The Effect of Dispersal on Permanence in a Predator-Prey Population Growth Model

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Abstract—In this paper, we consider a periodic predator-prey system where the prey can disperse between one patch with a low level of food and without predation and one patch with a higher level of food but with predation. We assume a Volterra within-patch dynamic. Under the assumption that the average of dispersal rate from Patch 1 to Patch 2 is less than that of the intrinsic growth rate of prey in Patch 1, we provide a sufficient and necessary condition to guarantee the prey and predator species to be permanent by using the main techniques in [1]. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Predator-prey system, Dispersal, Permanence, Extinction, Stability.

1. INTRODUCTION

The behavior of one biological species living in an environment where the need to forage and the need to avoid a predator are in conflict has been experimentally studied and discussed by many authors (e.g., [2-4]). This kind of biological problem has motivated many studies on a mathematical model for a one-prey, one-predator system in which the prey can diffuse between one patch with little food and no predation and one patch with much food and predation, such as [5-7].

The dispersal predator-prey systems which are described by autonomous ordinary differential equations have been well studied by many authors [5,7-20], and the references cited therein. However, realistic models often require the effects of the time delays and the changing environment. Recently, Song and Chen in [6,21] studied the effect of dispersal on the permanence and the stability of periodic predator-prey system with and without time delays, respectively. They provided a set of sufficient conditions for permanence of the following system:

\[
\begin{align*}
\dot{x}_1 &= x_1 [b_1(t) - a_1(t)x_1 - c(t)y] + D_1(t)(x_2 - x_1), \\
\dot{x}_2 &= x_2 [b_2(t) - a_2(t)x_2] + D_2(t)(x_1 - x_2), \\
\dot{y} &= y[-d(t) + e(t)x_1 - q(t)y].
\end{align*}
\]

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under the assumption that the prey species has a positive intrinsic growth rate in every patch. This condition indicates that their results cannot be chosen to satisfy the prey and predator species living in weak patch environment [22,23] where the prey species has negative intrinsic growth rate in some time intervals in some patches (not all patches). This kind of biological environment problem motivates our study on the following system:

\[
\begin{align*}
\dot{x}_1 &= x_1 [b_1(t) - a_1(t)x_1 - c_1(t)y] + D_{12}(t)x_2 - D_{21}(t)x_1, \\
\dot{x}_2 &= x_2 [b_2(t) - a_2(t)x_2] + D_{21}(t)x_1 - D_{12}(t)x_2, \\
\dot{y} &= y [-d(t) + e(t)x_1 - f(t)y].
\end{align*}
\]

We denote by \(x_1\) and by \(x_2\) the density of the prey in Patch 1 and in Patch 2, respectively, and by \(y\) the density of the predator in Patch 1. All coefficients in (1.2) are \(\omega\)-periodic and continuous for \(t \geq 0\), \(a_1(t), a_2(t), f(t), D_{12}(t),\) and \(D_{21}(t)\) are all positive, while \(d(t), b_1(t), c_1(t), e(t)\) are nonnegative. The function \(b_i(t)\) is the intrinsic growth rate for species \(x_i\) in patch \(i\), \(a_i(t)\) represents the self-inhibition coefficient, and \(D_{ij}(t)\) is the diffusion coefficient of species \(x_j\) from patch \(j\) to patch \(i\).

We suppose that in Patch 2 there is less food without predation, and in Patch 1 there is more food but risk of predation. The intrinsic growth rate \(b_2(t)\) of prey species \(x_2\) in Patch 2 may be negative on some time intervals to indicate that the prey species live in a weak patch environment.

The organization of this paper is as follows. In the next section, we agree on some notations, give some definitions, and state three lemmas which will be essential to our proofs. In Section 3, we consider the effect of dispersal on the permanence of prey-predator system (1.2). We get a sufficient and necessary condition to guarantee the prey and predator species to be permanent under the assumption that the prey species have a lower dispersal level.

2. NOTATIONS, DEFINITIONS AND PRELIMINARIES

In this section, we introduce some definitions and notations, and state some results which will be useful in the subsequent sections. Let \(C\) denote the space of all bounded continuous functions \(f : \mathbb{R} \to \mathbb{R}\), \(C^0_+\) is the set of nonnegative \(f \in C\), and \(C_+\) is the set of all \(f \in C\) such that \(f\) is bounded below by a positive constant. Given \(f \in C\), we denote

\[
\begin{align*}
\sup_{t \geq 0} f(t), & \quad \inf_{t \geq 0} f(t), \\
\end{align*}
\]

and define the lower average \(A_L(f)\) and upper average \(A_M(f)\) of \(f\) by

\[
A_L(f) = \lim_{r \to \infty} \inf_{t-s \geq r} (t-s)^{-1} \int_{s}^{t} f(\tau) \, d\tau
\]

and

\[
A_M(f) = \lim_{r \to \infty} \sup_{t-s \geq r} (t-s)^{-1} \int_{s}^{t} f(\tau) \, d\tau,
\]

respectively. If \(f \in C\) is \(\omega\)-periodic, then the average \(A_\omega(f)\) of \(f\) on a time interval \([0, \omega]\) can be defined as

\[
A_\omega(f) = \omega^{-1} \int_{0}^{\omega} f(t) \, dt.
\]

