$$\mathcal{E}_0(\mathbf{u})_i = -\partial_j a_{ijkl} \varepsilon_{kl}(\mathbf{u}) \quad \text{in } H^{-1}(\Omega), \quad 1 \leq i \leq 3.$$

If the closures of Γ_0 and Γ_t do not intersect, and if the boundary is sufficiently smooth, then the regularity theory for strongly elliptic systems shows that whenever $\mathcal{E}_0(\mathbf{u}) \in \mathbf{L}^2(\Omega)$ we have $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$ and then

$$\mathcal{E}(\mathbf{u})(\mathbf{v}) = (\mathcal{E}_0(\mathbf{u}), \mathbf{v})_{\mathbf{L}^2(\Omega)} + (a_{ijkl} \varepsilon_{kl}(\mathbf{u}) n_j, v_i)_{L^2(\Gamma_t)}, \quad \mathbf{v} \in \mathbf{V}. \quad (2.2)$$

This shows how \mathcal{E} decouples into the sum of its *formal* part \mathcal{E}_0 on Ω and its boundary part $\sigma_{ij} n_j$ on Γ_t . See [15, 19] for such regularity results on \mathcal{E} .

For the special case of an *isotropic medium* the elasticity tensor is given by

$$a_{ijkl} \equiv \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

with the positive Lamé constants λ and μ , and the stress takes the form

$$\sigma_{ij} = a_{ijkl} \varepsilon_{kl} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}.$$

Thus the weak form of the elasticity system is given in the isotropic case by

$$\int_{\Omega} (\lambda (\partial_k u_k) \overline{(\partial_j v_j)} + 2\mu \varepsilon_{ij}(\mathbf{u}) \overline{\varepsilon_{ij}(\mathbf{v})}) dx = \int_{\Omega} \mathbf{f} \cdot \bar{\mathbf{v}} dx + \int_{\Gamma_t} \mathbf{g} \cdot \bar{\mathbf{v}} ds.$$

If we write out in components the partial differential equations, we have

$$-\partial_j (\lambda \delta_{ij} (\partial_k u_k) + \mu (\partial_i u_j + \partial_j u_i)) = f_i$$

or

$$-(\lambda + \mu) \partial_i (\partial_k u_k) - \mu \Delta u_i = f_i,$$

and in vector form we have

$$-(\lambda + \mu) \nabla (\partial_k u_k) - \mu \Delta \mathbf{u} = \mathbf{f}.$$

The Diffusion Operator. Let Γ_1 and Γ_f be a second partition of the boundary Γ into complementary parts. The *diffusion system* is the initial-boundary-value problem

$$\begin{aligned} \frac{\partial}{\partial t} (c_0 p) - \partial_j (k \partial_j p) &= h_0 \quad \text{in } \Omega \\ p &= 0 \text{ on } \Gamma_1, \quad k \partial_j p n_j = h_1 \text{ on } \Gamma_f, \end{aligned} \quad (2.3)$$

in which Γ_1 is the *drained* portion of the boundary on which the pressure is set equal to zero, and the normal component of the *flux* $\mathbf{q} = -k\nabla p$ is prescribed on Γ_f . In order to prescribe the corresponding weak formulation of this problem, we set

$$\mathbf{V} \equiv \{p \in H^1(\Omega) : p = 0 \text{ on } \Gamma_1\}.$$

Then $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, and we define the sesquilinear form

$$A(p)(q) = \int_{\Omega} k\nabla p \cdot \overline{\nabla q} \, dx, \quad p, q \in V,$$

which gives a symmetric and monotone operator $A: V \rightarrow V'$. The *formal* part is defined to be the restriction of $A(p)$ to $C_0^\infty(\Omega)$, and it is given in $H^{-1}(\Omega)$ by the *Laplace* operator, $A_0 p = -\partial_j(k\partial_j p)$ for $p \in V$. If additionally $A_0 p \in L^2(\Omega)$, then the elliptic regularity theory implies that $p \in V \cap H^2(\Omega)$, and then Stokes' theorem yields

$$A(p)(q) = (A_0 p, q)_{L^2(\Omega)} + (k\partial p/\partial n, \gamma(q))_{L^2(\Gamma_f)}, \quad q \in V. \quad (2.4)$$

This provides the decoupling of A into a *formal* part on Ω and a boundary operator corresponding to *flux* on Γ_f , and we denote this representation by

$$A(p) = [A_0(p), k\partial p/\partial n] \in L^2(\Omega) \oplus L^2(\Gamma_f).$$

Finally, we note that the symmetric operator $A: V \rightarrow V'$ has the kernel $\text{Ker}(A) = \text{Ker}(\nabla)$ and closed range $\text{Rg}(A) = \text{Ker}(A)^\perp$, the indicated *annihilator* in V' . We shall denote the range of A by V'_a .

Coupling Terms. Denote by Γ_s that portion of the boundary on which neither pressure nor displacement are specified, i.e., $\Gamma_s = \Gamma_t \cap \Gamma_f$. Let the function $\beta(\cdot) \in L^\infty(\Gamma_s)$ be given; we shall assume that $0 \leq \beta(s) \leq 1$, $s \in \Gamma_s$. Then define the corresponding *gradient* operator, $\vec{\nabla}: V \rightarrow \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_s)$, by

$$\langle \vec{\nabla} p, [\mathbf{f}, \mathbf{g}] \rangle \equiv \int_{\Omega} \partial_j p \bar{f}_j \, dx - \int_{\Gamma_s} \beta p n_j \bar{g}_j \, ds, \quad (2.5)$$

$$p \in V, [\mathbf{f}, \mathbf{g}] \in \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_s).$$

This consists of the formal gradient ∇p in Ω and the boundary part $-\beta p \mathbf{n}$ on Γ_s , and we denote this representation by

$$\vec{\nabla} = [\nabla, -\beta p \mathbf{n}].$$

Define $\vec{\nabla} \cdot : \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_S) \rightarrow V'$ to be the negative of the corresponding dual operator. This is the *divergence* operator $\vec{\nabla} \cdot = -\vec{\nabla}'$ given by

$$\langle \vec{\nabla}[\mathbf{f}, \mathbf{g}], p \rangle \equiv -\overline{\langle \vec{\nabla} p, [\mathbf{f}, \mathbf{g}] \rangle}, \quad [\mathbf{f}, \mathbf{g}] \in \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_S), p \in V.$$

Each of these operators has closed range given as the annihilator of the kernel of the other.

The trace map gives a natural identification $\mathbf{v} \mapsto [\mathbf{v}, \gamma(\mathbf{v})|_{\Gamma_S}]$ of

$$\mathbf{V} \subset \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_S),$$

and this identification will be employed throughout the following. It also gives the identification $p \mapsto [p, \gamma(p)|_{\Gamma_S}]$ of

$$V \subset L^2(\Omega) \oplus L^2(\Gamma_S).$$

We note that both of these identifications have dense range, and so the corresponding duals can be identified. That is, we have

$$\mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_S) \subset \mathbf{V}', \quad L^2(\Omega) \oplus L^2(\Gamma_S) \subset V'.$$

These density conditions result from the respective requirements that $\Gamma_S \subset \Gamma_t$ and $\Gamma_S \subset \Gamma_f$.

For smoother functions $\mathbf{v} \in \mathbf{V} \subset \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_S)$ we have the *Stokes' Formula*

$$\begin{aligned} \langle \vec{\nabla} \cdot \mathbf{v}, p \rangle &= - \int_{\Omega} \overline{\partial_j p v_j} dx + \int_{\Gamma_S} \beta \bar{p} v_j n_j ds \\ &= \int_{\Omega} \partial_j v_j \bar{p} dx - \int_{\Gamma_S} (1 - \beta) \mathbf{v} \cdot \mathbf{n} \bar{p} ds, \quad p \in V. \end{aligned} \tag{2.6}$$

This shows the restriction satisfies

$$\vec{\nabla} \cdot : \mathbf{V} \rightarrow L^2(\Omega) \oplus L^2(\Gamma_S)$$

and that the divergence operator has a *formal* part in Ω as well as a boundary part on Γ_S . We denote the part in $L^2(\Omega)$ by $\nabla \cdot$, that is, $\nabla \cdot \mathbf{v} = \partial_j v_j$, and the identity (2.6) is indicated by

$$\vec{\nabla} \cdot \mathbf{v} = [\nabla \cdot \mathbf{v}, -(1 - \beta) \mathbf{v} \cdot \mathbf{n}] \in L^2(\Omega) \oplus L^2(\Gamma_S), \quad \mathbf{v} \in \mathbf{V}.$$

