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A definition of subjective possibility

Didier Dubois [a,*], Henri Prade [a], Philippe Smets [b,✠]

[a] IRIT, CNRS, Université de Toulouse, France

[b] IRIDIA Université Libre de Bruxelles, 50 av. Roosevelt, CP 194-6, 1050 Bruxelles, Belgium

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Abstract

Based on the setting of exchangeable bets, this paper proposes a subjectivist view of numerical possibility theory. It relies on the assumption that when an agent constructs a probability measure by assigning prices to lotteries, this probability measure is actually induced by a belief function representing the agent's actual state of knowledge. We also assume that the probability measure proposed by the agent in the course of the elicitation procedure is constructed via the so-called pignistic transformation (mathematically equivalent to the Shapley value in game theory). We pose and solve the problem of finding the least informative belief function having a given pignistic probability. We prove that it is unique and consonant, thus induced by a possibility distribution. This result exploits a simple informational ordering, in agreement with partial orderings between belief functions, comparing their information content. The obtained possibility distribution is subjective in the same sense as in the subjectivist school in probability theory. However, we claim that it is the least biased representation of the agent's state of knowledge compatible with the observed betting behaviour.

Keywords: Subjective probability; Belief functions; Probability–possibility transformation; Possibility theory

1. Introduction

Quantitative possibility theory was proposed as an approach to the representation of linguistic imprecision [43] and then as a theory of uncertainty of its own ([16,18,19], following an approach initiated by Shackle [27]). In order to sustain this claim for a different uncertainty theory, operational semantics are requested. In the subjectivist context, quantitative possibility theory competes with probability theory in its subjectivist or Bayesian views and with the Transferable Belief Model [31,33], both of which also intend to represent degrees of belief. The term subjectivist means that we consider probability, and other numerical set-functions proposed for the representation of uncertainty, as tools for quantifying an agent's beliefs in events without necessarily referring to their possible random nature and repeatability (still accepting the idea that beliefs may rely

* Corresponding author.
E-mail addresses: dubois@irit.fr (D. Dubois), prade@irit.fr (H. Prade).
✠ Deceased.

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on frequencies if the latter are available). An operational definition, and the assessment methods that can be derived from it, provide a meaning to the value .7 encountered in statements like "my degree of belief is .7". Bayesians claim that any state of incomplete knowledge of an agent can (and should) be modelled by a single probability distribution on the appropriate referential, and that degrees of belief coincide with probabilities that can be revealed by observing the betting behaviour of the agent (how much would the agent pay to enter a game). In such a betting experiment, the agent provides betting odds under an exchangeable bet assumption. A similar setting exists for imprecise probabilities [38], relaxing the assumption of exchangeable bets, and more recently for the Transferable Belief Model as well [32], introducing several betting frames corresponding to various partitions of the referential. In that sense, numerical values encountered in these three theories are well-defined.

Quantitative possibility theory seems to be worth exploring as well from this standpoint. Rejecting it because of an alleged lack of convincing semantics would be unfortunate, simply because it entertains close formal relationships with other theories: possibility measures are consonant Shafer [29] plausibility measures, and thus encode special families of probability functions. Since possibility theory is a special case of most existing non-additive uncertainty theories, be they numerical or not, progress in one of these theories usually has impact in possibility theory. The recent revival of a form of subjectivist possibility theory initiated by Giles [23] and pursued by De Cooman and Aeyels [5], along the lines of Walley's imprecise probabilities, and the development of possibilistic networks based on incomplete statistical data [1] also suggest that it is fruitful to investigate various operational semantics for possibility theory. Another major reason for studying possibility theory is that it is very simple, certainly the simplest challenger for probability theory, especially in the form of fuzzy intervals (e.g. [10]).

The aim of this paper is to propose subjectivist semantics for numerical possibility theory based on exchangeable bets.[1] In the next section, the basic conceptual framework of the proposal is presented, with comparison with other approaches to the semantics of possibility theory. Section 3 recalls belief functions, their informational comparison, and the pignistic transformation. Section 4 presents the main results of this paper, claiming that the least committed representation of the beliefs of an agent supplying a subjective probability distribution defined via betting rates is a possibility distribution.

2. Basic conceptual setting

In this paper, we assume, in contrast with the Bayesian tradition, that beliefs held by an agent are more naturally modelled by means of a belief function, thus leaving room for incomplete knowledge [21]. In the Bayesian setting of exchangeable bets, the agent is in some sense forced to produce a unique distribution. So, assuming belief states are faithfully modelled by belief functions implies that epistemic states and degrees of belief are not directly observable by means of the Bayesian elicitation procedure. We consider that the observed probability distribution is only a trace of the epistemic state of the agent. The questions raised by this view are then:

(1) what is the formal link, if any, between the belief function supposedly held by the agent and the subjective probability (s)he supplies through betting rates?
(2) how to reconstruct the epistemic state from an elicited subjective probability?

The first question was solved by Smets [30]. In previous works, he argued that there exists a natural transformation of a belief function into a (so-called pignistic) probability function such that if the agent's beliefs are modelled by the former, his betting rates are captured by the latter. He called it the pignistic transformation and proposed an axiomatisation thereof, justified later on as preserving the linearity of expected utility [36]. This transformation had been previously suggested by Dubois and Prade [13] in the setting of belief functions,

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as a generalization of Laplace principle of Insufficient Reason. It also formally coincides with the Shapley value in game theory [28] as pointed out in [18]. Denneberg and Grabisch [7] have generalized the Shapley value to so-called interaction weights attached to all subsets (not only to singletons), which have not found interpretation in the setting of uncertainty modelling so far.