DEFINITION 2.1. The system of differential equations

\[
\dot{x} = F(t, x), \quad x \in \mathbb{R}^n,
\]

is said to be permanent if there exists a compact set \(K\) in the interior of \(\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, 2, \ldots, n\}\), such that all solutions starting in the interior of \(\mathbb{R}^n_+\) ultimately enter \(K\).
**DEFINITION 2.2.** The system of differential equations

\[ \dot{x} = F(t, x), \quad x \in \mathbb{R}^n, \]

is said to be cooperative if the off-diagonal elements of \( D_x F(t, x) \) are nonnegative and competitive if the off-diagonal elements are nonpositive, where \( D_x F(t, x) \) is the \( n \times n \) matrix derivative of \( F \) with respect to \( x \).

**LEMMA 2.1.** (See [24].) Let \( x(t) \) and \( y(t) \) are solutions of

\[ \dot{x} = F(t, x) \]

and

\[ \dot{y} = G(t, y), \]

respectively, where both systems are assumed to have the uniqueness property for initial value problems. Assume both \( x(t) \) and \( y(t) \) belong to a domain \( D \subseteq \mathbb{R}^n \) for \([t_0, t_1]\) in which one of the two systems is cooperative and

\[ F(t, z) \leq G(t, z), \quad (t, z) \in [t_0, t_1] \times D. \]

If \( x(t_0) \leq y(t_0) \), then \( x(t_1) \leq y(t_1) \). If \( F = G \) and \( x(t_0) < y(t_0) \), then \( x(t_1) < y(t_1) \).

To prove the permanence of species in (1.2), we need the information on the periodic logistic models with and without dispersal.

**LEMMA 2.2.** (See [25].) The problem

\[ \dot{x} = x \left[ b(t) - a(t)x \right], \quad x \in \mathbb{C}^+, \tag{2.1} \]

has exactly one canonical solution \( U \) if \( a \in \mathbb{C}^+, b \in \mathbb{C} \) and \( A_L(b) > 0 \). Moreover, the following properties hold:

(a) \( U \) is \( \omega \)-periodic (almost periodic) if \( a, b \) are \( \omega \)-periodic (almost periodic);
(b) \( U \) is constant if \( b/a \) is constant, in this case, \( U = b/a \);
(c) \( u(t) - U(t) \to 0 \) as \( t \to \infty \), for any positive solution \( u(t) \) of equation (2.1);
(d) \( (b(t)/a(t))^L \leq U \leq (b(t)/a(t))^M \).

For the dispersal logistic equations

\[ \begin{align*}
\dot{x}_1 &= x_1 \left[ b_1(t) - a_1(t)x_1 \right] + D_{12}(t) x_2 - D_{21} x_1, \\
\dot{x}_2 &= x_2 \left[ b_2(t) - a_2(t)x_2 \right] + D_{21}(t) x_1 - D_{12} x_2, \tag{2.2}
\end{align*} \]

we can obtain the following results using similar proofs to those of Theorem 2 and 3 in [22].

**LEMMA 2.3.** Suppose that there exists an integer \( i \) (\( i = 1 \) or 2) such that

\[ A_\omega (b_i(t) - D_{ji}(t)) > 0 \quad (j \neq i), \]

then system (2.2) is permanent and there exist a unique positive \( \omega \)-periodic solution \((x^*_1(t), x^*_2(t))\), which is globally and asymptotically stable.
3. THE EFFECT OF DISPERSAL ON PREDATOR-PREY SPECIES

In this section, we consider the effect of dispersal on the permanence of the predator-prey species. For (1.2), we make the following assumptions.

(H1) \( A_0 [b_1(t) - D_{22}(t)] > 0 \).

**Theorem 3.1.** Under Assumption (H1), (1.2) is permanent if and only if

(H2) \( A_0 [-d(t) + e(t) x_2(t)] > 0 \).

To prove this theorem, we need the following several propositions. For the rest of this paper, we denote \((x_1(t), x_2(t), y(t))\) to any solution of (1.2) with positive initial conditions.

**Proposition 3.1.** Suppose (H1) holds, then there exist positive constants \( M_x \) such that

\[
\lim_{t \to \infty} \sup_{i=1,2} x_i(t) \leq M_x, \quad i = 1, 2. \tag{3.1}
\]

In addition, if (H2) holds, then there exists a positive constant \( M_y \) such that

\[
\lim_{t \to \infty} \sup_{i=1,2} y(t) \leq M_y. \tag{3.2}
\]