Now we can extend the definition of $\vec{\nabla}$ from V to $L^2(\Omega) \oplus L^2(\Gamma_S)$. This extension is obtained as $-(\vec{\nabla} \cdot)'$, the negative of the dual of the restriction to \mathbf{V} of the divergence. This dual operator

$$(\vec{\nabla} \cdot)': L^2(\Omega) \oplus L^2(\Gamma_S) \rightarrow \mathbf{V}'$$

is defined for each $[f, g] \in L^2(\Omega) \oplus L^2(\Gamma_S)$ by

$$\begin{aligned} \langle (\vec{\nabla} \cdot)' [f, g], \mathbf{v} \rangle &= \overline{(\vec{\nabla} \cdot \mathbf{v}, [f, g])}_{L^2(\Omega) \oplus L^2(\Gamma_S)} \\ &= \overline{(\partial_j v_j, f)}_{L^2(\Omega)} - \overline{((1 - \beta) \mathbf{v} \cdot \mathbf{n}, g)}_{L^2(\Gamma_S)} \\ &= (f, \nabla \cdot \mathbf{v})_{L^2(\Omega)} - (g, (1 - \beta) \mathbf{v} \cdot \mathbf{n})_{L^2(\Gamma_S)}, \quad \mathbf{v} \in \mathbf{V}. \end{aligned}$$

For the smoother case of $[f, g] = [w, w|_{\Gamma_S}]$, the indicated $w \in V$ identified as a function on Ω and its trace on Γ_S , the Stokes' formula shows that

$$\begin{aligned} -\langle (\vec{\nabla} \cdot)' [w, w|_{\Gamma_S}], \mathbf{v} \rangle &= -(w, \nabla \cdot \mathbf{v})_{L^2(\Omega)} + (w, (1 - \beta) \mathbf{v} \cdot \mathbf{n})_{L^2(\Gamma_S)} \\ &= (\partial_j w, v_j)_{L^2(\Omega)} - (\beta w, \mathbf{v} \cdot \mathbf{n})_{L^2(\Gamma_S)} \\ &= (\vec{\nabla} w, \mathbf{v})_{L^2(\Omega) \oplus L^2(\Gamma_S)}, \quad w \in V, \mathbf{v} \in \mathbf{V}, \end{aligned}$$

and this shows that it provides the desired extension of $\vec{\nabla}$ from V to $L^2(\Omega) \oplus L^2(\Gamma_S)$. Note that by taking $[f, g] = \vec{\nabla} \cdot \mathbf{v} = [\nabla \cdot \mathbf{v}, -(1 - \beta) \mathbf{v} \cdot \mathbf{n}]$ above, we obtain

$$\begin{aligned} \langle (\vec{\nabla} \cdot)' \vec{\nabla} \cdot \mathbf{v}, \mathbf{w} \rangle &= (\vec{\nabla} \cdot \mathbf{v}, \vec{\nabla} \cdot \mathbf{w})_{L^2(\Omega) \oplus L^2(\Gamma_S)} \\ &= (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{w})_{L^2(\Omega)} + ((1 - \beta) \mathbf{v} \cdot \mathbf{n}, (1 - \beta) \mathbf{w} \cdot \mathbf{n})_{L^2(\Gamma_S)}, \\ & \hspace{20em} \mathbf{v}, \mathbf{w} \in \mathbf{V}. \end{aligned}$$

The preceding constructions are summarized in the diagram

$$\begin{array}{ccc} L^2(\Omega) \oplus L^2(\Gamma_S) & \xrightarrow{\vec{\nabla} \cdot = -\vec{\nabla}'} & V' \\ \cup & & \cup \\ \mathbf{V} & \xrightarrow{\vec{\nabla} \cdot} & L^2(\Omega) \oplus L^2(\Gamma_S) \xrightarrow{\vec{\nabla} \cdot = -(\vec{\nabla} \cdot)'} \mathbf{V}' \\ & & \cup \\ & & V \xrightarrow{\vec{\nabla}} L^2(\Omega) \oplus L^2(\Gamma_S) \end{array}$$

It will be necessary to characterize the kernel of both the gradient operator $\vec{\nabla}$ and the formal gradient $\nabla: V \rightarrow L^2(\Omega)$. Recall that if $\nabla p = 0$ in $\mathbf{H}^{-1}(\Omega)$, then $p(x) = c$, a constant, for $x \in \Omega$. Let χ_Ω denote the characteristic function of the set Ω .

LEMMA 2.1. $\text{Ker}(\nabla) = \mathbb{R} \chi_\Omega$ if $\text{meas}(\Gamma_1) = 0$, and $\text{Ker}(\nabla) = \{0\}$ if $\text{meas}(\Gamma_1) > 0$.

Suppose that $[f, g] \in \text{Ker}(\vec{\nabla})$. Then we find first that $f(x) = c$ from above, and then from Stoke's theorem that

$$\int_{\Gamma_1 - \Gamma_S} c \mathbf{v} \cdot \mathbf{n} \, ds + \int_{\Gamma_S} (c - (1 - \beta)g) \mathbf{v} \cdot \mathbf{n} \, ds = 0$$

for all $\mathbf{v} \in \mathbf{V}$. Therefore, we have

$$f(x) = c, x \in \Omega, \quad c = (1 - \beta(s))g(s), s \in \Gamma_s, \\ c \cdot \text{meas}(\Gamma_t - \Gamma_s) = 0.$$

For the smoother case of $[f, g] = [w, w|_{\Gamma_s}] \in V \cap \text{Ker}(\vec{\nabla})$, we have

$$w(x) = c, x \in \Omega, \quad c\beta(s) = 0, s \in \Gamma_s, \\ c(\text{meas}(\Gamma_t - \Gamma_s) + \text{meas}(\Gamma_1)) = 0.$$

LEMMA 2.2. $\text{Ker}(\vec{\nabla}) \cap V = \mathbb{R}\chi_\Omega$ if $\text{meas}(\Gamma_1) = 0$ and $\beta \equiv 0$. Otherwise, $\text{Ker}(\vec{\nabla}) \cap V = \{0\}$.

Proof. Note that if $\text{meas}(\Gamma_1) = 0$, then $\text{meas}(\Gamma_t - \Gamma_s) = 0$. Also, if $\text{meas}(\Gamma_s) = 0$, then $\beta \equiv 0$ is vacuously satisfied.

The Quasi-Static System

In terms of the operators introduced above, the quasi-static case of the system (1.2) for diffusion in a general elastic medium takes the form

$$-\lambda^* \vec{\nabla} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \mathbf{u}(t)) + \mathcal{E}(\mathbf{u}(t)) + \alpha \vec{\nabla} p(t) = \mathbf{f}(t), \\ \frac{\partial}{\partial t} (c_0 Pp(t) + \alpha \vec{\nabla} \cdot \mathbf{u}(t)) + Ap(t) = h(t), \tag{2.7}$$

in which $P : L^2(\Omega) \oplus L^2(\Gamma_d) \rightarrow L^2(\Omega) \oplus \{0\}$ is the indicated *projection* operator. This system consists of the *equilibrium equation* for momentum and the *energy equation* for diffusion. The first holds in the space \mathbf{V}' and the second equation in V' . When $\lambda^* > 0$, the first of these equations is of *implicit* or *Sobolev* type, and for $\lambda^* = 0$ it is elliptic. The second is parabolic when $c_0 > 0$ and elliptic when $c_0 = 0$. These equations include the system (1.1) together with boundary conditions, and these are given in (3.2) below. The forcing terms $\mathbf{f}(\cdot)$ and $h(\cdot)$ represent any externally applied forces and fluid sources, respectively. In the usual case with $\lambda^* = 0$, we can eliminate the nonhomogeneous term $\mathbf{f}(t)$ from this system by a simple translation. For each $t \geq 0$, let $\mathbf{u}_0(t)$ be the solution of the stationary elasticity problem

$$\mathcal{E}(\mathbf{u}_0(t)) = \mathbf{f}(t),$$

and change variables in the above, i.e., replace $\mathbf{u}(t)$ by $\mathbf{u}(t) + \mathbf{u}_0(t)$, to obtain the equivalent system

$$\mathcal{E}(\mathbf{u}(t)) + \alpha \vec{\nabla} p(t) = \mathbf{0}, \\ \frac{\partial}{\partial t} (c_0 Pp(t) + \alpha \vec{\nabla} \cdot \mathbf{u}(t)) + Ap(t) = h(t) - \alpha \vec{\nabla} \cdot \dot{\mathbf{u}}_0(t). \tag{2.8}$$

Then rename $h(t)$ to be $h(t) - \alpha \vec{\nabla} \cdot \dot{\mathbf{u}}_0(t)$. Thus, any non-homogeneous internal or boundary distributed stresses can be replaced by corresponding null data.

