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Expansive homeomorphisms and indecomposable subcontinua[☆]

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Abstract

A continuum X is k -cyclic if given any $\varepsilon > 0$, there is a finite open cover \mathcal{U} of X such that $\text{mesh}(\mathcal{U}) < \varepsilon$ and the nerve $N(\mathcal{U})$ is has at most k distinct simple closed curves. A homeomorphism $h : X \rightarrow X$ is called *expansive* provided for some fixed $\varepsilon > 0$ and every $x, y \in X$ there exists an integer n such that $d(h^n(x), h^n(y)) > \varepsilon$. We prove that if X is a k -cyclic continuum that admits an expansive homeomorphism, then X must contain an nondegenerate indecomposable subcontinuum. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

One important idea that is studied in a dynamical system is what happens to points that are close together over a period of iterations or time. In dynamical continuum theory, one is interested in how far points move apart under repeated homeomorphism of a continuum onto itself. A *continuum* is defined to be a compact, connected metric space. A homeomorphism $h : X \rightarrow X$ is called *expansive* provided for some fixed $\varepsilon > 0$ and

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every $x, y \in X$ there exists an integer n such that $d(h^n(x), h^n(y)) > \varepsilon$, where if n is a positive integer, then

$$h^n(x) = \underbrace{h \circ h \circ \cdots \circ h}_n(x)$$

and if n is a negative integer, then

$$h^n(x) = \underbrace{h^{-1} \circ h^{-1} \circ \cdots \circ h^{-1}}_{|n|}(x).$$

Expansive homeomorphisms exhibit chaotic behavior in that no matter how close two points are, their images will eventually be a certain distance apart.

Plykin's attractors [6] and the dyadic solenoid [7] are examples of continua that admit an expansive homeomorphism. Both of these continua have the property of being indecomposable. A continuum is *decomposable* if it is the union of two proper subcontinuum. A continuum is *indecomposable* if it is not decomposable. Indecomposable continua are created by stretching and bending arcs an infinite number of times back and forth. Intuitively, it appears that in order to have expansiveness, the subset of the continuum between points that are close to each other would have to be continually stretched in order to move points away from each other. However, because of compactness, some folding or wrapping must also occur. Every known continuum that admits an expansive homeomorphism has an indecomposable subcontinuum.

One way to describe continua is through sequences of finite open covers. Let \mathcal{U} be a finite open cover. Since we are assuming that X is a one-dimensional continuum, we may also assume that each $x \in X$ is in at most two elements of \mathcal{U} . Define the *mesh* of \mathcal{U} to be $\text{mesh}(\mathcal{U}) = \sup\{\text{diam}(U) \mid U \in \mathcal{U}\}$ where $\text{diam}(U) = \sup\{d(x, y) \mid x, y \in U\}$. The *nerve* of \mathcal{U} is a graph $N(\mathcal{U})$ which has a vertex v_i that corresponds to each element U_i of \mathcal{U} and an edge between v_i and v_j if and only if $U_i \cap U_j \neq \emptyset$.

A continuum X is

- 1 *arc-like*,
- 2 *tree-like*,
- 3 *circle-like*,
- 4 *G-like*,
- 5 *k-cyclic*,

if given any $\varepsilon > 0$, there is a finite open cover \mathcal{U} of X such that $\text{mesh}(\mathcal{U}) < \varepsilon$ and the nerve $N(\mathcal{U})$ is

- 1 an arc,
- 2 a tree,
- 3 a circle,
- 4 homeomorphic to a fixed graph G ,
- 5 has exactly k distinct simple closed curves.

X is finitely cyclic if it is k -cyclic for some k . The double Warsaw circle or a double Warsaw circle with any number of stickers attached to it are examples of finitely cyclic continua. The Sierpinski curve is an example of an one-dimensional continuum that is not finitely cyclic.

G -like and tree-like continua are both k -cyclic. Every circle-like continuum is G -like, and every arc-like continuum is both tree-like and G -like.

A *refinement* of an open cover \mathcal{U} of X is any open cover \mathcal{V} of X whose elements are subsets of the elements of \mathcal{U} and whose union covers X . If U_i is an element of $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$, then the *core* of U_i is $U_i - (\bigcup_{j \neq i} U_j)$.

A *chain* is a collection of open subsets of X denoted by $[U_1, U_2, \dots, U_n]$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$ for each $1 \leq i, j \leq n$. A chain $[U_1, U_2, \dots, U_n]$ is a *chain from x to y* if $x \in U_1$ and $y \in U_n$. A *circle-chain* is a collection of open subsets of X denoted by $[U_1, U_2, \dots, U_n]$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$ or $|i - j| = n - 1$ for each $1 \leq i, j \leq n$.

H. Kato has shown [3] that arc-like continua do not admit expansive homeomorphisms. Further, Kato has also shown [1,2] that if X is an tree-like, circle-like or any G -like continuum that admits an expansive homeomorphism, then X must contain a nondegenerate indecomposable subcontinuum.

Theorem 1 (Kato [1]). *A nondegenerate tree-like or circle-like continuum must contain an indecomposable subcontinuum in order to admit an expansive homeomorphism.*

It follows that an arc and a simple closed curve do not admit an expansive homeomorphism.

Theorem 2 (Kato [2]). *Suppose that $h : X \rightarrow X$ is a homeomorphism of a continuum X and $f : Z \rightarrow Z$ is a homeomorphism of a continuum Z . Also, suppose that $\phi : X \rightarrow Z$ is an onto map from X onto Z such that $\phi^{-1}(z)$ is a hereditarily decomposable tree-like continuum (possibly degenerate) for every $z \in Z$, and for some z , $\phi^{-1}(z)$ is nondegenerate (i.e., ϕ is not a homeomorphism). If the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ \downarrow \phi & & \downarrow \phi \\ Z & \xrightarrow{f} & Z \end{array}$$

then h is not expansive.

A point x in X is *periodic* under h if there exists a positive integer k such that $x = h^k(x)$. Likewise, a closed subset E of X is *periodic* under h if there exists a positive integer k such that $E = h^k(E)$. If $k = 1$, then E is said to be *invariant*.

Proposition 3. *A homeomorphism $h : X \rightarrow X$ is expansive if and only if $h^k : X \rightarrow X$ is expansive for every integer $k \neq 0$.*

Proof. Let $g = h^k$. Therefore, $g^n = (h^k)^n = h^{nk}$. We will prove the proposition by showing the contrapositive for each direction.

Suppose that h is not expansive. Therefore, for every $\varepsilon > 0$ there exists $x_\varepsilon, y_\varepsilon \in X$ with $x_\varepsilon \neq y_\varepsilon$ such that $d(h^n(x_\varepsilon), h^n(y_\varepsilon)) < \varepsilon$ for every integer n . Therefore,

$d(g^n(x_\varepsilon), g^n(y_\varepsilon)) = d(h^{nk}(x_\varepsilon), h^{nk}(y_\varepsilon)) < \varepsilon$ for every integer n . Therefore, $g = h^k$ is not expansive.

Now suppose that $g = h^k$ is not expansive. Pick any $\varepsilon > 0$, and choose $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(h^i(x), h^i(y)) < \varepsilon$ for each $i \in \{0, 1, \dots, k\}$. Since g is not expansive there exists $x_\delta, y_\delta \in X$ with $x_\delta \neq y_\delta$ such that $d(h^{nk}(x_\delta), h^{nk}(y_\delta)) = d(g^n(x_\delta), g^n(y_\delta)) < \delta$ for every integer n . Thus, $d(h^{nk+i}(x_\delta), h^{nk+i}(y_\delta)) < \varepsilon$ for every integer n and every $i \in \{1, 2, \dots, k\}$. Since every integer can be represented by $nk + i$, h is not expansive. \square

Proposition 4. *No graph G admits an expansive homeomorphism.*

Proof. We may assume that G is not an arc or simple closed curve. Let V be the set of vertices of degree 1 or greater than 2. Since G is not a simple closed curve, $2 \leq |V| < \infty$. Let E be the collection of edges between the vertices in V . Since $2 \leq |V|$, E is nonempty, and since V is finite, E is finite and each edge in E is periodic under any homeomorphism h . Thus, since each edge is homeomorphic to an arc, by Proposition 3, h cannot be expansive. \square

2. Main result

In [2], Kato asked “If X admits an expansive homeomorphism, must X contain a nondegenerate indecomposable subcontinuum?”. Since every compact metric space of dimension greater than or equal to 2 contains a nondegenerate indecomposable continuum [5], only one-dimensional continua need to be considered. The next theorem is the main result of the paper and gives a partial answer in the affirmative to Kato’s question.