**Proof.** Obviously, \( R^3_+ = \{(x_1, x_2, y) \mid x_1 \geq 0, x_2 \geq 0, y \geq 0\} \) is a positively invariant set of system (1.2). Given any positive solution \((x_1(t), x_2(t), y(t))\) of (1.2), we have

\[
x_i \leq x_i [b_i(t) - a_i(t) x_i] + D_{ij}(t) x_j - D_{ji}(t) x_i, \quad i = 1, 2, \quad j \neq i;
\]
on the other hand, the following auxiliary equations:

\[
\dot{u}_i = u_i [b_i(t) - a_i(t) u_i] + D_{ij}(t) u_j - D_{ji}(t) u_i, \quad i = 1, 2, \quad j \neq i, \tag{3.3}
\]
have a globally asymptotically stable positive \( \omega \)-periodic solution \((x^*_1(t), x^*_2(t))\), under Assumption (H1). Let \((u_1(t), u_2(t))\) be the solution of (3.3) with \(u_i(0) = x_i(0)\), by Lemma 2.1, we have

\[
x_i(t) \leq u_i(t), \quad i = 1, 2,
\]
for \( t \geq 0 \). Moreover, from the global stability of \((x^*_1(t), x^*_2(t))\), for every given \( \epsilon > 0 \), there exists \( T_0 > 0 \), such that

\[
u_i(t) < x^*_i(t) + \epsilon, \quad \text{for} \ t > T_0,
\]
hence

\[
x_i(t) < x^*_i(t) + \epsilon, \quad i = 1, 2,
\]
for \( t \geq T_0 \). In addition, for \( t \geq T_0 \), we have

\[
\dot{y} \leq y [-d(t) + e(t) (x^*_1(t) + \epsilon) - f(t) y].
\]

By (H2) and Lemmas 2.1 and 2.2, there exists \( T_1 > T_0 \), such that

\[
y(t) < y^*(t) + \epsilon.
\]
for \( t \geq T_1 \), where \( y^*(t) \) is the positive and globally asymptotically stable \( \omega \)-periodic solution of the following auxiliary equation:

\[
\dot{v} = v [-d(t) + e(t) (x^*_1(t) + \epsilon) - f(t) v].
\]
Denote \( M_x = \max_{0 \leq t \leq \omega} \{x^*_i(t) + \epsilon : i = 1, 2\} \) and \( M_y = \max_{0 \leq t \leq \omega} \{y^*(t) + \epsilon\} \), then (3.1) and (3.2) hold for system (1.2).
PROPOSITION 3.2. Suppose that (H1) holds, then there exists a positive constant \( \eta_2 \) such that

\[
\lim_{t \to \infty} \sup_{x_1(t)} \geq \eta_2. \tag{3.4}
\]

PROOF. Suppose that (3.4) is not true, then there is a sequence \( \{z_m\} \subset \mathbb{R}_+^3 \), such that

\[
\lim_{t \to \infty} \sup_{x_1(t, z_m)} < \frac{1}{m}, \quad m = 1, 2, \ldots, \tag{3.5}
\]

where \( (x_1(t, z_m), x_2(t, z_m), y(t, z_m)) \) is the solution of (1.2) with initial values \( (x_1(t, 0), x_2(t, 0), y(t, 0)) = z_m \). Choose sufficiently small positive constants \( \varepsilon_x \) and \( \varepsilon_y \) such that \( \varepsilon_x < 1, \varepsilon_y < 1, \) and

\[
A_\omega (-d(t) + e(t)\varepsilon_y) < 0. \tag{3.6}
\]

and

\[
A_\omega \left[ h_1(t) - c_1(t) \varepsilon_y \exp \left( \alpha \omega \right) - a_1(t) \varepsilon_x - D_{21}(t) \right] > 0, \tag{3.7}
\]

where \( \alpha = \max_{0 \leq \omega \leq \omega} \{d(t) + e(t) + f(t)\} \). By (3.5), for the given \( \varepsilon_x > 0 \), there exists a positive integer \( N_0 \), such that

\[
\lim_{t \to \infty} \sup_{x_1(t, z_m)} < \frac{1}{m} < \varepsilon_x,
\]

for \( m > N_0 \). For the rest of this proof, we assume that \( m > N_0 \). Hence, there exists \( \tau_1^{(m)} > 0 \), such that

\[
x_1(t, z_m) < \varepsilon_x,
\]

for \( t \geq \tau_1^{(m)} \), and further,

\[
y(t, z_m) \leq y(t, z_m) \left[ -d(t) + e(t)\varepsilon_x - f(t)y(t, z_m) \right],
\]

for \( t \geq \tau_1^{(m)} \). By (3.6), any positive solution \( v(t) \) of the following equation:

\[
v = v \left[ -d(t) + e(t)\varepsilon_x - f(t)v \right]
\]

satisfies

\[
\lim_{t \to \infty} v(t) = 0.
\]

By Lemma 2.1, we have

\[
\lim_{t \to \infty} y(t, z_m) = 0.
\]

Therefore, there is a \( \tau_2^{(m)} > \tau_1^{(m)} \) such that

\[
y(t, z_m) < \varepsilon_y, \quad \text{for} \ m > N_0, \ t \geq \tau_2^{(m)}. \tag{3.8}
\]