In general, distinct belief functions may correspond to the same pignistic probability. However, in the case of possibility distributions, corresponding to consonant plausibility functions, the transformation is one-to-one. This restricted form of the pignistic transformation has been proposed by several authors in the fuzzy set context. Kaufmann [24] and Yager [40] proposed a scheme for the random simulation of a finite fuzzy set: picking a membership grade at random in the unit interval, and then randomly picking a value of the variable in the corresponding cut of the fuzzy set. In the continuous setting, Chanas and Nowakowski [2] proposed a more general probabilistic interpretation of fuzzy intervals based on a similar interpretation.

This paper addresses the second question: given a subjective pignistic probability distribution p provided by an agent under the form of betting rates, find a suitable least committed belief function whose pignistic transform is p. The principle we apply to this end is the one of minimal commitment: by default, the agent's knowledge is supposed to be minimal. Such a minimally committed belief function is a cautious representation of the agent's belief, assuming minimal statistical knowledge. For instance, if the agent supplies a uniform probability, it is assumed by default that the agent has no information. In that case, an unbiased representation is the vacuous belief function, or equivalently, the uniform possibility distribution, thus reversing Laplace's principle of Insufficient Reason. The main result of the paper is that the least committed belief function with prescribed pignistic transform is unique and consonant, that is, it can be modelled as a possibility distribution. This result was already announced by the authors in [22], but its proof was not provided. Since the pignistic transformation is one-to-one for possibility distributions, this result also provides the converse transform with a natural interpretation. This transformation from probability to possibility was first suggested with a different rationale by Dubois and Prade [14].

Our subjectivist semantics differs from the upper and lower probabilistic setting proposed by Giles [23], Walley [38] and followers, without questioning its merit. This school interprets the maximal acceptable buying price of a lottery ticket pertaining to the occurrence of an event as its lower probability, and the minimal sale price of the same lottery ticket as its upper probability, both prices being possibly distinct, in opposition with the exchangeable bets assumption of Bayesians. Walley's actually questions the "dogma of precision" as a major difference between the Bayesian approach and his own (see Chapter 5 of his book [38]), which leads him to give up exchangeable bets, as they enforce infinite precision for degrees of belief. Bayesians consider that a precise probability exists that faithfully represent the agent's beliefs even if in practice, precision may be limited. Here, we assume exchangeable bets, just like the Bayesian School, but we consider that betting rates only partially reflect an agent's beliefs. In other words, even if betting rates ideally produce a unique probability distribution, they are induced by the agent's beliefs without being in one-to-one correspondence with them. For instance, an agent may assign equal probabilities to the facets of a die, either because the fairness of the die has been experimentally validated, or, by symmetry, just because this agent does not know if the die is biased or not. Clearly, beliefs entertained by the agent in both situations are very distinct [17].

Besides, there exists a very different kind of semantics for possibility distributions, relying on the idea of similarity between a situation and a prototypical one, investigated by Ruspini [26]. This is a purely metric view of possibility, while we focus on probabilistic-like semantics. Yet another semantics of possibility theory is in terms of likelihood functions [12]. When all that is known about a probability measure is of the form $P(A|\omega)$, $\omega \in \Omega$ for some observed event A, it is clear that the probability $P(A|B)$ for another event B is upper-bounded by $\max_{\omega \in B} P(A|\omega)$. Moreover, if $P(A|B)$ is to be understood at all as the likelihood $\lambda_A(B)$ of B when observing A, it is legitimate to consider that $\lambda_A(B)$ should be monotonic with B in the sense of set inclusion. So, as noticed by Coletti and Scozzafava [4], the equality $\lambda_A(B) = \max_{\omega \in B} P(A|\omega)$ should be enforced by default, when no other information is available. So, the likelihood function $\lambda_A(\cdot)$ behaves like a possibility measure. This technical interpretation of possibility theory takes no side in the debate between subjectivist vs. objectivist probability, while this paper considers the modelling of subjective beliefs.

3. Formal setting

This section is a refresher on belief functions, the pignistic probabilities, and the informational comparison of belief functions.

3.1. Belief functions

Consider beliefs held by an agent on what is the actual value of a variable ranging on a set Ω, called the frame of discernment. It is assumed that such beliefs can be represented by a belief function. A belief function can be mathematically defined from a (generally finite) random set that has a very specific interpretation. A so-called basic belief mass $m(A)$ is assigned to each subset A of Ω, and is such that $m(A) \geqslant 0\ \forall A \subseteq \Omega$; moreover

$$\sum_{A \subseteq \Omega} m(A) = 1. \tag{1}$$

The degree $m(A)$ is understood as the weight given to the assumption that the agent knows that the value of the variable of interest lies somewhere in set A, and nothing else. In other words, the probability allocation $m(A)$ is potentially shared between elements of A, but remains suspended for lack of knowledge. A set E such that $m(E) > 0$ is called a focal set. In the absence of conflicting information it is generally assumed that $m(\emptyset) = 0$. This is what is assumed in the following. A belief function Bel as well as a plausibility function Pl, attached to each event (or each proposition of interest) can be bijectively associated with the basic mass function m [29]. They are defined by

$$\mathrm{Bel}(A) = \sum_{E \subseteq A} m(E) \quad \text{and} \quad \mathrm{Pl}(A) = 1 - \mathrm{Bel}(A^c) = \sum_{E: E \cap A \neq \emptyset} m(E), \tag{2}$$

where A^c is the complement of A. The belief function evaluates to what extent events are logically implied by the available evidence. The plausibility function evaluates to what extent events are consistent with the available evidence. A companion set-function, called commonality, and denoted by Q, is defined by reversing the direction of inclusion in the belief function expression:

$$Q(A) = \sum_{A \subseteq E} m(E). \tag{3}$$

$Q(A)$ is the share of belief free to potentially support any proposition in the context where the agent accepts that A holds true. It can be argued that $Q(A)$ is a measure of guaranteed plausibility of A because it clearly provides a lower bound of the plausibility of each element in A (and of each subset as well) [9]. When conditioning a mass function on event A, the mass $m(E)$ of each focal set E is allocated to the subset $A \cap E$. The overall (possibly subnormal) mass finally allocated to a subset C of A is denoted $m(C|A)$. Then $Q(A)$ coincides with the mass $m(A|A)$ assigned to set A before normalizing. So, up to normalization, $Q(A)$ is a measure of unassigned belief in the context where the agent accepts that A holds true.