Theorem 5. *If X is a k -cyclic continuum that admits an expansive homeomorphism, then X must contain a nondegenerate indecomposable subcontinuum.*

The proof of the main result begins by assuming that X is a hereditarily decomposable k -cyclic continuum and $h: X \rightarrow X$ is an expansive homeomorphism. A series of assumptions on X are then shown to be valid until a contradiction is shown.

Note. It is assumed in each lemma, claim and theorem that the continuum X satisfies all the assumptions that were stated before each.

First it is shown that we may assume that X contains no proper, nondegenerate subcontinuum with period less than or equal to $k + 1$, where k is the cyclic bound for X . Then it is shown that if A is a proper subcontinuum of X , then A must be tree-like. Then under the previous assumption, it is shown that there exists a monotone map Φ from X to the simple closed curve S and a homeomorphism $f: S \rightarrow S$ such that $f \circ \Phi = \Phi \circ h$.

Since each proper subcontinuum of X is tree-like, $\Phi^{-1}(y)$ is tree-like for each y in S . Thus, by Theorem 2, either X is a graph and h cannot be expansive, or X must contain a nondegenerate indecomposable subcontinuum.

Assumption 1. *Suppose that X is a hereditarily decomposable k -cyclic continuum and that $h : X \rightarrow X$ is an expansive homeomorphism.*

By Theorem 1, we may also assume that X is neither tree-like nor circle-like. Also, since X is not tree-like, we may assume that $k \geq 1$.

Assumption 2. *X contains no proper nondegenerate subcontinuum that has period less than or equal to $\max\{2, k + 1\}$.*

The next lemma gives the justification for Assumption 2 in that we may take X to be the minimal subcontinuum described.

Lemma 6. *If X admits an expansive homeomorphism h , then there exists a minimal nondegenerate subcontinuum of period less than or equal to $k + 1$ that contains no proper subcontinuum of period less than or equal to $k + 1$.*

Proof. Let \mathcal{Y} be the collection of all subcontinua of X with period less than or equal to $k + 1$. Partially order \mathcal{Y} by inclusion. Let \mathcal{P} be any maximal chain in the ordering.

There exists an $m \in \{1, 2, \dots, k + 1\}$ such that for every $A_\alpha \in \mathcal{P}$, there exists an $E \in \mathcal{P}$ such that $E \subset A_\alpha$ and E has period m . Let

$$\mathcal{P}_m = \{A_\alpha \in \mathcal{P} \mid A_\alpha \text{ has period } m\},$$

and let $A = \bigcap_{A_\alpha \in \mathcal{P}_m} A_\alpha = \bigcap_{A_\alpha \in \mathcal{P}} A_\alpha$. Since \mathcal{P}_m is ordered by inclusion, A must be a continuum. If A is degenerate, then for any $\varepsilon > 0$ there exists a nondegenerate continuum $A_{\alpha_m} \in \mathcal{P}_m$ such that $\text{diam}(A_{\alpha_m}) < \varepsilon$. Therefore, h^m is not expansive and thus, h is not expansive. Thus, A must be nondegenerate and have diameter at least as big as the expansive constant for h^m . Notice that

$$h^m(A) = h^m\left(\bigcap_{A_\alpha \in \mathcal{P}_m} A_\alpha\right) = \bigcap_{A_\alpha \in \mathcal{P}_m} h^m(A_\alpha) = \bigcap_{A_\alpha \in \mathcal{P}_m} A_\alpha = A.$$

Thus, A is a minimal periodic subcontinuum that contains no proper subcontinuum with period less than or equal to $k + 1$. \square

Assumption 3. *Every proper subcontinuum of X is tree-like.*

Let $x \in X$ and $A \subset X$. Define $d(x, A) = \inf\{d(x, y) \mid y \in A\}$ and $d(A, B) = \inf\{d(x, y) \mid y \in A \text{ and } x \in B\}$.

Theorem 7. *If H and K are two subcontinua of X such that $X = H \cup K$ and $H \cap K$ has at least two components, then X is not tree-like.*

Proof. Suppose that $X = H \cup K$ and $H \cap K = A \cup B$, where A and B are two nonempty, disjoint closed sets. Let U and V be open sets in X such that $A \subset U$, $B \subset V$, and $\bar{U} \cap \bar{V} = \emptyset$. $H - (U \cup V)$ and $K - (U \cup V)$ are both nonempty, disjoint closed sets in X . Let $\varepsilon > 0$ such that

$$\varepsilon < \min(d(H - (U \cup V), K - (U \cup V)), d(U, V)).$$

Let \mathcal{W} be a finite open cover of X such that $\text{mesh}(\mathcal{W}) < \varepsilon/4$.

Let W_i be an element of \mathcal{W} which intersects A and let W_j be an element of \mathcal{W} which intersects B . Since H and K are both connected, there exists a chain \mathcal{C}_H from W_i to W_j of elements \mathcal{W} each of which intersects H , and a chain \mathcal{C}_K from W_i to W_j of elements \mathcal{W} each of which intersects K . By our choice of ε , there must exist a $C_h \in \mathcal{C}_H$ and a $C_k \in \mathcal{C}_K$ such that $C_h \cap (H - (U \cup V)) \neq \emptyset$ and $C_k \cap (K - (U \cup V)) \neq \emptyset$. Pick $x \in C_h$ and $y \in C_k$. Then, $d(x, H - (U \cup V)) < \varepsilon/4$ and $d(y, K - (U \cup V)) < \varepsilon/4$. So,

$$d(x, y) > d(H - (U \cup V), K - (U \cup V)) - \varepsilon/4 - \varepsilon/4 > \varepsilon/2.$$

Therefore, $x \notin C_k$ and $y \notin C_h$. Thus, $C_h \cap C_k = \emptyset$ and the nerve of $\mathcal{C}_H \cup \mathcal{C}_K$ must contain a simple closed curve. Hence, X is not tree-like. \square

Theorem 8. *If H and K are two subcontinua of X such that $X = H \cup K$ and $H \cap K$ has at least $k + 2$ components, then X is not k -cyclic.*

Proof. The proof is similar to that of Theorem 7. \square

Corollary 9. *If H and K are two subcontinua of X such that $X = H \cup K$ and $H \cap K$ has an infinite number of components, then X is not k -cyclic for any k .*

Lemma 10. *If $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ is a finite open cover of a continuum X such that the nerve of \mathcal{U} is a tree and \mathcal{V} is a cover of X with $V_i \subset U_i$ for every $i \in \{1, 2, \dots, n\}$, then the nerve of $\mathcal{V} = \{V_1, V_2, \dots, V_n\}$ is also a tree.*