This leads to

\[
\dot{x}_1(t, z_m) \geq x_1(t, z_m) \left[ b_1(t) - c_1(t) x_1(t, z_m) - c_1(t) \varepsilon_x \right] + D_{12}(t) x_2(t, z_m) - D_{21}(t) x_1(t, z_m),
\]

\[
\dot{x}_2(t, z_m) = x_2(t, z_m) \left[ b_2(t) - a_2(t) x_2(t, z_m) \right] + D_{21}(t) x_1(t, z_m) - D_{12}(t) x_2(t, z_m),
\]

for \( t \geq \tau_2^{(m)} \). Let \( (u_1(t), u_2(t)) \) be any positive solution of the following auxiliary equations:

\[
\dot{u}_1 = u_1 \left[ b_1(t) - a_1(t) u_1 - c_1(t) \varepsilon_x \right] + D_{12}(t) u_2 - D_{21}(t) u_1,
\]

\[
\dot{u}_2 = u_2 \left[ b_2(t) - a_2(t) u_2 \right] + D_{21}(t) u_1 - D_{12}(t) u_2. \tag{3.9}
\]

By (3.7) and Lemma 2.3, system (3.9) has a unique positive solution \( (u_1^*(t), u_2^*(t)) \), which is globally asymptotically stable. So we have

\[
x_i(t, z_m) > \frac{u_i^*(t)}{2}, \quad i = 1, 2,
\]

for sufficiently large \( t > 0 \) and \( m > N_0 \), which contradicts (3.5). This completes the proof.
PROPOSITION 3.3. Suppose that (H1) holds, then there exist positive constants $\gamma_{\varepsilon_i}$ such that

$$
\lim \inf_{t \to \infty} x_i(t) \geq \gamma_{\varepsilon_i} \quad (i = 1, 2).
$$

PROOF. We first show that inequality (3.10) holds for $i = 1$. Otherwise, there exists a sequence $\{z_m\} \subset R^k$, such that

$$
\lim \inf_{t \to \infty} x_1(t, z_m) < \frac{\eta_x}{2m^2}, \quad m = 1, 2, \ldots.
$$

On the other hand, by Proposition 3.2,

$$
\lim \sup_{t \to \infty} x_1(t, z_m) > \eta_x, \quad m = 1, 2, \ldots.
$$

Hence, there are two sequences $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$ satisfying the following conditions:

$$
0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots, \quad s_q^{(m)} \to \infty, \quad t_q^{(m)} \to \infty, \quad \text{as } q \to \infty,
$$

and

$$
x_1(s_q^{(m)}, z_m) = \frac{\eta_x}{m}, \quad x_1(t_q^{(m)}, z_m) = \frac{\eta_x}{m^2}, \quad \frac{\eta_x}{m^2} < x_1(t, z_m) < \frac{\eta_x}{m}, \quad t \in \left(s_q^{(m)}, t_q^{(m)}\right).
$$

By Proposition 3.1, for a given integer $m > 0$, there is a $T_1^{(m)} > 0$, such that

$$
x_1(t, z_m) \leq M_x, \quad y(t, z_m) \leq M_y,
$$

for $t \geq T_1^{(m)}$. Because $s_q^{(m)} \to \infty$ as $q \to \infty$, there is a positive integer $K^{(m)}$, such that $s_q^{(m)} > T_1^{(m)}$ as $q \geq K^{(m)}$, hence

$$
\dot{x}_1(t, z_m) \geq \zeta(t)x_1(t, z_m),
$$

for $q \geq K^{(m)}$, $t \in [s_q^{(m)}, t_q^{(m)}]$, where $\zeta(t) = b_1(t) - D_2(t) - a_1(t)M_x - c_1(t)M_y$. Integrating (3.12) from $s_q^{(m)}$ to $t_q^{(m)}$ yields

$$
x_1(t_q^{(m)}, z_m) \geq x_1(s_q^{(m)}, z_m) \exp \int_{s_q^{(m)}}^{t_q^{(m)}} \zeta(t) dt,
$$

or

$$
- \int_{s_q^{(m)}}^{t_q^{(m)}} \zeta(t) dt \geq \ln m, \quad \text{for } q \geq K^{(m)}.
$$

If $A_\omega(\zeta(t)) \geq 0$, this leads to a contradiction; otherwise, $A_\omega(\zeta(t)) < 0$, we have

$$
t_q^{(m)} - s_q^{(m)} \to \infty, \quad \text{as } m \to \infty, \quad q \geq K^{(m)},
$$

according to the boundedness of $\zeta(t)$. There exist positive constant $P$, $N_0$, $\varepsilon_x$, and $\varepsilon_y$, $0 < \varepsilon_x < 1$, $0 < \varepsilon_y < 1$, such that (3.6) and (3.7) hold and

$$
\frac{\eta_x}{m} < \varepsilon_x, \quad \frac{t_q^{(m)} - s_q^{(m)}}{2P},
$$

$$
M_y \exp \int_0^P [-d(t) + e(t)\varepsilon_x - f(t)\varepsilon_y] dt < \varepsilon_y.
$$
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and

\[ \int_0^a \left[ b_1(t) - c_1(t) \varepsilon \exp(\alpha \omega) - a_1(t) \varepsilon_x - D_{21}(t) \right] dt > 0, \]  
for all \( m \geq N_0, q \geq K^{(m)}, \) and \( a \geq P. \) Inequality (3.13) implies

\[ x_i(t, z_m) < \varepsilon, \quad i = 1, 2, \quad t \in \left[ s_q^{(m)}, t_q^{(m)} \right], \]  
for \( m \geq N_0, q \geq K^{(m)}. \) For positive \( \varepsilon_y \) satisfying (3.7) and (3.14), we have the following two circumstances:

(i) \( y(t, z_m) \geq \varepsilon_y \) for all \( t \in [s_q^{(m)}, s_q^{(m)} + P]; \)
(ii) there exists \( \tau_q^{(m)} \in [s_q^{(m)}, s_q^{(m)} + P], \) such that \( y(\tau_q^{(m)}, z_m) < \varepsilon_y. \)

If (i) holds, by (3.16), we have

\[ \varepsilon_y \leq y \left( s_q^{(m)} + P, z_m \right) \]
\[ \leq y \left( s_q^{(m)}, z_m \right) \exp \int_{s_q^{(m)}}^{s_q^{(m)} + P} \left[ -d(t) + e(t)\varepsilon_x - f(t)\varepsilon_y \right] dt \]
\[ \leq M_y \exp \int_0^P \left[ -d(t) + e(t)\varepsilon_x - f(t)\varepsilon_y \right] dt \]
\[ < \varepsilon_y, \]
which is a contradiction.

If (ii) holds, we now claim that

\[ y \left( t, z_m \right) \leq \varepsilon_y \exp(\alpha \omega), \quad t \in \left( \tau_q^{(m)}, t_q^{(m)} \right). \]  
(3.17)

Otherwise, there exists \( \tau_q^{(m)} \in \left( \tau_q^{(m)}, \tau_q^{(m)} \right) \) such that

\[ y \left( \tau_q^{(m)}, z_m \right) > \varepsilon_y \exp(\alpha \omega). \]

By the continuity of \( y(t, z_m), \) there must exist \( \tau_q^{(m)} \in (\tau_q^{(m)}, \tau_q^{(m)}) \) such that

\[ y \left( \tau_q^{(m)}, z_m \right) = \varepsilon_y, \]
and

\[ y \left( t, z_m \right) > \varepsilon_y, \]
for \( t \in (\tau_q^{(m)}, \tau_q^{(m)}). \) Let \( P^{(m)} \) be the nonnegative integer such that \( \tau_q^{(m)} \in (\tau_q^{(m)} + P^{(m)}\omega, \tau_q^{(m)} + (P^{(m)} + 1)\omega), \) we obtain, by (3.4),

\[ \varepsilon_y \exp(\alpha \omega) < y \left( \tau_q^{(m)}, z_m \right) \]
\[ < y \left( \tau_q^{(m)}, z_m \right) \exp \int_{\tau_q^{(m)}}^{\tau_q^{(m)} + P^{(m)}\omega} \left[ -d(t) + e(t)\varepsilon_x - f(t)\varepsilon_y \right] dt \]
\[ = \varepsilon_y \exp \left\{ \int_{\tau_q^{(m)}}^{\tau_q^{(m)} + P^{(m)}\omega} + \int_{\tau_q^{(m)} + P^{(m)}\omega}^{\tau_q^{(m)} + P^{(m)}\omega} \right\} \left[ -d(t) + e(t)\varepsilon_x - f(t)\varepsilon_y \right] dt \]
\[ < \varepsilon_y \exp(\alpha \omega). \]
This contradiction establishes that (3.17) is true, particularly (3.17) holds for \( t \in [s_q^{(m)} + P, t_q^{(m)}] \).

By (3.11) and (3.15), we have

\[
\eta_x \frac{m^2}{M^2} = x_1 \left( \frac{1}{s_q^{(m)} + P}, z_m \right)
\geq x_1 \left( \frac{1}{s_q^{(m)} + P}, z_m \right) \exp \int_{s_q^{(m)} + P}^{t_q^{(m)}} [b_1(t) - c_1(t) \varepsilon_y \exp (\alpha \omega) - a_1(t) \varepsilon_x - D_{21}(t)] \, dt
\geq \eta_x \frac{m^2}{M^2} \exp \int_{s_q^{(m)} + P}^{t_q^{(m)}} [b_1(t) - c_1(t) \varepsilon_y \exp (\alpha \omega) - a_1(t) \varepsilon_x - D_{21}(t)] \, dt
> \eta_x \frac{m^2}{M^2},
\]

which is also a contradiction. Hence, there exists \( \gamma_{x1} \) such that (3.10) holds for \( i = 1 \).