The function Pl restricted to singletons, induced by a mass function m is called its contour function by Shafer [29], and is denoted π_m, defined by $\pi_m(\omega) = \mathrm{Pl}(\{\omega\})$. When the focal sets are nested, the plausibility function is called a possibility measure [43], and can be characterized, just like probability, by its contour function, then called a possibility distribution π. In such a situation, the primitive object can be the possibility distribution, and each of the functions m, Pl, Bel, can be reconstructed from it, noticing that

$$\mathrm{Pl}(A) = \max_{\omega \in A} \pi(\omega). \tag{4}$$

The set function Pl is then often denoted Π. If $\Omega = \{\omega_1, \ldots, \omega_n\}$, and letting $\pi_i = \pi(\omega_i)$, such that $1 = \pi_1 \geqslant \pi_2 \geqslant \cdots \geqslant \pi_n \geqslant \pi_{n+1} = 0$, then the mass function generating π is denoted m_π such that [13]:

$$m_\pi(A) = \pi_i - \pi_{i+1} \quad \text{if } A = \{\omega_1, \ldots, \omega_i\}\ \forall i = 1, \ldots, n, \tag{5}$$

$$= 0, \quad \text{otherwise.} \tag{6}$$

If the mass function m is not consonant the contour function π_m is not enough to recover it as in (5) since retrieving m then needs up to $2^{|\Omega|}$ terms to be determined, where $|\cdot|$ stands for cardinality.

3.2. Informational comparison of belief functions

There are several methods for comparing belief functions in terms of their informational contents. Some informational indices extend the probabilistic notion of entropy. Other ones generalize the notion of cardinality of a set representing incomplete knowledge, yet other ones combine both (see the recent survey by Klir and Smith [25], for instance). Besides, three partial orderings comparing the information content of two belief functions in terms of specificity have been proposed by Yager [42] and Dubois and Prade [15]: the precision ordering, the Q-information ordering, and the specialization ordering.

A first natural specificity ordering of belief functions compares intervals limited by belief and plausibility. Namely the interval $[\mathrm{Bel}(A), \mathrm{Pl}(A)]$ is all the wider as the information concerning A is scarce. So, a partial information order on the set of belief functions over Ω can be defined as follows: Bel_1 is at least as *precise* as Bel_2 if and only if $[\mathrm{Bel}_1(A), \mathrm{Pl}_1(A)] \subseteq [\mathrm{Bel}_2(A), \mathrm{Pl}_2(A)]\ \forall A \subseteq \Omega$; it corresponds to an inclusion relation between sets of probabilities dominating Bel_1 and Bel_2. In fact, this ordering can be defined equivalently and more simply as $\mathrm{Pl}_1(A) \leqslant \mathrm{Pl}_2(A)\ \forall A \subseteq \Omega$ due to the duality between Bel and Pl.

Interestingly, this partial ordering does not imply any relationship between the commonality functions Q_1 and Q_2 (see [15] and the counterexample below). Another partial informational ordering between belief functions has thus been defined by comparing the commonality functions: Bel_1 is at least as *Q-informed* as Bel_2 if and only if $Q_1(A) \leqslant Q_2(A)\ \forall A \subseteq \Omega$. This direction of inequality is natural since it ensures that for singletons, $\mathrm{Pl}_1(\{\omega\}) \leqslant \mathrm{Pl}_2(\{\omega\})$, due the identity of Pl and Q functions on singletons.

The third partial informational ordering can be described directly from the mass functions m_1 and m_2. The idea is that Bel_1 is at least as informed as Bel_2 whenever it is possible to turn m_2 into m_1 by consistently reassigning each weight $m_2(E)$ to subsets of E that are focal sets of m_1 (possibly sharing this weight among them). It is called the *specialization* ordering. Namely, m_1 is more specialized than m_2 if and only if there is a stochastic matrix W whose rows correspond to focal sets of m_1 and columns to focal sets of m_2, such that $m_1 = W \cdot m_2$. Here, mass functions are encoded as vectors and entry w_{ij} reflects the proportion of the mass $m_2(E_j)$ allocated to focal set F_i of m_1, with the condition that F_i must be a subset of E_j for w_{ij} to be positive.

This third ordering leads to more incomparabilities than the other ones and is refined by them. But the Q-informativeness and the precision orderings are not comparable.

Example 1. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3\},\ E = \{\omega_1, \omega_2\},\ F = \{\omega_1, \omega_3\},\ a \in (0.5, 1)$. Consider the mass function $m(E) = a$, $m(F) = 1 - a$, and the possibility measure π such that

$$\pi(\omega_1) = 1, \quad \pi(\omega_2) = a, \quad \pi(\omega_3) = 1 - a.$$

It is clear that $\mathrm{Pl}(\{\omega\}) = \pi(\omega)\ \forall \omega$; the consonant mass function associated to π by (5) is

$$m_\pi(\{\omega_1\}) = 1 - a, \quad m_\pi(E) = 2a - 1, \quad m_\pi(\Omega) = 1 - a.$$

It is obvious that none of the two mass functions m and m_π is a specialization of the other since m_π has a focal set contained in none of E or F, and a focal set containing none of them. Now it is obvious that m is at the same time less precise and more Q-informed than m_π. Indeed, $\mathrm{Pl}(A) \geqslant \Pi(A)\ \forall A$, and $\mathrm{Pl}(\{\omega_2, \omega_3\}) = 1 > \Pi(\{\omega_2, \omega_3\}) = a$. However, $Q_\pi(A) \geqslant Q(A)\ \forall A$, and $Q_\pi(\{\omega_2, \omega_3\}) = 1 - a > Q(\{\omega_2, \omega_3\}) = 0$.