Proof. Suppose that the nerve of \mathcal{V} is not a tree and has dimension less than or equal to 1. Thus, the nerve of \mathcal{V} must contain a simple closed curve. Let $\{V_{i_1}, V_{i_2}, \dots, V_{i_m}, V_{i_1}\}$ be elements of \mathcal{V} whose nerve is a simple closed curve, that is, $V_{i_j} \cap V_{i_{j+1}} \neq \emptyset$ if $j \in \{1, 2, \dots, m-1\}$, and $V_{i_m} \cap V_{i_1} \neq \emptyset$. But since $V_i \subset U_i$ for every $i \in \{1, 2, \dots, n\}$, that would imply that $U_{i_j} \cap U_{i_{j+1}} \neq \emptyset$ if $j \in \{1, 2, \dots, m-1\}$, and $U_{i_m} \cap U_{i_1} \neq \emptyset$. Thus, the nerve of \mathcal{U} must contain a simple closed curve. Thus, \mathcal{U} is not tree-like. This is a contradiction. \square

Theorem 11. *If H and K are tree-like and $H \cap K$ has only one component, then $H \cup K$ is tree-like. Likewise, if $H \cap K$ has $k + 1$ components, then $H \cup K$ is k -cyclic.*

Proof. Pick $\varepsilon > 0$ and let \mathcal{U}_H be any finite open cover of H such that $\text{mesh}(\mathcal{U}_H) < \varepsilon$, \mathcal{U}_H is open relative to $H \cup K$, and the nerve of \mathcal{U}_H is a tree. Let \mathcal{U}_K be a finite open cover of K such that $\text{mesh}(\mathcal{U}_K) < \varepsilon$, if $U_K \in \mathcal{U}_K$ and $U_K \cap (H \cap K) \neq \emptyset$, then U_K is a subset of some $U_H \in \mathcal{U}_H$, and the nerve of \mathcal{U}_K is a tree. Since $H \cap K$ is a continuum, the nerve of

$$\mathcal{U}_{H \cap K} = \{U \mid U \in \mathcal{U}_K \text{ and } U \cap (H \cap K) \neq \emptyset\}$$

must be connected and a tree. Let

$$\begin{aligned} \tilde{\mathcal{U}}_K &= \{U \mid U \in \mathcal{U}_K \text{ and } U \cap (H \cap K) = \emptyset\} \\ &\cup \left\{ \bigcup_{U \subset U_H} U \mid U \in \mathcal{U}_K, U \cap (H \cap K) \neq \emptyset \text{ and } U_H \in \mathcal{U}_H \right\}. \end{aligned}$$

Claim. *The nerve of $\tilde{\mathcal{U}}_K$ is a tree.*

Proof of Claim. Suppose not. Let $\{\tilde{U}_{i_1}, \tilde{U}_{i_2}, \dots, \tilde{U}_{i_m}, \tilde{U}_{i_1}\}$ be elements of $\tilde{\mathcal{U}}$ whose nerve is a simple closed curve. Let

$$\tilde{\mathcal{U}}_{H \cap K} = \{\tilde{U} \mid \tilde{U} \in \tilde{\mathcal{U}}_K \text{ and } \tilde{U} \cap (H \cap K) \neq \emptyset\}.$$

Then, the nerve of $\tilde{\mathcal{U}}_{H \cap K}$ is connected and by Lemma 10 is a tree. Suppose there exists a j such that $\tilde{U}_{i_j} \notin \tilde{\mathcal{U}}_{H \cap K}$. Then $\tilde{U}_{i_j} \cap (H \cap K) = \emptyset$, and therefore, $\tilde{U}_{i_j} \in \mathcal{U}_K$. If $\tilde{U}_{i_j} \notin \tilde{\mathcal{U}}_{H \cap K}$ for every $j \in \{1, 2, \dots, m\}$, then $\{\tilde{U}_{i_1}, \tilde{U}_{i_2}, \dots, \tilde{U}_{i_m}, \tilde{U}_{i_1}\} \subset \mathcal{U}_K$. However, that would imply that nerve of \mathcal{U}_K would contain a simple closed curve which is a contradiction. Thus, we may choose j, n such that $\{\tilde{U}_{i_j}, \dots, \tilde{U}_{i_n}\}$ is a subchain of $\{\tilde{U}_{i_1}, \tilde{U}_{i_2}, \dots, \tilde{U}_{i_m}, \tilde{U}_{i_1}\}$, $\tilde{U}_{i_p} \notin \tilde{\mathcal{U}}_{H \cap K}$ for every $\tilde{U}_{i_p} \in \{\tilde{U}_{i_j}, \dots, \tilde{U}_{i_n}\}$, $\tilde{U}_{i_{j'}} \in \tilde{\mathcal{U}}_{H \cap K}$, and $\tilde{U}_{i_{n'}} \in \tilde{\mathcal{U}}_{H \cap K}$, where $\tilde{U}_{i_{j'}}, \tilde{U}_{i_{n'}}$ are the unique elements of $\{\tilde{U}_{i_1}, \tilde{U}_{i_2}, \dots, \tilde{U}_{i_m}, \tilde{U}_{i_1}\} - \{\tilde{U}_{i_j}, \dots, \tilde{U}_{i_n}\}$ such that $\tilde{U}_{i_j} \cap \tilde{U}_{i_{j'}} \neq \emptyset$ and $\tilde{U}_{i_n} \cap \tilde{U}_{i_{n'}} \neq \emptyset$. Now, there exist $U_\alpha, U_\beta \in \mathcal{U}_K$ such that $U_\alpha \subset \tilde{U}_{i_{j'}}$, $U_\beta \subset \tilde{U}_{i_{n'}}$, $U_\alpha \cap \tilde{U}_{i_j} \neq \emptyset$, and $U_\beta \cap \tilde{U}_{i_n} \neq \emptyset$. Since $H \cap K$ is a continuum, there exists a chain $\{U_\alpha, U_\gamma, \dots, U_\beta\}$ of elements of $\mathcal{U}_{H \cap K}$ from U_α to U_β . Thus, $\{\tilde{U}_{i_j}, \dots, \tilde{U}_{i_n}, U_\beta, \dots, U_\alpha, \tilde{U}_{i_j}\} \subset \mathcal{U}_K$. However, the nerve of $\{U_{i_j}, \dots, U_{i_n}, U_\beta, \dots, U_\alpha, U_{i_j}\}$ is a simple closed curve. Thus, the nerve of \mathcal{U}_K is not a tree, which is a contradiction. Thus, $\tilde{U}_{i_j} \in \{\tilde{U}_{i_1}, \tilde{U}_{i_2}, \dots, \tilde{U}_{i_m}, \tilde{U}_{i_1}\}$ implies $\tilde{U}_{i_j} \in \tilde{\mathcal{U}}_{H \cap K}$. Hence, $\{\tilde{U}_{i_1}, \tilde{U}_{i_2}, \dots, \tilde{U}_{i_m}, \tilde{U}_{i_1}\} \subset \tilde{\mathcal{U}}_{H \cap K}$. But that is impossible since the nerve of $\tilde{\mathcal{U}}_{H \cap K}$ is a tree. Thus, the claim is proved.

Notice that if $V \in \tilde{\mathcal{U}}_K$ then either $V \in \mathcal{U}_K$ or $V \subset U_H$ where $U_H \in \mathcal{U}_H$. Thus, $\text{mesh}(\tilde{\mathcal{U}}_K) < \varepsilon$. Let $\mathcal{U}_H = \{U_{H_1}, U_{H_2}, \dots, U_{H_n}\}$ and $\tilde{\mathcal{U}}_K = \{U_{K_1}, U_{K_2}, \dots, U_{K_m}\}$.

Now compose \mathcal{W} of the following open sets:

- (1) $W_{H_i} = U_{H_i} - K$ if $U_{H_i} \in \mathcal{U}_H$ and $U_{H_i} \cap (H \cap K) = \emptyset$.
- (2) $W_{K_j} = U_{K_j} - H$ if $U_{K_j} \in \tilde{\mathcal{U}}_K$ and $U_{K_j} \cap (H \cap K) = \emptyset$.
- (3) $W_{H_i} = W_{K_j} = (U_{H_i} - (K - (H \cap K))) \cup (U_{K_j} - (H - (H \cap K)))$ if $U_{H_i} \in \mathcal{U}_H$,

$U_{K_j} \in \tilde{\mathcal{U}}_K$ and $U_{K_j} \subset U_{H_i}$.