Secondly, we show that (3.10) holds for \( i = 2 \). According to the above discussion, there exists \( T_2 \geq T_1 \) such that

\[ x_1(t) > \gamma_{x1}, \]
for \( t \geq T_2 \). So we have

\[
\dot{x}_2 \geq x_2 \left[ b_2(t) - D_{12}(t) - a_2(t)x_2 \right] + D_{21}(t)\gamma_{x1}
\geq -a_2 x_2^2 + (b_2^2 - D_{12}^2) x_2 + D_{21} \gamma_{x1} = F(x_2),
\]
for \( t \geq T_2 \). The algebraic equation \( F(x_2) = 0 \) give us one positive root

\[ \bar{x}_2 = \frac{b_2^2 - D_{12}^2 + \sqrt{(b_2^2 - D_{12}^2)^2 + 4D_{21}^2 a_2^2 \gamma_{x1}}}{2a_2^2}. \]

Clearly, \( F(x_2) > 0 \) for every positive number \( x_2 \) \( (0 < x_2 < \bar{x}_2) \). Choose \( \gamma_{x2}(0 < \gamma_{x2} < \bar{x}_2), \dot{x}_2|_{x_2=\gamma_{x2}} \geq F(\gamma_{x2}) > 0 \). If \( x_2(T_2) \geq \gamma_{x2} \), then it also holds for \( t \geq T_2 \); if \( x_2(T_2) < \gamma_{x2} \), then

\[ \dot{x}_2(T_2) \geq \inf \{ F(x_2) \mid 0 \leq x_2 < \gamma_{x2} \} > 0, \]
there must exist \( T_3 \geq T_2 \), such that \( x_2(t) > \gamma_{x2} \) for \( t \geq T_3 \). This completes the proof.

**Proposition 3.4.** Suppose that (H1) and (H2) hold, then there exists a positive constant \( \eta_y \) such that

\[ \lim_{t \to \infty} \sup y(t) > \eta_y. \]  

**Proof.** By Assumption (H2), we can choose constant \( \varepsilon_0 > 0 \) such that

\[ A_\omega (\psi_{\varepsilon_0}(t)) > 0, \]  

where

\[ \psi_{\varepsilon_0}(t) = -d(t) + e(t)x_1^*(t) - e(t)\varepsilon_0 - f(t)\varepsilon_0. \]

Consider the following equations with parameter \( \alpha > 0 \):

\[
\dot{x}_1 = x_1 \left[ b_1(t) - 2\alpha c_1(t) - a_1(t)x_1 \right] + D_{12}(t)x_2 - D_{21}(t)x_1,
\dot{x}_2 = x_2 \left[ b_2(t) - a_2(t)x_2 \right] + D_{21}(t)x_1 - D_{12}(t)x_2.
\]  

By Assumption (H1), we know that

\[ A_\omega \left[ b_1(t) - D_{21}(t) - 2\alpha c_1(t) \right] > 0. \]
holds for sufficiently small \( \alpha > 0 \). By Lemma 2.3, equation (3.20) has a unique positive \( \omega \)-periodic solution \((x_{1\alpha}(t), x_{2\alpha}(t))\), which is globally asymptotically stable. Let \((\bar{x}_{1\alpha}(t), \bar{x}_{2\alpha}(t))\) be the solution of (3.20) with initial condition \( \bar{x}_{i\alpha}(0) = x_{i\alpha}^*(0), i = 1, 2, \) then for the above \( \varepsilon_0 \), there exists \( T_4 \geq T_3 \), such that

\[
|x_{1\alpha}(t) - x_{1\alpha}(t)| < \frac{\varepsilon_0}{4}, \quad \text{for } t \geq T_4.
\]

By the continuity of the solution in the parameter, we have \((\bar{x}_{1\alpha}(t), \bar{x}_{2\alpha}(t)) \rightarrow (x_{1\alpha}^*(t), x_{2\alpha}^*(t))\) uniformly in \([T_4, T_4 + \omega]\) as \( \alpha \rightarrow 0 \). Hence, for \( \varepsilon_0 > 0 \), there exists \( \alpha_0 = \alpha_0(\varepsilon_0) > 0 \) such that

\[
|x_{1\alpha}(t) - x_{1\alpha}^*(t)| < \frac{\varepsilon_0}{4}, \quad \text{for } t \in [T_4, T_4 + \omega], \quad 0 < \alpha < \alpha_0.
\]

So we have

\[
|x_{1\alpha}(t) - x_{1\alpha}^*(t)| \leq |\bar{x}_{1\alpha}(t) - x_{1\alpha}(t)| + |\bar{x}_{1\alpha}(t) - x_{1\alpha}^*(t)| < \frac{\varepsilon_0}{2},
\]

for \( t \geq 0, 0 < \alpha < \alpha_0 \). Choosing constant \( \alpha_1 (0 < \alpha_1 < \alpha_0, 2\alpha_1 < \varepsilon_0) \), then

\[
x_{1\alpha_1}(t) \geq x_{1\alpha}^*(t) - \frac{\varepsilon_0}{2}, \quad t \geq 0.
\]

(3.21)

Suppose that conclusion (3.18) is not true, then there exists \( Z \in R_+^3 \) such that for the positive solution \((x_1(t), x_2(t), y(t))\) of (1.2) with initial condition \((x_1(0), x_2(0), y(0)) = Z\), we have

\[
\limsup_{t \rightarrow \infty} y(t) < \alpha_1.
\]