The pressure-deformation coupling involves the Biot–Willis constant α . This is given by $\alpha = 1 - K/K_s$, where K_s is the bulk modulus of the solid material and K the bulk modulus of the porous matrix, so it follows that $\alpha \approx 1$ in most situations. Hereafter we shall set $\alpha = 1$. This corresponds to an incompressible matrix material, and in the general case we can obtain this condition by rescaling the remaining constants.

3. THE STRONG SOLUTION

Here we consider the Cauchy problem for the degenerate quasi-static system

$$\mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) = \mathbf{0}, \quad (3.1.a)$$

$$\frac{d}{dt} (c_0 P p(t) + \vec{\nabla} \cdot \mathbf{u}(t)) + A(p(t)) = h(t), \quad (3.1.b)$$

$$c_0 P p(0) + \vec{\nabla} \cdot \mathbf{u}(0) = [v_0, -v_1], \quad (3.1.c)$$

and we show that it is essentially a *parabolic* system which has a *strong solution* under minimal smoothness requirements on the initial data $[v_0, -v_1]$ and source term $h(\cdot)$. Note that (3.1.b) requires that $p(t) \in V$, so both terms of (3.1.a) necessarily belong to $L^2(\Omega) \oplus L^2(\Gamma_t)$, and this forces additional regularity on $\mathbf{u}(t)$. Equation (3.1.b) holds in V' , so it contains a complementary boundary condition on Γ_f . For the solution obtained in Theorem 3.1 below, each term of Eq. (3.1.b) belongs to the smaller space $L^2(\Omega) \oplus L^2(\Gamma_f)$, so it has the additional regularity that is required to decouple it into a partial differential equation and boundary conditions. This is the sense in which the solution is *strong*.

We display the system (3.1) explicitly decoupled into its parts as an initial-boundary-value problem for the system of partial differential equations and boundary conditions. Let each $h(t) \equiv [h_0(t), h_1(t)]$ for $t \in (0, T]$ and the initial functions $[v_0, v_1]$ be given in $L^2(\Omega) \oplus L^2(\Gamma_S)$. Denote by χ_S the characteristic function of Γ_S . In the notation introduced in Section 2, the strong solution takes the form of the *initial-boundary-value problem*

$$\mathcal{E}_0(\mathbf{u}(t)) + \nabla p(t) = \mathbf{0} \quad (3.2.a)$$

and

$$\frac{\partial}{\partial t}(c_0 p(t) + \nabla \cdot \mathbf{u}(t)) + A_0(p(t)) = h_0(t) \quad \text{in } \Omega, \quad (3.2.b)$$

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_0, \quad a_{ijkl} \varepsilon_{kl}(\mathbf{u}(t)) n_j - \beta p(t) n_i \chi_S = 0, \quad 1 \leq i \leq 3, \text{ on } \Gamma_t, \quad (3.2.c)$$

$$p(t) = 0 \text{ on } \Gamma_1,$$

$$-\frac{\partial}{\partial t}((1 - \beta)\mathbf{u}(t) \cdot \mathbf{n}) \chi_S + k \frac{\partial p(t)}{\partial n} = h_1(t) \chi_S \text{ on } \Gamma_f, \quad (3.2.d)$$

$$\lim_{t \rightarrow 0^+} (c_0 p(t) + \nabla \cdot \mathbf{u}(t)) = v_0 \quad \text{in } L^2(\Omega), \quad (3.2.e)$$

and

$$\lim_{t \rightarrow 0^+} (1 - \beta)\mathbf{u}(t) \cdot \mathbf{n} = v_1 \quad \text{in } L^2(\Gamma_S).$$

Note that (3.1.a) is always equivalent to the pair (3.2.a) and (3.2.c) because $p(t) \in V$, while (3.1.b) is equivalent to (3.2.b) and (3.2.d) whenever we have sufficient additional regularity to guarantee that $A_0(p(t)) \in L^2(\Omega)$. This is the case for the strong solution.

The boundary conditions (3.2.c) consist of the complementary pair requiring null displacement on the *clamped boundary* Γ_0 and a balance of forces on the *traction boundary* Γ_t . The boundary conditions (3.2.d) require null pressure on the *drained boundary* Γ_1 and a balance of fluid mass on the complementary *flux boundary* Γ_f . The function $\beta(\cdot)$ is defined on that portion of the boundary $\Gamma_S = \Gamma_f \cap \Gamma_t$ which is neither drained nor clamped, and it specifies the surface fraction of the matrix pores which are *sealed* along Γ_S . For these the hydraulic pressure contributes to the total stress within the matrix. The remaining portion $1 - \beta(\cdot)$ of the pores are *exposed* along Γ_S , and these contribute to the flux. On any portion of Γ_t which is completely exposed, that is, where $\beta = 0$, only the *effective* or elastic component of stress is specified, since there the fluid pressure does not contribute to the support of the matrix. On the *flux* boundary Γ_f there is a transverse flow given by the input $h_1(\cdot)$ and the relative normal displacement of the medium. This input could be specified in the form $h_1(t) = -(1 - \beta)\mathbf{v}(t) \cdot \mathbf{n}$, where $\mathbf{v}(t)$ is the given velocity of fluid or boundary flux on Γ_S . The first term and right side of this flux balance are null where $\beta = 1$, so the same holds for the second term in (3.2.d), that is, we have the *impermeable* condition $k \frac{\partial p(t)}{\partial n} = 0$ on a completely sealed portion.

In order to establish the existence of the strong solution for the system (3.1), we first solve Eq. (3.1.a) for $\mathbf{u}(t)$ and then substitute it into (3.1.b) to

obtain an equivalent form as the single equation

$$\frac{d}{dt} \left(c_0 P p(t) - \vec{\nabla} \cdot \mathcal{E}^{-1}(\vec{\nabla} p(t)) \right) + A(p(t)) = h(t)$$

for which the time derivative of the solution occurs implicitly, an evolution equation of generalized *Sobolev type*, an *implicit evolution equation*. This leads to the following construction. Define an operator B on $L^2(\Omega) \oplus L^2(\Gamma_S)$ as

$$B(p) = \vec{\nabla} \cdot \mathbf{u}, \text{ where } \mathcal{E}(\mathbf{u}) = -\vec{\nabla} p, \quad p = [f, g] \in L^2(\Omega) \oplus L^2(\Gamma_S).$$

This operator is self-adjoint, that is,

$$(B(p), q)_{L^2(\Omega) \oplus L^2(\Gamma_S)} = \langle \vec{\nabla} q, \mathcal{E}^{-1}(\vec{\nabla} p) \rangle = \overline{(p, B(q))}_{L^2(\Omega) \oplus L^2(\Gamma_S)},$$

since \mathcal{E}^{-1} is symmetric on \mathbf{V} . Note that $(B(p), p)_{L^2(\Omega) \oplus L^2(\Gamma_S)} = \langle -\vec{\nabla} p, \mathbf{u} \rangle = \langle \mathcal{E}(\mathbf{u}), \mathbf{u} \rangle$, so if $B(p) = 0$ then we have $\mathbf{0} = \mathcal{E}(\mathbf{u}) = -\vec{\nabla} p$, hence, $\vec{\nabla} p = 0$. It follows that $\text{Ker}(B) = \text{Ker}(\vec{\nabla})$. (This uses the fact that \mathcal{E} is injective.)

LEMMA. *The operator $B = -\vec{\nabla} \cdot \mathcal{E}^{-1} \vec{\nabla} : L^2(\Omega) \oplus L^2(\Gamma_S) \rightarrow L^2(\Omega) \oplus L^2(\Gamma_S)$ is continuous and self-adjoint with $\text{Ker}(B) = \text{Ker}(\vec{\nabla})$ and $\text{Rg}(B) = \text{Ker}(\vec{\nabla})^\perp$, and each of the Sobolev spaces $(H^m(\Omega) \cap V) \oplus H^{m-1/2}(\Gamma_S)$ is invariant under B .*

In terms of the operator B , the system (3.1) has the form of an evolution equation,

$$\frac{d}{dt} (c_0 P + B)p(t) + A(p(t)) = h(t), \quad (3.3)$$

in which the time-derivative of the solution appears implicitly in the equation. Since $\text{Ker}(c_0 P + B) \cap V \subset \text{Ker}(A)$, we can apply the Theorem 6.2 with $\mathcal{B} \equiv c_0 P + B : L^2(\Omega) \oplus L^2(\Gamma_S) \rightarrow L^2(\Omega) \oplus L^2(\Gamma_S)$ and $\mathcal{A} \equiv A : V \rightarrow V'$ to obtain the following.