Notice that if $W_{H_i} = W_{K_j}$ and $W_{H_{i'}} = W_{K_{j'}}$, then $W_{K_{j'}} \cap W_{K_j} \neq \emptyset$ if and only if $U_{K_{j'}} \cap U_{K_j} \neq \emptyset$ if and only if $U_{H_{i'}} \cap U_{H_i} \neq \emptyset$. Thus, by Lemma 10, the nerves of both $\mathcal{W}_H = \{W_{H_i}\}_{i=1}^n$ and $\mathcal{W}_K = \{W_{K_j}\}_{j=1}^m$ are trees. Let

$$\mathcal{W}_{H \cap K} = \{W_{H_i} \mid W_{H_i} \cap (H \cap K) \neq \emptyset\} = \{W_{K_j} \mid W_{K_j} \cap (H \cap K) \neq \emptyset\}.$$

Then $\mathcal{W}_{H \cap K} \subset \mathcal{W}_H$ (and likewise, $\mathcal{W}_{H \cap K} \subset \mathcal{W}_K$). Thus, the nerve of $\mathcal{W}_{H \cap K}$ is a tree. Also, if $W_{H_i} \in \mathcal{W}_H$ and $W_{H_i} \cap (H \cap K) = \emptyset$, then $W_{H_i} \cap K = \emptyset$. Likewise, if $W_{K_j} \in \mathcal{W}_K$

and $W_{K_j} \cap (H \cap K) = \emptyset$, then $W_{K_j} \cap H = \emptyset$. Thus if the nerve of $\mathcal{W} = \mathcal{W}_H \cup \mathcal{W}_K$ is not a tree, then the nerve of

$$\mathcal{W}_{H \cap K} = \{W \mid W \cap (H \cap K) \neq \emptyset\}$$

is not a tree, which is a contradiction. Thus, the nerve of \mathcal{W} is a tree. Proof is similar when $H \cap K$ has $k + 1$ components. \square

Suppose that \mathcal{U} is a finite open cover of continuum A . Let B be a subcontinuum of A and let $\mathcal{U}(B) = \{U \in \mathcal{U} \mid U \cap B \neq \emptyset\}$.

Lemma 12. *Let X be a one-dimensional continuum and let A be a tree-like subcontinuum of X . If \mathcal{U} is a finite open cover of X , then there exists a finite open cover \mathcal{W} of X such that \mathcal{W} refines \mathcal{U} , the nerve of \mathcal{W} is one-dimensional, and the nerve of $\mathcal{W}(A)$ is a tree.*

Proof. Let \mathcal{U} be a finite open cover of X . There exists a finite open cover \mathcal{V} of X such that \mathcal{V} refines \mathcal{U} and the nerve of $\mathcal{V}(A)$ is a tree. Since X is one-dimensional, there exists a finite open cover \mathcal{D} of X such that \mathcal{D} refines \mathcal{V} and the nerve of \mathcal{D} is one-dimensional. Define

$$\mathcal{W} = (\mathcal{D} - \mathcal{D}(A)) \cup \left\{ \bigcup_{U \subset V} U \mid U \in \mathcal{D}(A) \text{ and } V \in \mathcal{V}(A) \right\}.$$

Then by Lemma 10, \mathcal{W} satisfies the conclusion of the lemma. \square

From here on out, we may assume that if \mathcal{W} is an open cover of X that refines open cover \mathcal{U} such that the nerve of $\mathcal{W}(A)$ is a tree, then the nerve of \mathcal{W} is one-dimensional.

If \mathcal{U} is a collection of sets, then \mathcal{U}^* is the union of the elements of \mathcal{U} . If $x \in \mathcal{U}$, then the *star of x in \mathcal{U}* , denoted by $\text{st}(x, \mathcal{U})$, is the collection of elements of \mathcal{U} that contain x (i.e., $\text{st}(x, \mathcal{U}) = \{U \in \mathcal{U} \mid x \in U\}$). If $A \subset \mathcal{U}^*$, then $\text{st}(A, \mathcal{U}) = \{U \in \mathcal{U} \mid A \cap U \neq \emptyset\}$. Inductively, $\text{st}^{i+1}(x, \mathcal{U}) = \text{st}(\text{st}^i(x, \mathcal{U})^*, \mathcal{U})$.

Lemma 13. *Let X be a k -cyclic continuum and let \mathcal{T} be a collection of tree-like subcontinua of X such that \mathcal{T} has p elements. If \mathcal{U}_1 is a finite open cover of X such that the nerve of \mathcal{U}_1 has at most k simple closed curves, then there exists a finite open cover \mathcal{W} of X such that \mathcal{W} refines \mathcal{U}_1 , the nerve of \mathcal{W} contains at most k simple closed curves, and the nerve of $\mathcal{W}(T)$ is a tree for each $T \in \mathcal{T}$.*

Proof. First, we must verify a claim.

Claim. *There exists a finite open cover \mathcal{V} of X such that \mathcal{V} refines \mathcal{U}_1 and the nerve of $\mathcal{V}(T)$ is a tree for each $T \in \mathcal{T}$.*

Proof of Claim. Pick $T_1 \in \mathcal{T}$. Since T_1 is tree-like, there exists a finite open cover \mathcal{V}_1 that refines \mathcal{U}_1 such that the nerve of $\mathcal{V}_1(T_1)$ is a tree.

Pick $T_2 \in \mathcal{T}$. Since T_2 is tree-like, there exists a finite open cover \mathcal{V}_2 that refines \mathcal{V}_1 such that the nerve of $\mathcal{V}_2(T_2)$ is a tree. Define

$$\mathcal{V}'_2 = (\mathcal{V}_2 - \mathcal{V}_2(T_1)) \cup \left\{ \bigcup_{U \subset V} U \mid V \in \mathcal{V}_1(T_1) \text{ and } U \in \mathcal{V}_2(T_1) \right\}.$$

Thus, \mathcal{V}'_2 refines \mathcal{V}_1 , and the nerves of $\mathcal{V}'_2(T_1)$ and $\mathcal{V}'_2(T_2)$ are trees.

Pick $T_3 \in \mathcal{T}$. Since T_3 is tree-like, there exists a finite open cover \mathcal{V}_3 that refines \mathcal{V}'_2 such that the nerve of $\mathcal{V}_3(T_3)$ is a tree. Define

$$\begin{aligned} \mathcal{V}'_3 = & (\mathcal{V}_3 - (\mathcal{V}_3(T_1) \cup \mathcal{V}_3(T_2))) \cup \left\{ \bigcup_{U \subset V} U \mid V \in \mathcal{V}'_2(T_1) \text{ and } U \in \mathcal{V}_3(T_1) \right\} \\ & \cup \left\{ \bigcup_{U \subset V} U \mid V \in \mathcal{V}'_2(T_2) \text{ and } U \in \mathcal{V}_3(T_2) \right\}. \end{aligned}$$

Thus, \mathcal{V}'_3 refines \mathcal{V}'_2 , and the nerves of $\mathcal{V}'_3(T_1)$, $\mathcal{V}'_3(T_2)$ and $\mathcal{V}'_3(T_3)$ are trees. Continue in the same manner inductively. Suppose that \mathcal{V}_p has been found. Define

$$\mathcal{V} = \left(\mathcal{V}_p - \left(\bigcup_{j=1}^p \mathcal{V}_p(T_j) \right) \right) \cup \left(\bigcup_{j=1}^p \left\{ \bigcup_{U \subset V} U \mid V \in \mathcal{V}'_{p-1}(T_j) \text{ and } U \in \mathcal{V}_p(T_j) \right\} \right).$$

Then \mathcal{V} satisfies the claim.