In view of this example, the respective interpretations of the Q-informativeness and precision ordering would deserve a more careful study. Nevertheless, all three orderings coincide for possibility measures and come down to the possibilistic ordering of specificity on singletons [41,16]: π_1 is at least as informed as π_2 if and only if $\pi_1 \leqslant \pi_2$.

3.3. The pignistic transformation

It is assumed that the actual beliefs of the agent can be faithfully modelled by a mass function on Ω. A probability measure induced by a mass function can be built by defining a uniform probability on each set with positive mass, and performing the convex mixture of these probabilities according to the mass function. This transformation, which, as pointed out earlier, recurrently appears in various contexts since the fifties, was called the pignistic transformation by Smets [30]. Let m be a mass function from 2^{Ω} to $[0,1]$. The pignistic transform of m is a probability distribution $\mathrm{BetP} = \mathrm{Pig}(m)$ such that

$$\mathrm{BetP}(\omega) = \sum_{A:\omega\in A} \frac{m(A)}{|A|}. \tag{7}$$

It can be viewed as an extension of Laplace indifference principle, according to which equally possible outcomes have equal probability. It looks like a weighted form thereof, since, by symmetry, each focal set is then interpreted as a uniform probability. According to Smets [30], the agent's beliefs cannot be directly assessed. All that can be known are the values of the "pignistic" probabilities the agent uses to bet on the frame Ω. Only the probability distribution BetP, not the belief function accounting for the agent's beliefs, is obtained by eliciting an agent's betting rates on the frame Ω.

The pignistic probability depends on the chosen betting frame. Changing Ω into one of its refinements, thus modifying the granularity, a different probability is obtained. It has been proved that for any event A, the minimal (resp. maximal) value of $\mathrm{BetP}(A) = \sum_{\omega\in A}\mathrm{BetP}(\omega)$ over all possible changes of granularity yields back $\mathrm{Bel}(A)$ (resp. $\mathrm{Pl}(A)$) [39]. So, the interval $[\mathrm{Bel}(A), \mathrm{Pl}(A)]$ contains all possible values of the pignistic probability of A, across all betting frames. This is related to the fact that all probability functions P dominating the belief function Bel induced by m (that is $P \geqslant \mathrm{Bel}$) can be generated by changing each focal set E into a probability distribution $p(\cdot|E)$ with support E. Namely

$$p(\omega) = \sum_{E} p(\omega|E)\cdot m(E). \tag{8}$$

In Bayesian terms, this is an application of the total probability theorem where $p(\omega|E)$ is the (subjective) probability of ω when all that is known is the piece of evidence E, and $m(E)$ is the probability of knowing this piece of evidence only. So, in terms of upper and lower probabilities, BetP is the centre of gravity of the set of probabilities dominating the belief function [20]. In terms of game theory, it corresponds to the Shapley value of a cooperative game [28], which is modelled by a set-function on a set Ω of agents. This set-function assigns to each subset of agents, viewed as a potential coalition, a number reflecting its power. In this setting, $\mathrm{BetP}(\omega)$ represent the power of agent ω across all potential coalitions (s)he may be part of.

In the special case of consonant belief functions, the pignistic transformation can be expressed in terms of the possibility distribution π such that $1 = \pi_1 \geqslant \pi_2 \geqslant \cdots \geqslant \pi_n \geqslant \pi_{n+1} = 0$ as follows, letting $p_i = \mathrm{BetP}(\omega_i)$:

$$p_i = \sum_{j=i,\dots,n} \frac{(\pi_j - \pi_{j+1})}{j} \quad \forall i = 1,\dots,n. \tag{9}$$

It can be checked that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$ and that the transformation is bijective between probabilities and possibilities. Its converse Pig^{-1} was independently suggested by Dubois and Prade [14]. It reconstructs the possibility distribution as follows:

$$\pi_i = \sum_{j=i,\dots,n} \min(p_i, p_j) \quad \forall i = 1,\dots,n \tag{10}$$

and we write $\pi = \mathrm{Pig}^{-1}(\mathrm{BetP})$. Note that another probability–possibility transformation exists, of the form [13,6]:

$$\sigma_i = \sum_{j=i,\dots,n} p_j \quad \forall i = 1,\dots,n. \tag{11}$$

The latter transformation of a probability distribution p yields the most specific (=restrictive) possibility distribution such that $\Pi(A) \geqslant P(A)\ \forall A$. When p stems from validated statistical data, one may argue that this transformation yields its most legitimate possibilistic representation [22] since p represents a complete model of the studied random phenomenon and (11) yields the most specific possibility distribution respecting the ordering of elements of Ω induced by p, in the sense that $\sum_{j=1,\dots,n}\sigma_j$ is minimal (minimal cardinality of the fuzzy set with membership grades σ_j). This transformation is easily interpretable in terms of finding the most specific confidence sets induced by p for various levels of confidence. On the real line, using a unimodal probability density, the α-*cut* of the possibility distribution thus obtained is the narrowest confidence interval with confidence level $1-\alpha$ [11].

In contrast, in the subjective probability case, it is questionable whether the expert possesses a complete model of the phenomenon referred to, even if the betting framework enforces it. If the parameter under concern is random, the agent may have only partial knowledge about it. If the parameter is not random (just ill-known), a complete model should come down to knowing its precise value. Hence the optimal (maximally specific) transformation (11) does not convincingly apply to subjective probabilities.