THEOREM 3.1. *Let $T > 0$, $v_0 \in L^2(\Omega)$, $v_1 \in L^2(\Gamma_S)$, and the pair of Hölder continuous functions $h_0(\cdot) \in C^\alpha([0, T], L^2(\Omega))$, $h_1(\cdot) \in C^\alpha([0, T], L^2(\Gamma_S))$ be given with*

$$\begin{aligned} \int_{\Omega} v_0(x) dx - \int_{\Gamma_S} v_1(s) ds &= 0, \\ \int_{\Omega} h_0(x, t) dx + \int_{\Gamma_S} h_1(s, t) ds &= 0, \quad t \in [0, T]. \end{aligned} \quad (3.4)$$

Then there exists a pair of functions $p(\cdot): (0, T] \rightarrow V$ and $\mathbf{u}(\cdot): (0, T] \rightarrow \mathbf{V}$ for which $c_0 p(\cdot) + \nabla \cdot \mathbf{u}(\cdot) \in C^0([0, T], L^2(\Omega)) \cap C^1((0, T], L^2(\Omega))$ and $(1 - \beta)\mathbf{u}(\cdot) \cdot \mathbf{n} \in C^0([0, T], L^2(\Gamma_S)) \cap C^1((0, T], L^2(\Gamma_S))$, and they satisfy the initial-boundary-value problem (3.2) with $t \mapsto tA(p(t)) = t[A_0(p(t)), k \frac{\partial p(t)}{\partial n}] \in L^\infty([0, T], L^2(\Omega) \oplus L^2(\Gamma_S)) \cap C^0((0, T], L^2(\Omega) \oplus L^2(\Gamma_S))$ and

$$\int_{\Omega} (c_0 p(t) + \nabla \cdot \mathbf{u}(t)) \, dx - \int_{\Gamma_S} (1 - \beta)\mathbf{u}(t) \cdot \mathbf{n} \, ds = 0, \quad t \in (0, T]. \tag{3.5}$$

The function $\mathbf{u}(\cdot)$ is unique. When $\text{Ker}(c_0 P + B + A) = \{0\}$, $p(\cdot)$ is unique, and if $\text{Ker}(c_0 P + B) \cap V = \{0\}$ we delete the integral constraints (3.4) and (3.5).

Note that $\text{Ker}(c_0 P + B) \cap V = \{0\}$ unless $\text{meas}(\Gamma_1) = 0$, $\beta \equiv 0$, and $c_0 = 0$. It follows from (3.1.a) and the continuity of $\vec{\nabla}$ and \mathcal{E}^{-1} that

$$\mathbf{u}(\cdot) \in C^0([0, T], \mathbf{V}) \cap C^1((0, T], \mathbf{V}),$$

and from regularity of A and \mathcal{E} we obtain

$$\|p(t)\|_{H^2(\Omega)}, \|\mathbf{u}(t)\|_{H^2(\Omega)^3} \leq C/t, \quad 0 < t \leq T.$$

Here we have required the data to be given in $L^2(\Omega) \oplus L^2(\Gamma_S)$ in order to get this strong solution, whereas in the case below (for a weak solution) it will be given in the larger space V' . The initial condition is equivalent to specifying $p(0)$ and $\vec{\nabla} \cdot \mathbf{u}(0)$ independently. This condition determines the combination $c_0 Pp(t) + B(p(t)) = c_0 Pp(t) + \vec{\nabla} \cdot \mathbf{u}(t) = [c_0 p(t) + \nabla \mathbf{u}(t), -(1 - \beta)\mathbf{u}(t) \cdot \mathbf{n}]$, and this is just the fluid content in the poro-elasticity problem or the entropy for the thermoelasticity problem.

Finally, from the regularity theory for A and \mathcal{E} , we can show that the operator \mathbb{C} appearing in the proof of Theorem 6.2 satisfies $(I + \mathbb{C})^{-1} : H^m(\Omega) \oplus H^{m-1/2}(\Gamma_S) \rightarrow H^{m+2}(\Omega) \oplus H^{m+3/2}(\Gamma_S)$ for each $m \geq 1$. Specifically, if v and $(I + \mathbb{C})v \in H^m(\Omega) \oplus H^{m-1/2}(\Gamma_S)$, then $\mathbb{C}(v) = A(p) \in H^m(\Omega) \oplus H^{m-1/2}(\Gamma_S)$ so the regularity theory for A implies $p \in H^{m+2}(\Omega) \oplus H^{m+3/2}(\Gamma_S)$. But then the preceding lemma shows that $v = c_0 Pp(t) + B(p(t)) \in H^{m+2}(\Omega) \oplus H^{m+3/2}(\Gamma_S)$. When the data $h_0(\cdot), h_1(\cdot)$ are smooth, this implies by a standard argument [32] that the solution $p(\cdot)$ is $C^\infty(\Omega \times (0, T])$. Thus, the system is parabolic, even if $c_0 = 0$.

4. THE WEAK SOLUTION

We shall first extend the preceding construction to obtain a weak solution of the system (3.1) in the holomorphic situation of Theorem 6.4 under appropriately weaker assumptions on the data. Then we develop an

alternative method for integrating the system as an example of Theorem 6.3 for the strongly continuous case and again for the holomorphic case, but with the added restriction that $c_0 > 0$. A third method for obtaining a weak solution of (3.1) from a well-posed abstract Cauchy problem will be given in Section 5.

For our first and most elementary approach, we apply Theorem 6.4 directly to the implicit evolution equation (3.3). Recall that the range of A is denoted by $\text{Rg}(A) = V'_a$. This yields immediately the following.

THEOREM 4.1. *Let $T > 0$, $v_0 \in V'_a$, and $h(\cdot) \in C^\alpha([0, T], V'_a)$ be given. Then there exists a unique pair of functions $p(\cdot): (0, T] \rightarrow V$ and $\mathbf{u}(\cdot): (0, T] \rightarrow \mathbf{V}$ for which $c_0 Pp(\cdot) + \vec{\nabla} \cdot \mathbf{u}(\cdot) \in C^0([0, T], V'_a) \cap C^1((0, T], V'_a)$, and they satisfy the initial-value problem*

$$\mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) = \mathbf{0}, \quad (4.1.a)$$

$$\frac{d}{dt} (c_0 Pp(t) + \vec{\nabla} \cdot \mathbf{u}(t)) + A(p(t)) = h(t), \quad t \in (0, T], \quad (4.1.b)$$

$$\lim_{t \rightarrow 0^+} (c_0 Pp(t) + \vec{\nabla} \cdot \mathbf{u}(t)) = v_0 \quad \text{in } V'_a. \quad (4.1.c)$$

Remark. The functions $p(\cdot): (0, T] \rightarrow V/\text{Ker}(A)$ and $\mathbf{u}(\cdot): (0, T] \rightarrow \mathbf{V} \cap \mathbf{H}^2(\Omega)$ are bounded by $\frac{C}{t}$ on $(0, T]$.

As before, Eq. (4.1.a) holds in $\mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_t)$ and is therefore equivalent to the pair of equations

$$\mathcal{E}_0(\mathbf{u}(t)) + \nabla p(t) = \mathbf{0} \quad (4.2.a)$$

and

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_0, \quad a_{ijkl} \varepsilon_{kl}(\mathbf{u}(t)) n_j - \beta p(t) n_i \chi_S = 0, \quad 1 \leq i \leq 3, \text{ on } \Gamma_t. \quad (4.2.b)$$

However, (4.1.b) cannot be similarly decomposed into a partial differential equation and boundary condition, since it holds in V'_a . We call such a solution a *weak solution* of (4.1). It is not sufficiently smooth to apply Green's theorem to (4.1.b), but note that by integrating that component in time we obtain

$$c_0 Pp(t) + \vec{\nabla} \cdot \mathbf{u}(t) + A\left(\int_0^t p\right) = \int_0^t h + v_0,$$

where $\int_0^t p$ is understood in the space $V/\text{Ker}(\nabla)$. The weak solution satisfies $c_0 Pp(t) + \vec{\nabla} \cdot \mathbf{u}(t) \in L^2(\Omega) \oplus L^2(\Gamma_S)$ for each $t > 0$, so if we

require that $\int_0^t h + v_0$ belong to $L^2(\Omega) \oplus L^2(\Gamma_S)$ at each $t > 0$, then the same holds for $A(\int_0^t p)$. In this situation, it follows that $\int_0^t p(t) \in V \cap H^2(\Omega)$ and the integrated form of (4.1.b) is equivalent to the pair of equations

$$c_0 p(t) + \nabla \cdot \mathbf{u}(t) + A_0 \left(\int_0^t p(t) \right) = \int_0^t h_0 + v_{00} \quad \text{in } \Omega, \quad (4.2.c)$$

$$-(1 - \beta) \mathbf{u}(t) \cdot \mathbf{n} \chi_S + k \frac{\partial \int_0^t p(t)}{\partial n} = \left(\int_0^t h_1 + v_{01} \right) \chi_S \quad \text{on } \Gamma. \quad (4.2.d)$$

Thus, with the indicated additional regularity of the data, the weak solution of (4.1) satisfies the system (4.2) of partial differential equations and boundary conditions.