So there exists \( T_5 \geq T_4 \) such that

\[
y(t) < 2\alpha_1,
\]

(3.22)

for \( t \geq T_5 \), and hence,

\[
\begin{align*}
\dot{x}_1 &\geq x_1 [b_1(t) - 2\alpha_1 c_1(t) - a_1(t)]x_1 + D_{12}(t)x_2 - D_{21}(t)x_1, \\
\dot{x}_2 &\geq x_2 [b_2(t) - a_2(t)]x_2 + D_{21}(t)x_1 - D_{12}(t)x_2.
\end{align*}
\]

Let \((u_1(t), u_2(t))\) be the solution of (3.20) with \( \alpha = \alpha_1 \) and condition \( u_i(T_5) = x_i(T_5), i = 1, 2, \) by Lemma 2.1, we know that

\[
x_i(t) \geq u_i(t), \quad t \geq T_5, \quad i = 1, 2.
\]

By the global asymptotic stability of \((x_{1\alpha_1}(t), x_{2\alpha_1}(t))\), for given \( \varepsilon = \varepsilon_0/2 \), there exists \( T_6 \geq T_5 \) such that

\[
|u_i(t) - x_{1\alpha_1}(t)| < \frac{\varepsilon_0}{2}, 
\]

for \( t \geq T_6 \). So we have

\[
x_1(t) \geq u_1(t) > x_{1\alpha_1}(t) - \frac{\varepsilon_0}{2}, \quad t \geq T_6,
\]

and hence,

\[
x_1(t) \geq x_{1\alpha}^*(t) - \varepsilon_0, \quad t \geq T_6.
\]

This implies

\[
y(t) \geq \psi_{\varepsilon_0}(t)y(t), \quad t \geq T_6,
\]

integrating the above inequality from \( T_6 \) to \( t \) yields

\[
y(t) \geq y(T_6)\exp \int_{T_6}^t \psi_{\varepsilon_0}(t) dt.
\]

By (3.19) we know that \( y(t) \rightarrow \infty \) as \( t \rightarrow \infty \), which is a contradiction. This completes the proof.
PROPOSITION 3.5. Under Assumptions (H1) and (H2), there exists a positive constant $\gamma_y$ such that

$$\liminf_{t \to \infty} y(t) \geq \gamma_y.$$  \hfill (3.23)

PROOF. Otherwise, there must exist a sequence $\{z_m\} \subset \mathbb{R}_+^4$, such that

$$\liminf_{t \to \infty} y(t, z_m) < \frac{\eta_y}{(m+1)^2}, \quad m = 1, 2, \ldots;$$

but

$$\limsup_{t \to \infty} y(t, z_m) > \eta_y, \quad m = 1, 2, \ldots,$$

from Proposition 3.4. Hence, there are two time sequences $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$ satisfying the following conditions:

$$0 < s_1^{(m)} < s_2^{(m)} < \cdots < s_q^{(m)} < t_1^{(m)} < \cdots < t_q^{(m)} < \cdots,$$

$$s_q^{(m)} \to \infty, \quad t_q^{(m)} \to \infty, \quad \text{as } q \to \infty,$$

and

$$y\left(s_q^{(m)}, z_m\right) = \frac{\eta_y}{m+1}, \quad y\left(t_q^{(m)}, z_m\right) = \frac{\eta_y}{(m+1)^2},$$

$$\frac{\eta_y}{(m+1)^2} < y(t, z_m) < \frac{\eta_y}{m+1}, \quad t \in \left(s_q^{(m)}, t_q^{(m)}\right).$$  \hfill (3.24)

By Proposition 3.1, for a given integer $m > 0$, there is a $T_1^{(m)} > 0$, such that

$$y(t, z_m) \leq M_y, \quad \text{for } t \geq T_1^{(m)}.$$  

Because $s_q^{(m)} \to \infty$ as $q \to \infty$, there is a positive integer $K^{(m)}$, such that $s_q^{(m)} > T_1^{(m)}$ as $q \geq K^{(m)}$, hence,

$$y(t, z_m) > y(t, z_m) \left[-d(t) - f(t)M_y\right].$$

for $q \geq K^{(m)}$, $t \in [s_q^{(m)}, t_q^{(m)}]$. Integrating the above inequality from $s_q^{(m)}$ to $t_q^{(m)}$, we get

$$y\left(t_q^{(m)}, z_m\right) \geq y\left(s_q^{(m)}, z_m\right) \exp\int_{s_q^{(m)}}^{t_q^{(m)}} \left[-d(t) - f(t)M_y\right] \, dt.$$  

So we have

$$\int_{s_q^{(m)}}^{t_q^{(m)}} [d(t) + f(t)M_y] \, dt \geq \ln(m+1),$$

for $q \geq K^{(m)}$. According to the boundedness of the function $d(t) + f(t)M_y$, we know that

$$t_q^{(m)} - s_q^{(m)} \to \infty, \quad \text{as } m \to \infty, \quad q \geq K^{(m)}.$$  \hfill (3.25)