If the nerve of \mathcal{V} has at most k simple closed curves, let $\mathcal{W} = \mathcal{V}$ and we are done. Otherwise, since X is k -cyclic, there exists a finite open cover \mathcal{U}_2 of X such that \mathcal{U}_2 refines \mathcal{V} and the nerve of \mathcal{U}_2 contains n simple closed curves, where $n \leq k$. Let $\{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n\}$ be the n distinct subsets of \mathcal{U}_2 such that the nerve of each \mathcal{C}_i is a simple closed curve. If, for each $j \in \{1, \dots, p\}$, no \mathcal{C}_i^* is a subset of $\mathcal{V}(T_j)^*$, then no \mathcal{C}_i is a subset of $\mathcal{U}_2(T_j)$, and thus, the nerve of $\mathcal{U}_2(T_j)$ is a tree for each $T_j \in \mathcal{T}$, and we are done. Otherwise, suppose that $\{\mathcal{C}_{i_1}, \mathcal{C}_{i_2}, \dots, \mathcal{C}_{i_{m_j}}\}$ are the elements of $\{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n\}$ such that $\mathcal{C}_{i_l}^* \subset \mathcal{V}(T_j)^*$ for every $l \in \{1, \dots, m_j\}$, where $m_j \leq n$. Let $\mathcal{A}_j = \bigcup_{l=1}^{m_j} \mathcal{C}_{i_l}$. Define

$$\mathcal{W} = \left(\mathcal{U}_2 - \left(\bigcup_{j=1}^p \mathcal{A}_j \right) \right) \cup \left(\bigcup_{j=1}^p \left\{ \bigcup_{U \subset V} U \mid V \in \mathcal{V}(T_j) \text{ and } U \in \mathcal{A}_j \right\} \right).$$

Thus, \mathcal{W} is a finite open cover of X such that \mathcal{W} refines \mathcal{U}_1 , the nerve of \mathcal{W} contains at most k simple closed curves and the nerve of $\mathcal{W}(T)$ is a tree for each $T \in \mathcal{T}$. \square

Theorem 14. *If A is a proper subcontinuum of X , then A is tree-like.*

Proof. Let \mathcal{A} be the set of all subcontinua of X that are not tree-like. Let \mathcal{P} be a partial ordering of \mathcal{A} by inclusion and let \mathcal{M} be a maximal chain of \mathcal{P} . Let

$$M = \bigcap_{A \in \mathcal{M}} A.$$

Claim. M is not tree-like.

Proof of Claim. The proof is by induction.

Base case. Suppose that there exists a subset \mathcal{M}_1 of \mathcal{M} such that each $A \in \mathcal{M}_1$ is 1-cyclic, $M = \bigcap_{A \in \mathcal{M}} A = \bigcap_{A \in \mathcal{M}_1} A$, and that M is tree-like. Pick any $A_1 \in \mathcal{M}_1$. By Lemma 13, there exists a sequence of finite open covers, $\{\mathcal{U}_n\}_{n=1}^\infty$, of A_1 such that $\text{mesh}(\mathcal{U}_n) \rightarrow 0$ as $n \rightarrow \infty$, \mathcal{U}_{n+1} refines \mathcal{U}_n for each n , the nerve of \mathcal{U}_n has exactly 1 simple closed curve, and the nerve of $\mathcal{U}_n(M)$ is a tree.

Since $M = \bigcap_{A \in \mathcal{M}_1} A$, there exist an $A_2 \in \mathcal{M}_1$ which is a proper subcontinuum of A_1 such that the nerve of $\mathcal{U}_1(A_2)$ is a tree. Since A_2 is not tree-like, there exists an integer N_2 such that for every $n \geq N_2$, the nerve of $\mathcal{U}_n(A_2)$ is not a tree. Thus, the nerve of $\mathcal{U}_{N_2}(A_2)$ must contain a simple closed curve. Let

$$\mathcal{V}_1 = (\mathcal{U}_{N_2} - \mathcal{U}_{N_2}(A_2)) \cup \left\{ \bigcup_{U \subset V} U \mid U \in \mathcal{U}_{N_2}(A_2) \text{ and } V \in \mathcal{U}_1(A_2) \right\}.$$

\mathcal{V}_1 is a finite open cover of A_1 , and by Lemma 10, the nerve of $\mathcal{V}_1(A_2)$ is a tree. Since the nerve of \mathcal{U}_{N_2} contains exactly 1 simple closed curve, the nerve of \mathcal{V}_1 must be a tree. Notice that $\text{mesh}(\mathcal{V}_1) \leq \text{mesh}(\mathcal{U}_1)$.

Suppose that \mathcal{U}_{N_i} has been found. Since $M = \bigcap_{A \in \mathcal{M}_1} A$, there exists an $A_{i+1} \in \mathcal{M}_1$ which is a proper subcontinuum of A_1 such that the nerve of $\mathcal{U}_{N_i}(A_{i+1})$ is a tree. Since A_{i+1} is not tree-like, there exists an integer N_{i+1} such that for every $n \geq N_{i+1}$, the nerve of $\mathcal{U}_n(A_{i+1})$ is not a tree. Thus, the nerve of $\mathcal{U}_{N_{i+1}}(A_{i+1})$ must contain a simple closed curve. Let

$$\mathcal{V}_i = (\mathcal{U}_{N_{i+1}} - \mathcal{U}_{N_{i+1}}(A_{i+1})) \cup \left\{ \bigcup_{U \subset V} U \mid U \in \mathcal{U}_{N_{i+1}}(A_{i+1}) \text{ and } V \in \mathcal{U}_{N_i}(A_{i+1}) \right\}.$$

\mathcal{V}_i is a finite open cover of A_1 , and the nerve of $\mathcal{V}_i(A_2)$ is a tree. Since the nerve of $\mathcal{U}_{N_{i+1}}$ contains exactly 1 simple closed curve, the nerve of \mathcal{V}_i must be a tree. Notice that $\text{mesh}(\mathcal{V}_i) \leq \text{mesh}(\mathcal{U}_{N_i})$. Thus, $\text{mesh}(\mathcal{V}_i) \rightarrow 0$ as $i \rightarrow \infty$. Thus, A_1 must be tree-like. This is a contradiction. Thus, M is not tree-like.

Induction step. Suppose that there exists a subset \mathcal{M}_j of \mathcal{M} such that each $A \in \mathcal{M}_j$ is j -cyclic, but not $(j-1)$ -cyclic, $M = \bigcap_{A \in \mathcal{M}} A = \bigcap_{A \in \mathcal{M}_j} A$, and that M is tree-like. Pick any $A_1 \in \mathcal{M}_j$. There exists a sequence of finite open covers, $\{\mathcal{U}_n\}_{n=1}^\infty$, of A_1 such that $\text{mesh}(\mathcal{U}_n) \rightarrow 0$ as $n \rightarrow \infty$, \mathcal{U}_{n+1} refines \mathcal{U}_n for each n , the nerve of \mathcal{U}_n has exactly j simple closed curves, and the nerve of $\mathcal{U}_n(M)$ is a tree.

Since $M = \bigcap_{A \in \mathcal{M}_j} A$, there exist an $A_2 \in \mathcal{M}_j$ which is a proper subcontinuum of A_1 such that the nerve of $\mathcal{U}_1(A_2)$ is a tree. Now, since A_2 is not $(j-1)$ -cyclic, there exists an integer N_2 such that for every $n \geq N_2$, the nerve of $\mathcal{U}_n(A_2)$ contains exactly j simple closed curves. Let

$$\mathcal{V}_1 = (\mathcal{U}_{N_2} - \mathcal{U}_{N_2}(A_2)) \cup \left\{ \bigcup_{U \subset V} U \mid U \in \mathcal{U}_{N_2}(A_2) \text{ and } V \in \mathcal{U}_{N_1}(A_2) \right\}.$$

\mathcal{V}_1 is a finite open cover of A_1 , and the nerve of $\mathcal{V}_1(A_2)$ is a tree. Since the nerve of \mathcal{U}_{N_2} contains exactly j simple closed curves, and the circular chains of $\mathcal{U}_{N_2}(A_2)$ have been collapsed, the nerve of \mathcal{V}_1 must contain no more than $j-1$ simple closed curves. Notice that $\text{mesh}(\mathcal{V}_1) \leq \text{mesh}(\mathcal{U}_1)$.