We next obtain the weak solution by another approach. Proceeding formally for the moment, we differentiate the first equation in the system to obtain

$$\frac{d}{dt} \left(\mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) \right) = \mathbf{0}, \quad (4.3.a)$$

$$\frac{d}{dt} c_0 P p(t) + A(p(t)) + \frac{d}{dt} \vec{\nabla} \cdot \mathbf{u}(t) = h(t), \quad (4.3.b)$$

in which the anti-symmetric first order *coupling* terms are *both* acting on time derivatives. This puts the system in the form of an implicit evolution equation

$$\frac{d}{dt} \begin{pmatrix} \mathcal{E} & \vec{\nabla} \\ \vec{\nabla} \cdot & c_0 P \end{pmatrix} \begin{bmatrix} \mathbf{u}(t) \\ p(t) \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \begin{bmatrix} \mathbf{u}(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ h(t) \end{bmatrix}$$

with the indicated matrix operators which clearly display the symmetric as well as anti-symmetric terms in the system, and it suggests the second structure which we use to obtain the weak solution.

On the product space $\mathcal{V} \equiv \mathbf{V} \times V$ we define the *sesquilinear* form

$$b([\mathbf{u}, p], [\mathbf{v}, q]) = e(\mathbf{u}, \mathbf{v}) + \langle \vec{\nabla} p, \mathbf{v} \rangle + \langle \vec{\nabla} \cdot \mathbf{u}, q \rangle + \int_{\Omega} (c_0 p \bar{q}) \, dx$$

and the corresponding operator $\mathcal{B} : \mathcal{V} \rightarrow \mathcal{V}'$. Although $b(\cdot, \cdot)$ is not symmetric, by using the definition of the dual operator in the second term we find that its values on the diagonal are given by

$$b([\mathbf{u}, p], [\mathbf{u}, p]) = e(\mathbf{u}, \mathbf{u}) + \langle \vec{\nabla} \cdot \mathbf{u}, p \rangle - \overline{\langle \vec{\nabla} \cdot \mathbf{u}, p \rangle} + \int_{\Omega} (c_0 |p|^2) \, dx.$$

Similarly we define on \mathcal{V} the sesquilinear form

$$\mathcal{A}([\mathbf{u}, p], [\mathbf{v}, q]) = A(p)(q) = \int_{\Omega} k \nabla p \cdot \overline{\nabla q} \, dx,$$

and this gives a symmetric and monotone operator $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{V}'$ in terms of the component operator $A: V \rightarrow V'$. They are related by

$$\mathcal{B} = \begin{pmatrix} \mathcal{E} & \vec{\nabla} \\ \vec{\nabla} \cdot & c_0 P \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},$$

and it was precisely these operators that arose in our representation of the quasi-static system (4.3) in the form

$$\frac{d}{dt} \mathcal{B}(\vec{u}(t)) + \mathcal{A}(\vec{u}(t)) = \vec{f}(t). \quad (4.4)$$

If $c_0 > 0$ it follows that $b(\cdot, \cdot)$ is a *sectorial* form. (See Section 6.) That is, we have an estimate of the form

$$\operatorname{Re}(b([\mathbf{u}, p], [\mathbf{u}, p])) \geq k_0(c_0) |\operatorname{Im}(b([\mathbf{u}, p], [\mathbf{u}, p]))|$$

for the numerical range, where $k_0(c_0) > 0$. It follows that \mathcal{B} is sectorial when $c_0 > 0$.

Consider the *kernel condition* on the operators, \mathcal{B}, \mathcal{A} . If $c_0 > 0$, then we have $\operatorname{Ker}(\mathcal{B}) = \{\mathbf{0}\} \times \{0\}$, so there is no issue. If $c_0 = 0$, then $\operatorname{Ker}(\mathcal{B}) = \{\mathbf{0}\} \times (\operatorname{Ker}(\vec{\nabla}) \cap V)$, so we have $\operatorname{Ker}(\mathcal{B}) \subset \{\mathbf{0}\} \times \operatorname{Ker}(A) \subset \operatorname{Ker}(\mathcal{A})$ as desired. As before, this depends on the inclusion $\operatorname{Ker}(\vec{\nabla}) \cap V \subset \operatorname{Ker}(A)$.

Recall that $A: V \rightarrow V'$ has the kernel $\operatorname{Ker}(A) = \operatorname{Ker}(\nabla)$ and range $\operatorname{Rg}(A) = V'_a = \operatorname{Ker}(A)^\perp$, the indicated *annihilator* in V' . Note that $\operatorname{Rg}(\mathcal{A}) = \mathcal{V}'_a = \{\mathbf{0}\} \times V'_a \subset \mathbf{V}' \times V' = \mathcal{V}'$.

PROPOSITION. *If $c_0 \geq 0$, then $\operatorname{Rg}(\mathcal{B} + \mathcal{A}) = \mathcal{V}'_a$.*

Proof. The system

$$\begin{pmatrix} \mathcal{E} & \vec{\nabla} \\ \vec{\nabla} \cdot & c_0 P + A \end{pmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ h \end{bmatrix}$$

is equivalent to the single equation

$$(c_0 P + A)(p) - \vec{\nabla} \cdot \mathcal{E}^{-1}(\vec{\nabla} p) = h \in V',$$

and this has a solution if $h \in \operatorname{Rg}(A) = V'_a$.

Next we consider the *strongly continuous* case of $c_0 \geq 0$. We shall apply Theorem 6.3. Denote $\vec{u}(t) = [\mathbf{u}(t), p(t)] \in \mathcal{V}$.

THEOREM 4.2. *Let $T > 0$, $v_0 = (c_0P + B)p_0 \in V'_a$ for some $p_0 \in V$, and $h(\cdot) \in C^1([0, T], V'_a)$ be given. Then there exists a pair of functions $p(\cdot) : [0, T] \rightarrow V$ and $\mathbf{u}(\cdot) : [0, T] \rightarrow \mathbf{V}$ for which $c_0Pp(\cdot) + \vec{\nabla} \cdot \mathbf{u}(\cdot) \in C^1([0, T], V'_a)$, and they satisfy the initial-value problem (4.1). The function $\mathbf{u}(\cdot)$ is unique. When $\text{Ker}(c_0P + B + A) = \{0\}$, the function $p(\cdot)$ is unique.*

Remark. The functions $p(\cdot) : [0, T] \rightarrow V/\text{Ker}(A)$ and $\mathbf{u}(\cdot) : [0, T] \rightarrow \mathbf{V} \cap \mathbf{H}^2(\Omega)$ are *continuous*, and $(B + c_0)p(\cdot) = \nabla \cdot \mathbf{u}(\cdot) + c_0p(\cdot) = v(\cdot) : [0, T] \rightarrow V'_a$ is *differentiable*.

Finally, note that we can apply the holomorphic Theorem 6.4 to the system (4.3) to obtain Theorem 4.1, but only with the (unnecessary) assumption that $c_0 > 0$.

5. SECONDARY CONSOLIDATION

We shall obtain a weak solution for the quasi-static poro-elasticity system with *secondary consolidation*, $\lambda^* \geq 0$,

$$-\lambda^* \vec{\nabla} \frac{d}{dt} (\vec{\nabla} \cdot \mathbf{u}(t)) + \mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) = \mathbf{h}(t), \tag{5.1.a}$$

$$\frac{d}{dt} c_0 P p(t) + A(p(t)) + \frac{d}{dt} \vec{\nabla} \cdot \mathbf{u}(t) = h(t). \tag{5.1.b}$$