There are yet other transforms that change belief functions to probabilities, like the probabilistic renormalization of the contour function (called plausibility transformation by Cobb and Shenoy [3]). Even, if this transform is consistent with the application of Dempster's combination rule, the resulting probability measure is generally not in agreement with the belief function it comes from, namely $P(A)$ may fail to lie in the interval $[\mathrm{Bel}(A), \mathrm{Pl}(A)]$ for some event A.

4. The most cautious belief function inducing a subjective probability

The knowledge of the values of the probability p allocated to the elements of Ω by the agent is not sufficient to reconstruct a unique underlying belief function whose pignistic transform is p. Many belief functions induce the same pignistic probability distribution. As already said, for instance, uniform betting rates on Ω either correspond to complete ignorance on the values of the variable, or to the knowledge that the variable is random and uniformly distributed. So, all that is known about the mass function that represents the agent's beliefs is that it belongs to the ones that induce the available subjective probability. Under this scheme, we do not question the exchangeability of bets, as done by Walley [38], Giles [23] and others. What we question is the assumption of a one-to-one correspondence between the betting rates produced by the agent, and the actual beliefs entertained by this agent. Betting rates do not tell if the uncertainty of the agent results from the perceived randomness of the phenomenon under study or from a simple lack of information about it.

The belief functions whose pignistic transform is p are called *isopignistic* belief functions and form the set $\mathrm{IP}(p)$. A cautious approach among isopignistic belief functions is to obey the least commitment principle. It states that one should never presuppose more beliefs than justified. Then, a reasonable choice is to select the least committed element, that is, the least informed one, in the family of isopignistic belief functions corresponding to the pignistic probability function prescribed by the obtained betting rates.

One may try to define the least debatable representation of an agent's belief as a minimally informative isopignistic mass function according to information content comparison techniques presented in Section 3.2. While the merit of these partial ordering relations is to provide an ordinal foundation to the comparison of belief functions, they often lead to incomparability. So, unicity may easily fail for least informative mass functions, as the corresponding optimization problem comes down to a kind of vector-maximization.

4.1. Using expected cardinality

An easier problem is to maximize an information index. A natural measure of non-commitment of a belief function is its expected cardinality viewed as the average of its focal sets, weighted by the mass function m:

$$I(m) = \sum_{A \subseteq \Omega} m(A) \cdot |A|. \tag{12}$$

It is the simplest imprecision measure. It is easy to see that $I(m)$ is the cardinality of the fuzzy set whose membership function coincides with the contour function of m [8], namely, $I(m) = \sum_{\omega \in \Omega} \pi_m(\omega)$.

It is clear that this index is compatible with the specialization ordering, namely that if m_1 is more specialized than m_2 then $I(m_1) \leqslant I(m_2)$. However, as it only depends on the contour function, it cannot discriminate between belief functions that share the same contour function, while the commonality and plausibility orderings might discriminate among them (not always in the same way as shown by Example 1).

We define a least biased belief representation, for an agent supplying a pignistic probability p, as any belief function whose mass function m^* maximizes $I(m)$ among isopignistic belief functions whose pignistic transform according to Eq. (10) is p. The following result is now established. It compares the respective specificities of m and the possibility distribution obtained from $\mathrm{Pig}(m)$ via (10):

Lemma 1. *For any belief function with mass function m,* $I(\mathrm{Pig}^{-1}(\mathrm{Pig}(m))) \geqslant I(m)$.

Proof. Let $p = \mathrm{Pig}(m)$, such that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$ and $\pi = \mathrm{Pig}^{-1}(p)$. It is such that $1 = \pi_1 \geqslant \pi_2 \geqslant \cdots \geqslant \pi_n \geqslant \pi_{n+1} = 0$. It can be checked that $I(\mathrm{Pig}^{-1}(p))$ is the sum of entries in the $n \times n$ matrix with coefficients $\min(p_i, p_j)$[2]:

$$I(\mathrm{Pig}^{-1}(p)) = \sum_{i=1,\ldots,n} \pi_i = \sum_{i=1,\ldots,n} \sum_{j=1,\ldots,n} \min(p_i, p_j) = \sum_{i=1,\ldots,n} (2i-1) \cdot p_i$$

(since there is only one entry containing p_1, three entries containing p_2, etc.).

Now, since $p_i = \sum_{E:\omega_i \in E} \frac{m(E)}{|E|}$, it all boils down to proving that

$$\sum_{i=1,\ldots,n} (2i-1) \cdot \sum_{E:\omega_i \in E} \frac{m(E)}{|E|} \geqslant \sum_{i=1,\ldots,n} \sum_{E:\omega_i \in E} m(E).$$

Subtracting the right-hand side from the left-hand side, and factoring $m(E)$, it is enough to prove that the multiplicative coefficient of each $m(E)$ is positive, that is, denoting by μ_E the indicator function of E:

$$c(E) = \sum_{i=1,\ldots,n} (2i-1) \cdot \frac{\mu_E(\omega_i)}{|E|} - \sum_{i=1,\ldots,n} \mu_E(\omega_i) \geqslant 0.$$

Let $E = \{\omega_{i_1}, \ldots, \omega_{i_k}\}$ such that $p(\omega_{i_1}) \geqslant p(\omega_{i_2}) \geqslant \cdots \geqslant p(\omega_{i_k})$. Note that by construction, $i_j \geqslant j$.