Suppose that \mathcal{U}_{N_i} has been found. Since $M = \bigcap_{A \in \mathcal{M}_j} A$, there exists an $A_{i+1} \in \mathcal{M}_j$ which is a proper subcontinuum of A_1 such that the nerve of $\mathcal{U}_{N_i}(A_{i+1})$ is a tree. Now since A_{i+1} is not $(j - 1)$ -cyclic, there exists an integer N_{i+1} such that for every $n \geq N_{i+1}$, the nerve of $\mathcal{U}_n(A_{i+1})$ has exactly j simple closed curves. Let

$$\mathcal{V}_i = (\mathcal{U}_{N_{i+1}} - \mathcal{U}_{N_{i+1}}(A_{i+1})) \cup \left\{ \bigcup_{U \subset V} U \mid U \in \mathcal{U}_{N_{i+1}}(A_{i+1}) \text{ and } V \in \mathcal{U}_{N_i}(A_{i+1}) \right\}.$$

\mathcal{V}_i is a finite open cover of A_1 , and the nerve of $\mathcal{V}_i(A_2)$ is a tree. Since the nerve of $\mathcal{U}_{N_{i+1}}$ contains exactly j simple closed curves, and the circular chains of $\mathcal{U}_{N_{i+1}}(A_2)$ have been collapsed, the nerve of \mathcal{V}_i must contain no more than $j - 1$ simple closed curves. Notice that $\text{mesh}(\mathcal{V}_i) \leq \text{mesh}(\mathcal{U}_{N_i})$. Thus, $\text{mesh}(\mathcal{V}_i) \rightarrow 0$ as $i \rightarrow \infty$, and hence, A_1 must be $(j - 1)$ -cyclic. This is a contradiction. Thus, M is not tree-like, and the claim is proved.

If B is a proper subcontinuum of M , then B is tree-like. Also, the same properties that hold for M must hold for $h^j(M)$ for every j . Consider the collection

$$\{M, h(M), \dots, h^k(M), h^{k+1}(M)\}$$

and suppose there exist $i, j \in \{0, 1, \dots, k + 1\}$ such that $i \neq j$ and $h^i(M) = h^j(M)$. Then M has period $\leq k + 1$ which is a contradiction. Thus, either $M = X$ or $h^i(M) \neq h^j(M)$ for every $i \neq j$. Suppose $h^i(M) \cap h^j(M)$ has a component C that is not tree-like. Since C is a continuum, and $h^i(M)$ and $h^j(M)$ are minimal, $h^i(M) = C = h^j(M)$. This is a contradiction. Thus, every component of $h^i(M) \cap h^j(M)$ is tree-like. Also, since X is k -cyclic, the number of components of $h^i(M) \cap h^j(M)$ must be finite by Corollary 9.

Let $\mathcal{C} = \{C \mid C \text{ is a component of } h^i(M) \cap h^j(M) \text{ for some } i \neq j\}$. We may suppose that X is k -cyclic but not $(k - 1)$ -cyclic. Since each $h^i(M)$ is not tree-like, there exists an $\varepsilon > 0$ such that if \mathcal{U} is a finite open cover of X such that $\text{mesh}(\mathcal{U}) < \varepsilon$, then the nerve of $\mathcal{U}(h^i(M))$ is not a tree for each i . Let $\{\mathcal{U}_n\}$ be a sequence of finite open covers of X such that $\text{mesh}(\mathcal{U}_n) < \varepsilon$ for each n , the nerve of each \mathcal{U}_n has exactly k simple closed curves, $\lim_{n \rightarrow \infty} (\text{mesh}(\mathcal{U}_n)) = 0$, each \mathcal{U}_{n+1} refines \mathcal{U}_n , and the nerve of $\mathcal{U}_n(C)$ is a tree for each $C \in \mathcal{C}$.

Claim. *There exists an integer N such that for every $n \geq N$ and each $i \neq j$, the nerves of $\mathcal{U}_n(h^i(M))$ and $\mathcal{U}_n(h^j(M))$ contain distinct simple closed curves.*

Proof of Claim. Suppose not. Thus, suppose for every p , there exists $n \geq p$ such that $i_n \neq j_n$ and the nerves of $\mathcal{U}_n(h^{i_n}(M))$ and $\mathcal{U}_n(h^{j_n}(M))$ do not have distinct simple closed curves. Since there is only a finite number of combinations of $\{i, j\}$, there exists a subsequence $\{n_m\}$ of $\{n\}$ such that $i_m = i_l$ and $j_m = j_l$ for all positive integers m, l . For ease of notation, let $\{n_m\} = \{n\}$, $i = i_m$ and $j = j_m$.

The nerves of $\mathcal{U}_n(h^i(M))$ and $\mathcal{U}_n(h^j(M))$ must both contain a collection of circle-chains, say \mathcal{S}_n^i and \mathcal{S}_n^j . Since the nerves of $\mathcal{U}_n(h^i(M))$ and $\mathcal{U}_n(h^j(M))$ do not have distinct simple closed curves, either $\mathcal{S}_n^i \subset \mathcal{S}_n^j$ or $\mathcal{S}_n^j \subset \mathcal{S}_n^i$ for each n . Thus, without loss of generality, we may assume that there exists an infinite subsequence $\{n_m\}$ of $\{n\}$ such that $\mathcal{S}_{n_m}^i \subset \mathcal{S}_{n_m}^j$ for each m . Again, for ease of notation, let $\{n\}$ represent $\{n_m\}$.

Pick $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < d(h^i(M) - \mathcal{U}_1(h^i(M) \cap h^j(M)), h^i(M) - \mathcal{U}_1(h^i(M) \cap h^j(M))).$$

There exists integer N_1 such that $\text{mesh}(\mathcal{U}_n) < \varepsilon_1/4$ for every $n \geq N_1$. Consider the circle-chains of $\mathcal{U}_{N_1}(h^i(M))$ and $\mathcal{U}_{N_1}(h^j(M))$. By the choice of $\varepsilon_1/4$, if $(\mathcal{S}_{N_1}^i)^* \not\subset \mathcal{U}_1(h^i(M) \cap h^j(M))^*$, then $\mathcal{S}_{N_1}^i$ would contain a circle-chain distinct from any circle-chain of $\mathcal{S}_{N_1}^j$. This contradicts our hypothesis. Thus, $(\mathcal{S}_{N_1}^i)^* \subset \mathcal{U}_1(h^i(M) \cap h^j(M))^*$.

Define

$$\mathcal{V}_1 = (\mathcal{U}_{N_1} - \mathcal{S}_{N_1}^i) \cap \left\{ \bigcup_{U \subset V} U \mid U \in \mathcal{S}_{N_1}^i \text{ and } V \in \mathcal{U}_1(h^i(M) \cap h^j(M)) \right\}.$$

Notice that the nerve of \mathcal{V}_1 is a tree. Let δ_1 be the Lebesgue number of \mathcal{V}_1 . There exists an integer N'_2 such that for every $n \geq N'_2$, $\text{mesh}(\mathcal{U}_n) < \delta_1$ and thus, $\mathcal{U}_{N'_2}$ refines \mathcal{V}_1 .

Let

$$\varepsilon_2 < d(h^i(M) - \mathcal{U}_{N'_2}(h^i(M) \cap h^j(M)), h^i(M) - \mathcal{U}_{N'_2}(h^i(M) \cap h^j(M))).$$

Continue in this manner inductively.