For the case of $\lambda^* = 0$, we have already shown above that one may assume without loss of generality that $\mathbf{h}(\cdot)$ has been replaced by $\mathbf{0}$, so we denote this by inserting a factor of λ^* on $\mathbf{h}(\cdot)$ in (5.1.a). Introduce the variable $q(t) = \int_0^t p(s) ds + g(t)$, where $g(t) \in V$ is a solution of

$$g(t) \in V: -A(g(t)) = \int_0^t h(s) ds + (c_0 P p(0) + \nabla \cdot \mathbf{u}(0));$$

we have assumed here that $h(\cdot)$ and $c_0 P p(0) + \vec{\nabla} \cdot \mathbf{u}(0)$ belong to V'_a . Then we integrate (5.1.b) and write the system (5.1) in the equivalent form

$$-\lambda^* \vec{\nabla} \frac{d}{dt} (\vec{\nabla} \cdot \mathbf{u}(t)) + \mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) = \lambda^* \mathbf{h}(t), \tag{5.2.a}$$

$$\vec{\nabla} \cdot \mathbf{u}(t) + c_0 P p(t) + A(q(t)) = 0, \tag{5.2.b}$$

$$\frac{d}{dt} A q(t) - A p(t) = -h(t). \tag{5.2.c}$$

This can be expressed in the matrix form as

$$\frac{d}{dt} \begin{pmatrix} -\lambda^* \vec{\nabla}(\vec{\nabla} \cdot) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix} \begin{bmatrix} \mathbf{u}(t) \\ p(t) \\ q(t) \end{bmatrix} + \begin{pmatrix} \mathcal{E} & \vec{\nabla} & 0 \\ \vec{\nabla} \cdot & c_0 P & A \\ 0 & -A & 0 \end{pmatrix} \begin{bmatrix} \mathbf{u}(t) \\ p(t) \\ q(t) \end{bmatrix} \\ = \begin{bmatrix} \lambda^* \mathbf{h}(t) \\ 0 \\ -h(t) \end{bmatrix},$$

so we see the corresponding matrix operators are given by

$$\mathcal{B} = \begin{pmatrix} -\lambda^* \vec{\nabla}(\vec{\nabla} \cdot) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{E} & \vec{\nabla} & 0 \\ \vec{\nabla} \cdot & c_0 P & A \\ 0 & -A & 0 \end{pmatrix}$$

on the product space $\mathcal{W} = \mathcal{V} = \mathbf{V} \times V \times V$. This system is *degenerate* and *weakly dissipative*. The operator \mathcal{B} is symmetric and monotone, and so according to Theorem 6.1 and Theorem 6.2 the dynamics takes place in the Hilbert space $\mathcal{W}'_b = \text{Rg } \mathcal{B} = \lambda^* \text{Rg}(\nabla) \times \{0\} \times V'_a$. Thus, we consider the restriction of \mathcal{A} to the domain $D = \{\vec{u} = [\mathbf{u}, p, q] \in \mathcal{V} : \mathcal{A}\vec{u} \in \mathcal{W}'_b\}$. Here it is essential that we made an appropriate choice of $q(\cdot)$ to get 0 on the right side of (5.2.b). Since this formulation has the form of an abstract Sobolev equation in which the two elliptic operators are of the *same* order, hence, the system is *pseudo-parabolic*, any regularizing effects will be rather limited [32]. However, we will identify below conditions for which \mathcal{A} is *monotone* or *sectorial*, so we shall obtain *strongly continuous* or *holomorphic* dynamics, respectively.

In order to check the monotonicity of \mathcal{A} for $\lambda^* \geq 0$, $c_0 \geq 0$, we compute

$$\langle \mathcal{A}\vec{u}, \vec{u} \rangle = \mathcal{E} \mathbf{u}(\mathbf{u}) + c_0 \|p\|_{L^2}^2 + \langle \vec{\nabla} p, \mathbf{u} \rangle + \langle \vec{\nabla} \cdot \mathbf{u}, p \rangle,$$

and this shows additionally that \mathcal{A} is sectorial when $c_0 > 0$. The remaining point is to show the range of $\lambda \mathcal{B} + \mathcal{A}$ contains $\mathcal{W}'_b = \text{Rg } \mathcal{B}$. To this end, we consider the *stationary equation*

$$\begin{pmatrix} -\lambda \lambda^* \vec{\nabla}(\vec{\nabla} \cdot) + \mathcal{E} & \vec{\nabla} & 0 \\ \vec{\nabla} \cdot & c_0 P & A \\ 0 & -A & \lambda A \end{pmatrix} \begin{bmatrix} \mathbf{u} \\ p \\ q \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \\ g \end{bmatrix}$$

for a pair $(\mathbf{f}, g) \in \mathbf{V}' \times V'_a$. If we eliminate q there follows the equivalent system

$$\begin{aligned}
 -\lambda^2 \lambda^* \vec{\nabla}(\vec{\nabla} \cdot \mathbf{u}) + \lambda \mathcal{E}(\mathbf{u}) + \lambda \vec{\nabla} p &= \lambda \mathbf{f}, \\
 \lambda \vec{\nabla} \cdot \mathbf{u} + \lambda c_0 Pp + A(p) &= -g.
 \end{aligned}$$

But it follows easily from a closed range argument that this system has a solution for each such pair $(\mathbf{f}, g) \in \mathbf{V}' \times V'_a$.

Consider the *kernel condition*. For comparison with the cases in Section 4, consider first the case of $\lambda^* = 0$. Then we have $\mathcal{W}'_b = \{\mathbf{0}\} \times \{0\} \times V'_a$, and so from the inclusion $\mathcal{A}[\mathbf{u}, p, q] \in \mathcal{W}'_b$ one obtains

$$\mathcal{E}(\mathbf{u}) + \vec{\nabla} p = 0, \quad \vec{\nabla} \cdot \mathbf{u} + c_0 Pp + A(q) = 0, \quad A(p) \in V'_a.$$

Thus from $[\mathbf{u}, p, q] \in \text{Ker}(\mathcal{B})$ we obtain successively $A(q) = 0$ and then $\mathbf{u} = \mathbf{0}$, $p \in \text{Ker}(\vec{\nabla})$. Hence, it suffices again to have $\text{Ker}(\vec{\nabla}) \subset \text{Ker}(A)$. From the preceding calculations it then follows that an application of Theorem 6.1 and Theorem 6.2 to (5.2) gives alternate proofs of Theorem 4.1 and Theorem 4.2, respectively.

Now we continue with the case of $\lambda^* > 0$. Here $\mathcal{W}'_b = \text{Rg}(\vec{\nabla}) \times \{0\} \times V'_a$, and so from the inclusion $\mathcal{A}[\mathbf{u}, p, q] \in \mathcal{W}'_b$ there follows

$$\mathcal{E}(\mathbf{u}) \in \text{Rg}(\nabla), \quad \nabla \cdot \mathbf{u} + c_0 Pp + A(q) = 0.$$

Thus, whenever $[\mathbf{u}, p, q] \in \text{Ker}(\mathcal{B})$, it follows that $\vec{\nabla} \cdot \mathbf{u} = 0$ and $A(q) = 0$. But then we have $\mathcal{E}(\mathbf{u})(\mathbf{u}) = 0$, since $\text{Rg}(\vec{\nabla})$ annihilates $\text{Ker}(\vec{\nabla} \cdot)$, and so it follows successively that $\mathbf{u} = \mathbf{0}$ and $c_0 Pp = 0$. But we need to get $p = 0$, so it appears necessary to have $c_0 > 0$ in order to obtain the kernel condition. In that case \mathcal{A} is sectorial, and we have again an *holomorphic* situation.

THEOREM 5.1. *Assume $\lambda^* > 0$ and $c_0 > 0$. Let $T > 0$, $v_0 \in V'_a$, $\mathbf{w}_0 \in \mathbf{V}$, and*

$$\mathbf{H}(\cdot) \in C^\alpha([0, T], L^2(\Omega) \oplus L^2(\Gamma_S)), \quad h(\cdot) \in C^\alpha([0, T], V'_a),$$

be given. Then there exists a unique pair of functions $p(\cdot): (0, T] \rightarrow V$, $\mathbf{u}(\cdot): (0, T] \rightarrow \mathbf{V}$ for which

$$\begin{aligned}
 c_0 Pp(\cdot) + \vec{\nabla} \cdot \mathbf{u}(\cdot) &\in C^0([0, T], V'_a) \cap C^1((0, T], V'_a) \\
 [\nabla \cdot \mathbf{u}(\cdot), \mathbf{u}(\cdot) \cdot \mathbf{n}] &\in C^0([0, T], L^2(\Omega) \oplus L^2(\Gamma_S)) \\
 &\cap C^1((0, T], L^2(\Omega) \oplus L^2(\Gamma_S)),
 \end{aligned}$$

and they satisfy the initial-value problem

$$-\lambda^* \vec{\nabla} \left(\frac{d}{dt} \vec{\nabla} \cdot \mathbf{u}(t) \right) + \mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) = \lambda^* \vec{\nabla} \mathbf{H}(t) \quad \text{in } \mathbf{V}', \quad (5.3.a)$$

$$\frac{d}{dt} (c_0 Pp(t) + \vec{\nabla} \cdot \mathbf{u}(t)) + A(p(t)) = h(t) \quad \text{in } V'_a, t \in (0, T], \quad (5.3.b)$$

$$\lim_{t \rightarrow 0^+} (c_0 Pp(t) + \vec{\nabla} \cdot \mathbf{u}(t)) = v_0 \quad \text{in } V'_a, \quad (5.3.c)$$

$$\lim_{t \rightarrow 0^+} \vec{\nabla} \cdot \mathbf{u}(t) = \vec{\nabla} \cdot \mathbf{w}_0 \quad \text{in } L^2(\Omega) \oplus L^2(\Gamma_S). \quad (5.3.d)$$

Remark. Again we obtain bounds of the form $\frac{c}{t}$ on $(0, T]$ for the function $p(\cdot): (0, T] \rightarrow V/\text{Ker}(A)$.