Then, $c(E) = \left(\frac{2i_1 - 1}{k} + \frac{(2i_2 - 1)}{k} + \cdots + \frac{(2i_k - 1)}{k}\right) - k$. It is minimal for $i_j = j$ for all $j = 1, \ldots, k$. Hence

$$c(E) \geqslant \left(\frac{2-1}{k} + \frac{4-1}{k} + \cdots + \frac{2k-1}{k}\right) - k = \frac{2 \cdot \left(\sum_{j=1,\ldots,k} j\right)}{k} - k - 1 = 0.$$

Hence $c(\{\omega_1, \omega_2, \ldots, \omega_k\}) = 0 \ \forall k$, and $c(E) > 0$ otherwise. □

The next result proves the consonance of the maximally imprecise isopignistic belief function:

Lemma 2. $I(\mathrm{Pig}^{-1}(\mathrm{Pig}(m))) = I(m)$ *only if m is consonant.*

Proof. For suppose m is not consonant. Then m has at least two non-nested focal sets E and F. Hence at least one of them, say E, is not of the form $\{\omega_1, \omega_2, \ldots, \omega_k\}$. Hence $c(E) > 0$, so $m(E) \cdot c(E) > 0$, hence $I(\mathrm{Pig}^{-1}(\mathrm{Pig}(m))) > I(m)$, using Lemma 1. □

Since there is only one consonant belief function in $\mathrm{Pig}^{-1}(\mathrm{Pig}(m))$, the following result is obtained:

Theorem 3. *The unique mass function which maximizes $I(m)$ under the constraint* $\mathrm{Pig}(m) = p$ *exists and is consonant. It is the possibility distribution π defined by the converse of the pignistic transform applied to the pignistic transform of m.*

Proof. Since function Pig is a bijection from possibility to probability measures, it follows, using the above lemmas, that the consonant mass function associated to $\mathrm{Pig}^{-1}(p)$ is the unique maximum of $I(m)$. □

4.2. Comparing commonality functions

Smets [34] suggested that the least specific isopignistic belief function according to the commonality ordering is also $\mathrm{Pig}^{-1}(\mathrm{Pig}(m))$. There is indeed a unique minimally Q-informative belief function in $\mathrm{IP}(p)$, and it is precisely the one found by maximizing $I(m)$. In order to prove it, we first prove that, for ensuring comparability in the sense of the Q-informativeness ordering between a consonant belief function and a belief function, it is enough to rely on contour functions:

[2] In fact, this is the Gini index of p.

Lemma 4. *Consider a belief function with mass function m and a possibility distribution* π *with respective commonality functions Q and* Q_π. *Then* $Q_\pi(A) \geqslant Q(A)\ \forall A \subseteq \Omega$ *if and only if* $\pi(\omega) \geqslant \mathrm{Pl}(\{\omega\})\ \forall \omega \in \Omega$.

Proof. It is obviously enough to prove the "if" part since $Q(\{\omega\}) = \mathrm{Pl}(\{\omega\})$. Besides, note that for possibility measures $Q_\pi(A) = \min_{\omega\in A}\pi(\omega)$. Now assume $\pi(\omega) \geqslant \mathrm{Pl}(\{\omega\})\ \ \forall\omega \in \Omega$. Then, $Q_\pi(A) = \min_{\omega\in A}\pi(\omega) = \pi(\omega^*) \geqslant \mathrm{Pl}(\{\omega^*\}) = Q(\{\omega^*\}) \geqslant Q(A)$ since function Q is antimonotonic with respect to inclusion. □

The following result is a strong form of Lemma 1.

Lemma 5. *Consider a belief function with mass function m,* $p = \mathrm{Pig}(m)$, *and* $\pi = \mathrm{Pig}^{-1}(p)$. *Then,* π *is not more specific than the contour function of m, i.e.* $\pi \geqslant \pi_m$.

Proof. Consider $p = \mathrm{Pig}(m)$, such that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$ and $\pi = \mathrm{Pig}^{-1}(p)$. It is such that $1 = \pi_1 \geqslant \pi_2 \geqslant \cdots \geqslant \pi_n \geqslant \pi_{n+1} = 0$. Using Eq. (10), note that $\pi_k = \pi(\omega_k)$ is defined in terms of m as

$$\pi_k = k\cdot p_k + \sum_{j=k+1,\ldots,n} p_j = k\cdot \sum_{E:\omega_k\in E}\frac{m(E)}{|E|} + \sum_{j=k+1,\ldots,n}\ \sum_{E:\omega_j\in E}\frac{m(E)}{|E|}.$$

We must show that this expression is not less than $\sum_{E:\omega_k\in E} m(E) = \mathrm{Pl}(\{\omega_k\}) = \pi_m(\omega_k)$. To this end we proceed focal set by focal set, with fixed cardinality. Denote by $c_k(E)$ the multiplicative coefficient of $m(E)$ in the expression of π_k, namely, denoting by μ_E the indicator function of E:

$$c_k(E) = \frac{k\cdot\mu_E(\omega_k)}{|E|} + \sum_{j=k+1,\ldots,n}\frac{\mu_E(\omega_j)}{|E|}.$$

Let us show that $c_k(E) \geqslant 1$ whenever $\omega_k \in E$ (otherwise $m(E)$ does not contribute to $\pi_m(\omega_k)$).

First, assume $|E| = n$. It means that $E = \Omega$. The coefficient $c_n(\Omega)$ of $m(\Omega)$ is $\frac{k}{n} + \frac{n-k}{n} = 1$ since all terms in the second summand of the expression of $c_n(E)$ are present.

Now, assume $|E| = i > k$. There are at least $i - k$ terms in the second summand of the expression of $c_k(E)$. Then $c_k(E) \geqslant \frac{k}{i} + \frac{i-k}{i} = 1$.