Suppose that the Lebesgue number, δ_{l-1} , of \mathcal{V}_{l-1} has been found. There exists an integer N'_l such that for every $n \geq N'_l$, $\text{mesh}(\mathcal{U}_n) < \delta_{l-1}$ and thus, $\mathcal{U}_{N'_l}$ refines \mathcal{V}_{l-1} .

Let

$$\varepsilon_l < d(h^i(M) - \mathcal{U}_{N'_l}(h^i(M) \cap h^j(M)), h^i(M) - \mathcal{U}_{N'_l}(h^i(M) \cap h^j(M))).$$

There exists an integer N_l such that $\text{mesh}(\mathcal{U}_n) < \varepsilon_l/4$ for every $n \geq N_l$. Consider the circle-chains of $\mathcal{U}_{N_l}(h^i(M))$ and $\mathcal{U}_{N_l}(h^j(M))$. By the choice of $\varepsilon_l/4$, if $(\mathcal{S}_{N_l}^i)^* \not\subset \mathcal{U}_l(h^i(M) \cap h^j(M))^*$, then $\mathcal{S}_{N_l}^i$ would contain a circle-chain distinct from any circle-chain of $\mathcal{S}_{N_l}^j$. This contradicts our hypothesis. Thus, $(\mathcal{S}_{N_l}^i)^* \subset \mathcal{U}_l(h^i(M) \cap h^j(M))^*$.

Define

$$\mathcal{V}_l = (\mathcal{U}_{N_l} - \mathcal{S}_{N_l}^i) \cap \left\{ \bigcup_{U \subset V} U \mid U \in \mathcal{S}_{N_l}^i \text{ and } V \in \mathcal{U}_l(h^i(M) \cap h^j(M)) \right\}.$$

Notice that the nerve of \mathcal{V}_l is a tree. Thus, $\{\mathcal{V}_l\}_{l=1}^\infty$ is a sequence of finite open covers of X such that $\text{mesh}(\mathcal{V}_l) \rightarrow 0$ as $l \rightarrow \infty$, \mathcal{V}_{l+1} refines \mathcal{V}_l and the nerve of $\mathcal{V}_l(h^i(M))$ is a tree. Thus, $h^i(M)$ is tree-like. However, this contradicts the fact that $h^i(M)$ is not tree-like. Thus, there exist an integer N such that for every $n \geq N$, the nerves of $\mathcal{U}_n(h^i(M))$ and $\mathcal{U}_n(h^j(M))$ contain distinct simple closed curves. Hence, the claim is proved.

Thus, for each n , the nerves of $\{\mathcal{U}_n(M), \mathcal{U}_n(h(M)), \dots, \mathcal{U}_n(h^k(M)), \mathcal{U}_n(h^{k+1}(M))\}$ all contain a distinct simple closed curve. Thus, for each $n \geq N$, the nerve of \mathcal{U}_n contains more than k simple closed curves. This is a contradiction. Thus, M must equal X , and every proper subcontinuum of X is tree-like. \square

3. Irreducible continua, tranches and Kuratowski's theorem

A subset A of a metric space X is ε -dense in X if for every $x \in X$, $d(x, A) < \varepsilon$. A map $g : X \rightarrow Y$ is *monotone* if $g^{-1}(y)$ is connected for each $y \in Y$. If X is a continuum, then

each $g^{-1}(y)$ is a continuum (possibly degenerate). Also, if A is any subcontinuum of $g(X)$, then $g^{-1}(A)$ is a subcontinuum of X .

A continuum X is *irreducible* between a and b if $a, b \in X$ and if A is a proper subcontinuum of X , then $a \notin A$ or $b \notin A$. The notation, $I(a, b)$, will be used to imply that $I(a, b)$ is a continuum irreducible between a and b . If K is a subcontinuum of X and $a, b \in K$, then let $I_K(a, b)$ be a subcontinuum of K irreducible between a and b .

Theorem 15 (Kuratowski [4]). *If X is hereditarily decomposable and is irreducible between a and b , then there exists a monotone map $\psi: X \rightarrow [0, 1]$ such that $\psi(a) = 0$ and $\psi(b) = 1$. In fact, there exists a minimal monotone onto map $\phi: X \rightarrow [0, 1]$ such that if $\psi: X \rightarrow [0, 1]$ is any other monotone onto map and $\phi^{-1}(z) \cap \psi^{-1}(y) \neq \emptyset$, then $\phi^{-1}(z) \subset \psi^{-1}(y)$.*

If $\phi: X \rightarrow [0, 1]$ is a minimal monotone onto map, then each $\phi^{-1}(y)$ is called a *tranche* of X . Tranches are *nowhere dense* subcontinua of X . That is, $\text{int}(\phi^{-1}(y)) = \emptyset$. Also, if $I(a, x) = I(a, y)$, then x and y are in the same tranche of $I(a, b)$.

The following well-known lemma is needed:

Lemma 16. *Continuum X is indecomposable if there exists distinct points a, b and c in X such that X is irreducible between each pair of a, b and c .*

Theorem 17. *Suppose that X is neither 1-cyclic nor tree-like and that every proper subcontinuum of X is tree-like, then X is either indecomposable or 2-indecomposable.*

Proof. If X is decomposable, there exist minimal proper subcontinua A, B such that $A \cup B = X$. That is, if A' is a proper subcontinuum of A then $A' \cup B \neq X$ and if B' is a proper subcontinuum of B then $A \cup B' \neq X$. Also, since X is not 1-cyclic, it follows from Theorems 7, 8, and 11 that $A \cap B$ must have at least 3 components. Let C_x, C_y and C_z be 3 distinct components of $A \cap B$ that contain the points x, y and z , respectively. Let $I_A(x, y), I_A(x, z)$ and $I_A(y, z)$ be subcontinua of A irreducible between x and y , x and z , and y and z , respectively. Suppose that $I_A(x, y)$ is a proper subcontinuum of A . Then from Theorem 7, $B \cap I_A(x, y)$ is a proper subcontinuum of X that is not tree-like which is impossible. Hence, $I_A(x, y) = A$. Similarly, it can be shown that $I_A(x, y) = I_A(x, z) = I_A(y, z) = A$. Thus, it may be concluded that A is indecomposable. Proof is similar to show that B is indecomposable. \square

Assumption 4. *Suppose that X is hereditarily decomposable, 1-cyclic but not tree-like and that every proper subcontinuum of X is tree-like.*

Lemma 18. *Under Assumption 4, there exists $a, b \in X$ and distinct minimal subcontinua $I(a, b)$ and $I'(a, b)$ irreducible about a, b such that $X = I(a, b) \cup I'(a, b)$.*

Proof. Since X is decomposable, there exist minimal proper subcontinua A, B such that $A \cup B = X$. Since A and B are proper subcontinua, they must be tree-like. Also, it follows from Theorems 7, 8, and 11 that since X is 1-cyclic but not tree-like, $A \cap B$ must have

exactly 2 components. Let C_a, C_b be the distinct components of $A \cup B$ where $a \in C_a$ and $b \in C_b$. Let $I_A(a, b)$ be a subcontinuum of A irreducible about a, b and let $I_B(a, b)$ be a subcontinuum of B irreducible about a, b . Since $I_A(a, b) \cap I_B(a, b)$ has exactly 2 components, $I_A(a, b) \cup I_B(a, b)$ cannot be tree-like. Hence, $I_A(a, b) \cup I_B(a, b) = X$. Also, since A and B are both minimal with respect to the decomposition of X , $I_A(a, b) = A$ and $I_B(a, b) = B$. \square

Theorem 19. *Under Assumption 4, there exists a minimal monotone map $\Phi : X \rightarrow S$, where S is a simple closed curve.*