Finally, we note that if $c_0 = 0$, then we can formally eliminate $p(t) = A^{-1}(h(t) - \frac{d}{dt} \vec{\nabla} \cdot \mathbf{u}(t))$ from the system to obtain a single equation

$$\begin{aligned} & -\lambda^* \vec{\nabla} \left(\frac{d}{dt} \vec{\nabla} \cdot \mathbf{u}(t) \right) - \nabla A^{-1} \left(\frac{d}{dt} \vec{\nabla} \cdot \mathbf{u}(t) \right) + \mathcal{E}(\mathbf{u}(t)) \\ & = \lambda^* \vec{\nabla} \mathbf{H}(t) - \nabla A^{-1}(h(t)) \end{aligned}$$

which is treated as in Section 3. In particular, if we assume $\text{Rg}(A) = \text{Rg}(\vec{\nabla} \cdot)$ in V' and (equivalently) $\text{Ker}(A) = \text{Ker}(\vec{\nabla})$, then the composite operator $\vec{\nabla} A^{-1} \vec{\nabla} \cdot$ is continuous and self-adjoint on $L^2(\Omega) \oplus L^2(\Gamma_g)$. By applying Theorem 6.2 directly to (5.4), we obtain the following.

THEOREM 5.2. *Let $T > 0$, $\mathbf{w}_0 \in \mathbf{V}$, and*

$$\mathbf{H}(\cdot) \in C^\alpha([0, T], L^2(\Omega) \oplus L^2(\Gamma_S)), \quad h(\cdot) \in C^\alpha([0, T], V'_a),$$

be given. Assume $\text{Ker}(A) = \text{Ker}(\vec{\nabla})$ in V . Then there exists a unique function $\mathbf{u}: (0, T] \rightarrow \mathbf{V}$ for which

$$\vec{\nabla} \cdot \mathbf{u}(\cdot) \in C^0([0, T], L^2(\Omega) \oplus L^2(\Gamma_S)) \cap C^1((0, T], L^2(\Omega) \oplus L^2(\Gamma_S)),$$

and with any choice of $p(t) \in A^{-1}(h(t) - \frac{d}{dt} \vec{\nabla} \cdot \mathbf{u}(t))$ satisfies the initial-value problem

$$-\lambda^* \frac{d}{dt} \vec{\nabla} (\vec{\nabla} \cdot \mathbf{u}(t)) + \mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) = \lambda^* \vec{\nabla} \mathbf{H}(t) \quad \text{in } \mathbf{V}', \quad (5.5.a)$$

$$\frac{d}{dt} (\vec{\nabla} \cdot \mathbf{u}(t)) + A(p(t)) = h(t) \quad \text{in } V'_a, t \in (0, T], \quad (5.5.b)$$

$$\lim_{t \rightarrow 0^+} \vec{\nabla} \cdot \mathbf{u}(t) = \vec{\nabla} \cdot \mathbf{w}_0 \quad \text{in } L^2(\Omega) \oplus L^2(\Gamma_S). \quad (5.5.c)$$

The solution in Theorem 5.1 is *less regular* than the weak solutions of Section 4. Specifically, not only is the diffusion equation (5.3.b) in V'_a , but the momentum equation (5.3.a) is in \mathbf{V}' , so neither of them has the appropriate regularity to be decoupled into a system of partial differential equations and boundary conditions. However, the solution in Theorem 5.2 has *more regularity* than is available in Theorem 5.1. Specifically, if we require that $h(t) = [h_0(t), (1 - \beta)h_1(t)] \in L^2(\Omega) \oplus L^2(\Gamma_S)$, then each term of (5.5.b) belongs to this space, and this forces the additional regularity, $p(t) \in V \cap H^2(\Omega)$, and the decoupling of (5.5.b) into the pair of equations (cf. (4.1.c))

$$\frac{d}{dt} \nabla \cdot \mathbf{u}(t) + A_0(p(t)) = h_0(t) \quad \text{in } \Omega, \tag{5.6.a}$$

$$-\frac{d}{dt} ((1 - \beta)\mathbf{u}(t) \cdot \mathbf{n}) \chi_S + k \frac{\partial p(t)}{\partial n} = (1 - \beta)h_1(t) \chi_S \quad \text{on } \Gamma. \tag{5.6.b}$$

A similar decoupling of (5.5.a) is not possible because of the lack of regularity of the leading term with λ^* . When $\lambda^* = 0$ we are back in the parabolic regularizing situation of Theorem 3.1. Thus we see that the degeneracy arising from either $c_0 = 0$ or from $\lambda^* = 0$ leads to a smoothing of the corresponding component of the solution.

6. IMPLICIT EVOLUTION EQUATIONS

We shall consider the Cauchy problem for implicit evolution equations

$$\frac{d}{dt} \mathcal{B}(\vec{u}(t)) + \mathcal{A}(\vec{u}(t)) = \vec{f}(t), \quad t \in [0, T],$$

in appropriate spaces. They arise in various formulations of the quasi-static system (1.2), and there either of the two operators may be non-symmetric or degenerate. We shall outline the well-posedness theory of the dynamics of such systems in two ways, depending on which of the two operators is symmetric. All of these results will be obtained as direct extensions of the fundamental semigroup theory for the special case of $\mathcal{B} = I$ as presented in [21]. Also see [32, 34].

Abstract Cauchy Problem, I. First we consider the situation in which the *first* operator is symmetric. Assume $\mathcal{B} : \mathcal{W} \rightarrow \mathcal{W}'$ is a continuous linear *symmetric* monotone operator, where \mathcal{W} is a Hilbert space, and denote the space \mathcal{W} with the seminorm $\mathcal{B}(\cdot, \cdot)^{1/2}$ by \mathcal{W}_b . Then the injection $\mathcal{W} \rightarrow \mathcal{W}_b$

is continuous, and we have $\mathscr{W}'_b \subset \mathscr{W}'$. \mathscr{W}'_b is a Hilbert space, for which we have the identity

$$\vec{f}(\vec{u}) = (\vec{f}, \mathscr{B}\vec{u})_{\mathscr{W}'_b}, \quad \vec{f} \in \mathscr{W}'_b, \vec{u} \in \mathscr{W}.$$

DEFINITIONS. The linear operator $\mathscr{A}: D \rightarrow W'_b$ with domain $D \subset W$ is *monotone* (or *non-negative*) if

$$\operatorname{Re} \mathscr{A}x(x) \geq 0, \quad x \in D.$$

It is called *sectorial* if it satisfies

$$\operatorname{Re} \mathscr{A}x(x) \geq c_0 \|m \mathscr{A}x(x)\|, \quad x \in D$$

for some $c_0 > 0$. When $D \subset \mathscr{W}'_b$ and the duality pairing is replaced by the scalar product, the corresponding property is called *accretive* instead of monotone. If, in addition, we have $\operatorname{Rg}(I + \mathscr{A}) = \mathscr{W}'_b$, then the operator \mathscr{A} is called *m-accretive* or *m-sectorial*, respectively.

Assume that $\operatorname{Ker}(\mathscr{B}) \cap D \subset \operatorname{Ker}(\mathscr{A})$. Define the operator \mathbb{C} in the Hilbert space \mathscr{W}'_b by

$$\mathbb{C}(\vec{v}) = \mathscr{A}\vec{u} \Leftrightarrow \vec{v} = \mathscr{B}(\vec{u}) \quad \text{for some } \vec{u} \in D.$$

Then for any $\vec{v} \in \operatorname{Dom} \mathbb{C} \equiv \mathscr{B}[D]$ we obtain

$$(\mathbb{C}\vec{v}, \vec{v})_{\mathscr{W}'_b} = (\mathscr{A}\vec{u}, \mathscr{B}\vec{u})_{\mathscr{W}'_b} = (\mathscr{A}\vec{u})(\vec{u})$$

from the identity above, so if \mathscr{A} is monotone (or sectorial), then \mathbb{C} is likewise \mathscr{W}'_b -accretive (or sectorial).