Assume $|E| = i \leqslant k$. Then the second summand of the expression of $c_k(E)$ may be zero since E may fail to contain any ω_j for $j > k$. It is no problem since then $c_k(E) \geqslant \frac{k}{i} \geqslant 1$ by assumption. □

Theorem 6. *The unique consonant mass function in* IP(p) (*induced by the possibility distribution defined by* (10)), *is minimally Q-informative.*

Proof. Based on the lemma above, we know that $\pi \geqslant \pi_m$ for $\pi = \mathrm{Pig}^{-1}(\mathrm{Pig}(m))$. Due to Lemma 5, it implies that π is not more Q-informative than m. Setting $p = \mathrm{Pig}(m)$, this property holds for all belief functions in IP(p), and $\pi \in$ IP(p), by construction. Hence π is not more Q-informative than any belief function in IP(p). □

Note that Lemma 5 implies Lemma 1 since the latter compares the sum $\sum_{i=1,\ldots,n}\pi_i$ to the sum $\sum_{i=1,\ldots,n}\pi_m(\omega_i)$. However, the proof of Lemma 1 is more direct. Moreover, Lemma 5 shows that when comparing mass functions in terms of commonality, one of them being consonant, commonality functions play no particular role. Only contour functions matter. So, the optimality of the possibility measure in IP(p) is really in the sense of the pointwise comparison, in the fuzzy set inclusion sense, of the plausibility functions on singletons, i.e. the contour functions.

Let us now turn to the issue of unicity of the least informative mass function in the sense of the pointwise comparison of contour functions. The unicity problem can be stated as follows: given a possibility distribution π on Ω, whose pignistic transform is a probability distribution $p = \mathrm{Pig}(\pi)$, is there another (non-consonant) mass function $m \neq m_\pi$ such that $p = \mathrm{Pig}(m)$ and $\pi = \pi_m$? The following result shows this is not the case.

Theorem 7. *For any probability distribution p on* Ω, *the mass function m with the least specific contour function* π_m *such that* $p = \mathrm{Pig}(m)$ *is unique, consonant and is such that* $\pi_m = \mathrm{Pig}^{-1}(p)$.

Proof. Fix the probability distribution p such that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$ and $\pi = \mathrm{Pig}^{-1}(p)$. It is such that $1 = \pi_1 \geqslant \pi_2 \geqslant \cdots \geqslant \pi_n \geqslant \pi_{n+1} = 0$. From Lemma 5, the condition $\pi = \pi_m$ must be enforced. The mass function m must then satisfy the following constraints, for $k = 1, \ldots, n$:

$$\sum\nolimits_{E:\omega_k \in E} \frac{m(E)}{|E|} = p_k; \tag{13}$$

$$\sum\nolimits_{E:\omega_k \in E} m(E) = k \cdot p_k + \sum_{j=k+1,\ldots,n} p_j \ (= \pi_k) \tag{14}$$

and moreover, $\sum_E m(E) = 1$.

Let us show, using backward recursion on the size of subsets E, that $\forall k = n$ downward, the only focal set E with cardinality at most k such that $\omega_k \in E$ is $E_k = \{\omega_1, \omega_2, \ldots, \omega_k\}$.

For $k = n$, it holds that $\pi_n = n \cdot p_n$ so that Eqs. (13) and (14) lead to

$$\sum_{E:\omega_n \in E} m(E) = n \cdot \left(\sum_{E:\omega_n \in E} \frac{m(E)}{|E|} \right).$$

It reads $\sum_{E:\omega_n \in E} m(E) \cdot \left(1 - \frac{n}{|E|}\right) = 0$, hence $m(E) = 0$ whenever $\omega_n \in E$, and $|E| < n$. So the only focal set containing ω_n is Ω. So, all such masses $m(E)$ in the pair of equations number $k = n$ are zero except $m(\Omega) = n \cdot p_n$.

Denote $E_j = \{\omega_1, \ldots, \omega_j\}$. Suppose all masses $m(E) = 0$ whenever $\omega_j \in E$, $|E| < j$ in the pairs of equations $j = k+1, \ldots, n$, except $m(\Omega) = n \cdot p_n$, and $m(E_j) = j(p_j - p_{j+1})$. Consider the pair of equations number k. Eq. (13) reads

$$\sum_{\omega_k \in E \subset E_k} \frac{m(E)}{|E|} + \frac{m(E_k)}{k} + \sum_{j=k+1,\ldots,n} \frac{m(E_j)}{j} = p_k.$$

Since the only focal sets with more than k elements are of the form E_j for $j > k$. Note that $\sum_{j=k+1,\ldots,n} \frac{m(E_j)}{j} = p_{k+1}$ by definition. So (13) reads

$$k \cdot \sum_{\omega_k \in E \subset E_k} \frac{m(E)}{|E|} + m(E_k) = k(p_k - p_{k+1}).$$

Now (14) reads

$$\sum_{\omega_k \in E \subset E_k} m(E) + m(E_k) + \sum_{j=k+1,\ldots,n} m(E_j) = k \cdot p_k + \sum_{j=k+1,\ldots,n} p_j.$$

But since $m(E_j) = j(p_j - p_{j+1})$ for $j > k$, it holds that

$$\sum_{j=k+1,\ldots,n} m(E_j) = (k+1)(p_{k+1} - p_{k+2}) + (k+2)(p_{k+2} - p_{k+3}) + \cdots + n(p_n - 0) = (k+1)p_{k+1} + \sum_{j>k+1} p_j.$$

Simplifying, it yields

$$\sum_{\omega_k \in E \subset E_k} m(E) + m(E_k) = k(p_k - p_{k+1}).$$

For equalities (13) and (14) to hold simultaneously, it requires the equality

$$k \cdot \sum_{\omega_k \in E \subset E_k} \frac{m(E)}{|E|} = \sum_{\omega_k \in E \subset E_k} m(E).$$

That is, $\sum_{\omega_k \in E \subset E_k} m(E)\left(1 - \frac{k}{|E|}\right) = 0$. But since $|E| < k$, it enforces $m(E) = 0 \ \forall E \subset E_k$ such that $\omega_k \in E$. So the only focal set E with cardinality k is $E_k = \{\omega_1, \omega_2, \ldots, \omega_k\}$ with mass $k(p_k - p_{k+1})$.