Proof. Let $I_A(a, b)$ and $I_B(a, b)$ be described as in Lemma 18. Since $I_A(a, b)$ and $I_B(a, b)$ are both hereditarily decomposable and irreducible between a and b , there exist minimal monotone maps $\Phi_A : I_A(a, b) \rightarrow [0, 1]$ and $\Phi_B : I_B(a, b) \rightarrow [1, 2]$ such that $\Phi_A(a) = 0$, $\Phi_A(b) = 1 = \Phi_B(b)$, and $\Phi_B(a) = 2$. Now let S be formed by identifying 0 to 2 on the interval $[0, 2]$. Let $\Phi : X \rightarrow S$ be defined in the following manner:

$$\begin{aligned}\Phi(x) &= \Phi_A(x) & \text{if } x \in I_A(a, b), \\ \Phi(x) &= \Phi_B(x) & \text{if } x \in I_B(a, b).\end{aligned}$$

Now, if there exists a $y \in (0, 1)$ such that $\Phi_A^{-1}(y) \cap I_B(a, b) \neq \emptyset$, then $I_A(a, b)$ and $I_B(a, b)$ are not both minimal with respect to the decomposition of X . Same is true if there exists a $y \in (1, 2)$ such that $\Phi_B^{-1}(y) \cap I_A(a, b) \neq \emptyset$. Clearly, if $y \in (0, 1)$ then $\Phi^{-1}(y) = \Phi_A^{-1}(y)$ and is therefore connected. Similarly, if $y \in (1, 2)$ then $\Phi^{-1}(y) = \Phi_B^{-1}(y)$ and is also connected. Now, $\Phi^{-1}(0) = \Phi^{-1}(2) = \Phi_A^{-1}(0) \cup \Phi_B^{-1}(2)$. Since both $\Phi_A^{-1}(0)$ and $\Phi_B^{-1}(2)$ are connected and $a \in \Phi_A^{-1}(0) \cap \Phi_B^{-1}(2)$, it follows that $\Phi^{-1}(0)$ (equivalently $\Phi^{-1}(2)$, $\Phi_A^{-1}(0) \cup \Phi_B^{-1}(2)$) is connected. Proof is similar to show that $\Phi^{-1}(1)$ is connected.

Next, it must be shown that Φ is minimal. Let $\Psi : X \rightarrow S$ be another monotone map. If $y \in (0, 1)$ and $\Phi^{-1}(y) \cap \Psi^{-1}(z) \neq \emptyset$, then $\Phi^{-1}(y) = \Phi_A^{-1}(y) \subset \Psi^{-1}(z)$ since Φ_A is minimal. Likewise, if $y \in (1, 2)$ and $\Phi^{-1}(y) \cap \Psi^{-1}(z) \neq \emptyset$, then $\Phi^{-1}(y) = \Phi_B^{-1}(y) \subset \Psi^{-1}(z)$ since Φ_B is minimal. So, suppose that $\Phi^{-1}(0) \cap \Psi^{-1}(z) \neq \emptyset$. Since Φ_A and Φ_B are both minimal, $\Phi_A^{-1}(0) \subset \Psi^{-1}(z)$ and $\Phi_B^{-1}(2) \subset \Psi^{-1}(z)$. Hence, $\Phi^{-1}(0) = \Phi^{-1}(2) = \Phi_A^{-1}(0) \cup \Phi_B^{-1}(2) \subset \Psi^{-1}(z)$. Proof is similar to show that if $\Phi^{-1}(1) \cap \Psi^{-1}(z) \neq \emptyset$, then $\Phi^{-1}(1) \subset \Psi^{-1}(z)$. Thus, Φ is a minimal, monotone map. \square

Theorem 20. *Under Assumption 1, if h is any homeomorphism of X onto itself, there exists a homeomorphism f of the circle, S , onto itself such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ \downarrow \Phi & & \downarrow \Phi \\ S & \xrightarrow{f} & S \end{array}$$

Proof. First, it will be shown that if $y, z \in S$ such that $h(\Phi^{-1}(z)) \cap \Phi^{-1}(y) \neq \emptyset$, then $h(\Phi^{-1}(z)) = \Phi^{-1}(y)$.

Let $\Psi = \Phi \circ h$ and $\Gamma = \Phi \circ h^{-1}$. Notice that Ψ and Γ are monotone maps. Suppose $y, z \in S$ such that $\Gamma^{-1}(z) \cap \Phi^{-1}(y) \neq \emptyset$. Then, since Φ is minimal, $\Phi^{-1}(y) \subset \Gamma^{-1}(z) = h(\Phi^{-1}(z))$.

Also,

$$\Phi^{-1}(z) \cap \Psi^{-1}(y) = \Phi^{-1}(z) \cap h^{-1}(\Phi^{-1}(y)) = h^{-1} \circ h(\Phi^{-1}(z)) \cap h^{-1}(\Phi^{-1}(y)) = h^{-1}(h(\Phi^{-1}(z)) \cap h^{-1}(\Phi^{-1}(y))) = h^{-1}(\Gamma^{-1}(z) \cap \Phi^{-1}(z)) \neq \emptyset.$$

Thus, since Φ is minimal, $\Phi^{-1}(z) \subset \Psi^{-1}(y) = h^{-1}(\Phi^{-1}(y))$. Therefore, $h(\Phi^{-1}(z)) \subset \Phi^{-1}(y)$. Hence, $\Phi^{-1}(y) = h(\Phi^{-1}(z))$.

Let $f(y) = \Phi \circ h \circ \Phi^{-1}(y)$. To show that f is an one-to-one function, pick any distinct $z_1, z_2 \in S$. There exists $y_1, y_2 \in S$ such that $h(\Phi^{-1}(z_1)) \cap \Phi^{-1}(y_1) \neq \emptyset$ and $h(\Phi^{-1}(z_2)) \cap \Phi^{-1}(y_2) \neq \emptyset$. So by the previous argument, $\Phi^{-1}(y_1) = h(\Phi^{-1}(z_1))$ and $\Phi^{-1}(y_2) = h(\Phi^{-1}(z_2))$. Since $\Phi^{-1}(y_1) \cap \Phi^{-1}(y_2) = h(\Phi^{-1}(z_1)) \cap h(\Phi^{-1}(z_2)) = h(\Phi^{-1}(z_1) \cap \Phi^{-1}(z_2)) = \emptyset$, y_1 and y_2 must be distinct. Hence, $y_1 = \Phi(\Phi^{-1}(y_1)) = \Phi \circ h(\Phi^{-1}(z_1)) = f(z_1)$ and $y_2 = \Phi(\Phi^{-1}(y_2)) = \Phi \circ h(\Phi^{-1}(z_2)) = f(z_2)$ are distinct points of S . Thus, f is a one-to-one function.

To show that f is continuous, let E be closed in S . Then $\Phi^{-1}(E)$ is closed and therefore $h^{-1}(\Phi^{-1}(E))$ is closed and hence compact. Therefore, $f^{-1}(E) = \Phi \circ h^{-1}(\Phi^{-1}(E))$ is compact and therefore closed. Thus, f is continuous.

Since f is a continuous, one-to-one mapping from a compact space to a Hausdorff space, f is a homeomorphism. Also, since $f \circ \Phi(x) = \Phi \circ h(x)$, the diagram commutes. \square

Since it has already been determined that each proper subcontinuum of X is tree-like, each point inverse $\Phi^{-1}(s)$ must either be degenerate or a tree-like subcontinuum. If $\Phi^{-1}(s)$ is degenerate for each $s \in S$, then Φ is a homeomorphism and h cannot be expansive from Theorem 1. If for some s , $\Phi^{-1}(s)$ is nondegenerate, then h cannot be expansive from Theorem 2. Hence, the main result (Theorem 5) now follows.

The following questions remain open:

Question 1 (Kato). *If X admits an expansive homeomorphism, must X contain a non-degenerate indecomposable subcontinuum?*

A continuum X is *hereditarily indecomposable* if every proper subcontinuum is indecomposable.

Question 2 (Kato). *Does there exist a hereditarily indecomposable continuum that admits an expansive homeomorphism?*

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