The equation $\vec{v} + \mathbb{C}(\vec{v}) = \vec{f}$ in \mathscr{W}'_b is equivalent to

$$\vec{u} \in D: \mathscr{B}\vec{u} + \mathscr{A}\vec{u} = \vec{f},$$

so $\operatorname{Rg}(I + \mathbb{C}) = \operatorname{Rg}(\mathscr{A} + \mathscr{B})$ in \mathscr{W}'_b . This shows that

LEMMA. The accretive (or sectorial) operator \mathbb{C} is *m-accretive* (respectively, *m-sectorial*) on \mathscr{W}'_b if $\operatorname{Rg}(\mathscr{A} + \mathscr{B}) = \mathscr{W}'_b$.

THEOREM 6.1 (Strongly Continuous Case). Assume that the operator \mathbb{C} is *m-accretive*. Let $T > 0$, $\vec{v}_0 \in \mathscr{W}'_b$ with $\vec{v}_0 = \mathscr{B}(\vec{u}_0)$ for some $\vec{u}_0 \in D$ and $\vec{f}(\cdot) \in C^1([0, T], \mathscr{W}'_b)$ be given. Then there exists a function $\vec{u}: (0, T] \rightarrow D$ with $\mathscr{B}(\vec{u}) \in C^1([0, T], \mathscr{W}'_b)$ and $\mathscr{A}(\vec{u}) \in C^0([0, T], \mathscr{W}'_b)$ for which

$$\frac{d}{dt} \mathscr{B}(\vec{u}(t)) + \mathscr{A}(\vec{u}(t)) = \vec{f}(t), \quad t \in [0, T],$$

$$\lim_{t \rightarrow 0^+} \mathscr{B}(\vec{u}(t)) = \vec{v}_0 \quad \text{in } \mathscr{W}'_b.$$

If $u_1(\cdot)$ and $u_2(\cdot)$ are two such solutions, then

$$u_1(t) - u_2(t) \in \text{Ker}(\mathcal{B}) \cap \text{Ker}(\mathcal{A}), \quad 0 \leq t \leq T.$$

THEOREM 6.2 (Holomorphic Case). Assume that the operator \mathbb{C} is m -sectorial. Let $T > 0$, $\vec{v}_0 \in \mathcal{W}'_b$, and Hölder continuous $\vec{f} \in C^\alpha([0, T], \mathcal{W}'_b)$ be given. Then there exists a function $\vec{u} : (0, T] \rightarrow D$ with $\mathcal{B}\vec{u} \in C^0([0, T], \mathcal{W}'_b) \cap C^1((0, T], \mathcal{W}'_b)$ for which

$$\begin{aligned} \frac{d}{dt} \mathcal{B}(\vec{u}(t)) + \mathcal{A}(\vec{u}(t)) &= \vec{f}(t), \quad t \in (0, T], \\ \lim_{t \rightarrow 0^+} \mathcal{B}(\vec{u}(t)) &= \vec{v}_0 \quad \text{in } \mathcal{W}'_b, \end{aligned}$$

and it satisfies $\|\mathcal{A}(\vec{u}(t))\|_{\mathcal{W}'_b} \leq \frac{C}{t}$, $0 < t \leq T$. Any two such solutions, $u_1(\cdot)$ and $u_2(\cdot)$, satisfy

$$u_1(t) - u_2(t) \in \text{Ker}(\mathcal{B}) \cap \text{Ker}(\mathcal{A}), \quad 0 \leq t \leq T.$$

Abstract Cauchy Problem, II. Here we consider the case in which the second operator is symmetric. Assume $\mathcal{B} : D \rightarrow \mathcal{V}'$ is a linear monotone operator with domain $D \subset \mathcal{V}$, where \mathcal{V} is a Hilbert space, and that $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ is a continuous linear symmetric monotone operator. Denote the space \mathcal{V} with the seminorm $\mathcal{A}(\cdot, \cdot)^{1/2}$ by \mathcal{V}'_a . Then the injection $\mathcal{V} \rightarrow \mathcal{V}'_a$ is continuous, and we have $\mathcal{V}'_a \subset \mathcal{V}'$. \mathcal{V}'_a is a Hilbert space, for which we have the identity

$$\vec{f}(\vec{u}) = \left(\vec{f}, \mathcal{A}\vec{u} \right)_{\mathcal{V}'_a}, \quad \vec{f} \in \mathcal{V}'_a, \vec{u} \in \mathcal{V}.$$

Assume that $\text{Ker}(\mathcal{B}) \subset \text{Ker}(\mathcal{A})$. Define the operator \mathbb{C} on the Hilbert space \mathcal{V}'_a by

$$\mathbb{C}(\vec{v}) = \mathcal{A}\vec{u} \Leftrightarrow \vec{v} = \mathcal{B}(\vec{u}) \quad \text{for some } \vec{u} \in D.$$

Then for any $\vec{v} \in \text{Dom } \mathbb{C} \equiv \mathcal{B}[D]$ we obtain

$$(\mathbb{C}\vec{v}, \vec{v})_{\mathcal{V}'_a} = (\mathcal{A}\vec{u}, \mathcal{B}\vec{u})_{\mathcal{V}'_a} = \overline{(\mathcal{B}\vec{u})(\vec{u})}$$

from the identity above, so if \mathcal{B} is monotone (or sectorial), then \mathbb{C} is likewise \mathcal{V}'_a -accretive (or sectorial).

The equation $\vec{v} + \mathbb{C}(\vec{v}) = \vec{f}$ in \mathcal{V}'_a is equivalent to

$$\vec{u} \in \mathcal{V} : \vec{v} + \mathcal{A}\vec{u} = \vec{f}, \quad \vec{v} = \mathcal{B}(\vec{u}),$$

so $\text{Rg}(I + \mathbb{C}) = \text{Rg}(\mathcal{A} + \mathcal{B}) \cap \mathcal{V}'_a$. This shows that

LEMMA. *The accretive (or sectorial) operator \mathbb{C} is m -accretive (respectively, m -sectorial) on \mathcal{V}'_a if $\text{Rg}(\mathcal{A} + \mathcal{B}) \supset \mathcal{V}'_a$.*

THEOREM 6.3 (Strongly Continuous Case). *Assume that the operator \mathbb{C} is m -accretive. Let $T > 0$, $\vec{v}_0 \in \mathcal{V}'_a$, with $\vec{v}_0 = \mathcal{B}(\vec{u}_0)$ for some $\vec{u}_0 \in \mathcal{V}$ and $\vec{f}(\cdot) \in C^1([0, T], \mathcal{V}'_a)$. Then there exists a function $\vec{u}: (0, T] \rightarrow \mathcal{V}$ with $\mathcal{B}\vec{u} \in C^1([0, T], \mathcal{V}'_a)$ and $\mathcal{A}\vec{u} \in C^0([0, T], \mathcal{V}'_a)$ for which*

$$\frac{d}{dt} \mathcal{B}(\vec{u}(t)) + \mathcal{A}(\vec{u}(t)) = \vec{f}(t), \quad t \in [0, T],$$

$$\lim_{t \rightarrow 0^+} \mathcal{B}(\vec{u}(t)) = \vec{v}_0 \quad \text{in } \mathcal{V}'_a.$$

If $u_1(\cdot)$ and $u_2(\cdot)$ are two such solutions, then

$$u_1(t) - u_2(t) \in \text{Ker}(\mathcal{B}) \cap \text{Ker}(\mathcal{A}), \quad 0 \leq t \leq T.$$

THEOREM 6.4 (Holomorphic Case). *Assume that the operator \mathbb{C} is m -sectorial. Let $T > 0$, $\vec{v}_0 \in \mathcal{V}'_a$, and Hölder continuous $\vec{f} \in C^\alpha([0, T], \mathcal{V}'_a)$ be given. Then there exists a function $\vec{u}: (0, T] \rightarrow \mathcal{V}$ with $\mathcal{B}\vec{u} \in C^0([0, T], \mathcal{V}'_a) \cap C^1((0, T], \mathcal{V}'_a)$ for which*

$$\frac{d}{dt} \mathcal{B}(\vec{u}(t)) + \mathcal{A}(\vec{u}(t)) = \vec{f}(t), \quad t \in (0, T],$$

$$\lim_{t \rightarrow 0^+} \mathcal{B}(\vec{u}(t)) = \vec{v}_0 \quad \text{in } \mathcal{V}'_a,$$

and it satisfies $\|\mathcal{A}(\vec{u}(t))\|_{\mathcal{V}'_a} \leq \frac{C}{t}$, $0 < t \leq T$. Any two such solutions, $u_1(\cdot)$ and $u_2(\cdot)$, satisfy

$$u_1(t) - u_2(t) \in \text{Ker}(\mathcal{B}) \cap \text{Ker}(\mathcal{A}), \quad 0 \leq t \leq T.$$

See [13, 32, 34] for alternative treatments of these results and applications to initial-boundary-value problems.

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