Finally, since $\pi_1 = 1$, $m(E) = 0$ as soon as $\omega_1 \notin E$.

Overall only subsets of the form $E = E_k$, $k = 1, \ldots, n$ may receive positive mass if the mass function has pignistic transform p and contour function $\pi = \mathrm{Pig}^{-1}(p)$. Hence, m is consonant, and because there is only one consonant mass function in $\mathrm{IP}(p)$, it precisely yields the one underlying $\mathrm{Pig}^{-1}(p)$. □

Putting together Theorems 6 and 7, the minimally Q-informative mass function with pignistic probability p exists, is unique and is consonant. It is actually the mass function having the least specific (i.e. pointwisely maximal in Ω) contour function, hence also least precise in the sense of the comparison of Bel–Pl intervals restricted to singletons.

Remark. Theorem 7 can also be viewed as a corollary of Lemma 5 and Theorem 3 conjointly. Note that the total ordering induced by the non-specificity index I refines the partial specificity order according to the pointwise comparison of contour functions. So, in theory the optimal isopignistic solutions in the sense of I form a subset of the maximal isopignistic solutions according to the latter partial ordering, and Theorem 7 does not follow from Theorem 3 alone. But Lemma 5 says such maximal isopignistic solutions have the same contour functions as $\mathrm{Pig}^{-1}(p)$, hence cannot be discriminated by index I. As Theorem 3 shows that the isopignistic belief function maximizing I is unique, it is also the unique maximal isopignistic maximal solution according to the pointwise comparison of contour functions. However, it is interesting to provide a self-contained proof of Theorem 7.

Our results also suggest that most of the time, a unique least precise non-consonant mass function in $\mathrm{IP}(p)$ in the sense of the comparison of Bel–Pl intervals for all events will not exist. Indeed if $m \in \mathrm{IP}(p)$ is a least precise mass function different from the one inducing $\pi = \mathrm{Pig}^{-1}(p)$, then $\pi(\omega) < \pi_m(\omega)$, for some $\omega \in \Omega$, due to the unicity result in Theorem 3. Since m is among minimally precise ones, it must also hold that $\mathrm{Pl}(A) > \Pi(A)$ for some non-singleton event A. So m and m_π are not comparable in the sense of Bel–Pl intervals. That this non-unicity situation does occur can be checked from Example 1.

Example 1 (continued). Assume $a = \frac{1}{2}$. So, $m(\{\omega_1, \omega_2\}) = m(\{\omega_1, \omega_3\}) = \frac{1}{2}$. The pignistic probability p induced by m is clearly: $p(\omega_1) = \frac{1}{2}, p(\omega_2) = \frac{1}{4}, p(\omega_3) = \frac{1}{4}$. Then, $\pi = \mathrm{Pig}^{-1}(p)$ is $\pi(\omega_1) = 1, \pi(\omega_2) = \frac{3}{4}, \pi(\omega_3) = \frac{3}{4}$. The contour function of m is $\pi_m(\omega_1) = 1, \pi_m(\omega_2) = \frac{1}{2}, \pi_m(\omega_3) = \frac{1}{2}$. It is more specific than $\mathrm{Pig}^{-1}(p)$ as expected. Note that $\mathrm{Pl}(\{\omega_2, \omega_3\}) = 1$, while $\Pi(\{\omega_2, \omega_3\}) = \frac{3}{4}$. Hence m and $\mathrm{Pig}^{-1}(p)$ are not comparable in the sense of the comparison of Bel–Pl intervals; they are both minimally precise in $\mathrm{IP}(p)$.

Remarks

- There is no point carrying out the similar study on the plausibility transform by Cobb and Shenoy [3] as for the pignistic transform, since the belief functions having the same plausibility transform also have the same contour function.
- Besides, in the case of objective probabilities, when p is frequentist and represents the available (rich) information, a counterpart to the above approach would be to look for the *most specific* belief function dominating p, but this is p itself, so we cannot follow this line in order to justify the probability–possibility transformation (11).

5. Conclusion

The main result of this paper is that, on finite sets, the least committed mass function, in the sense of the pointwise comparison between contour functions, among the ones which share the same pignistic transform, is unique and consonant. That is, the corresponding plausibility function is a possibility function. It is the unique one in the set of plausibility functions having this prescribed pignistic probability, because the pignistic transformation is a bijection between possibilities and probabilities. So this possibility function corresponds to the least committed mass function whose transform is equal to the subjective probability supplied by an agent. This consonant belief function is also the least informative isopignistic one in the sense of the commonality information ordering. It suggests a new justification to a probability–possibility transform previously suggested by two of the authors [14]. An adaptation of this result to the continuous setting is outlined by Smets [37].

This result provides an operational basis for defining subjective possibility degrees, hence the membership function of (discrete) fuzzy numbers. It tentatively addresses objections raised by Bayesian subjectivists against the use of fuzzy numbers and numerical possibility theory in decision-making and uncertainty modelling tasks. Interestingly, our approach refutes neither the Bayesian operational setting (unlike Walley [38] and De Cooman and Aeyels [5]) nor the use of standard expected utility for decisions (since the pignistic probability is tailored to respect the expected utility criterion as recently shown by Smets [36]). It only questions the interpretation of betting rates as full-fledged degrees of belief. Bayesians may then claim that our approach makes no contribution, since the underlying possibility distribution is not used for selecting decisions. However, the proposed subjective possibility approach, just like the Transferable Belief Model, does differ from the Bayesian approach in a dynamic environment. In our non-classical setting, when an event is known to have occurred, the revision of information takes place by modifying the possibility distribution underlying the pignistic probability, not this probability directly. It means that the new probability distribution obtained from the agent is no longer assumed to coincide with the result of conditioning the original pignistic probability, but that the agent would bet again based on a different frame supporting the revised knowledge (see e.g. [21,35], on this matter).

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