By (3.19), there are constants $P > 0$, $a \geq P$ and an integer $N_0 > 0$ such that

$$\frac{\eta_y}{m+1} < \alpha_1 < \varepsilon_0, \quad t_q^{(m)} - s_q^{(m)} > 2P,$$  \hfill (3.26)

and

$$\int_0^a \psi_{\varepsilon_0}(t) \, dt > 0.$$  \hfill (3.27)
for \( m \geq N_0, q \geq K^{(m)} \). Further, we have

\[ y(t, z_m) < \alpha_1, \quad t \in \left[ s_q^{(m)}, t_q^{(m)} \right], \]

for \( m \geq N_0, q \geq K^{(m)} \). In addition, for \( t \in \left[ s_q^{(m)}, t_q^{(m)} \right] \), we have

\[
\begin{align*}
\dot{x}_1(t, z_m) &\geq x_1(t, z_m) \left[ b_1(t) - 2\alpha_1 c_1(t) \alpha_1(t) x_1(t, z_m) \right] - D_{12}(t) x_2(t, z_m) - D_{21}(t) x_1(t, z_m), \\
\dot{x}_2(t, z_m) &= x_2(t, z_m) \left[ b_2(t) - a_2(t) x_2(t, z_m) \right] + D_{21}(t) x_1(t, z_m) - D_{12}(t) x_2(t, z_m).
\end{align*}
\]

Let \((u_1(t), u_2(t))\) be the solution of (3.20) with \( \alpha = \alpha_1 \) and \( u_i(s_q^{(m)}) = x_i(s_q^{(m)}, z_m) \), by Lemma 2.1, we have

\[ x_i(t, z_m) \geq u_i(t), \quad t \in \left[ s_q^{(m)}, t_q^{(m)} \right]. \]

Further, by Propositions 3.1 and 3.3 and \( s_q^{(m)} \to \infty \) as \( q \to \infty \), we can choose \( \gamma_{\omega, 1} > \gamma_{\omega, m} \), such that

\[ \gamma_{\omega, i} \leq x_i(s_q^{(m)}, z_m) \leq M_x, \quad i = 1, 2, \]

holds for \( q \geq K^{(m)}_1 \). For \( \alpha = \alpha_1 \), (3.20) has a unique positive \( \omega \)-periodic solution \((x_{1\alpha_1}(t), x_{2\alpha_1}(t))\) which is globally asymptotically stable. In addition, by the periodicity of (3.20), the periodic solution \((x_{1\alpha_1}(t), x_{2\alpha_1}(t))\) is uniformly asymptotically stable with respect to the compact set \( \Omega = \{(x_1, x_2) : \gamma_x \leq x_i \leq M_x, \ i = 1, 2\} \). Hence, for the given \( \varepsilon_0 \) in Proposition 3.4, there exists \( T_0 > P \) which is independent on \( m \) and \( q \), such that

\[ u_1(t) \geq x_{1\alpha_1}(t) - \frac{\varepsilon_0}{2}, \quad t \geq T_0 + s_q^{(m)}. \]

By (3.21), we have

\[ u_1(t) \geq x_1^*(t) - \varepsilon_0, \quad t \geq T_0 + s_q^{(m)}. \]

From (3.25), there exists a positive integer \( N_1 \geq N_0 \), such that \( t_q^{(m)} > s_q^{(m)} + 2T_0 > s_q^{(m)} + 2P \) for \( m \geq N_1 \) and \( q \geq K^{(m)}_1 \). So we have

\[ x_1(t, z_m) \geq x_1^*(t) - \varepsilon_0, \quad t \in \left[ s_q^{(m)} + T_0, t_q^{(m)} \right], \]

as \( m \geq N_1 \) and \( q \geq K^{(m)}_1 \). Hence,

\[ y(t, z_m) \geq \psi_{\varepsilon_0}(t) y(t, z_m), \]

for \( t \in \left[ s_q^{(m)} + T_0, t_q^{(m)} \right] \). Integrating the above inequality from \( s_q^{(m)} + T_0 \) to \( t_q^{(m)} \) yields

\[ y\left(t_q^{(m)}, z_m\right) \geq y\left(s_q^{(m)} + T_0, z_m\right) \exp \int_{s_q^{(m)} + T_0}^{t_q^{(m)}} \psi_{\varepsilon_0}(t) \, dt, \]

that is to say

\[ \frac{\eta_y}{(m + 1)^2} \geq \frac{\eta_y}{(m + 1)^2} \exp \int_{s_q^{(m)} + T_0}^{t_q^{(m)}} \psi_{\varepsilon_0}(t) \, dt > \frac{\eta_y}{(m + 1)^2}, \]

which is a contradiction. This completes the proof.

**Proof of Theorem 3.1.** Combining Propositions 3.1–3.5 completes the proof of the sufficiency of Theorem 3.1.

Next we prove the necessity of Theorem 3.1. We will show that, under Assumption (H1), if

\[ A_\omega [-d(t) + e(t) z_1^*(t)] \leq 0, \]

(3.28)
then
\[ \lim_{t \to \infty} y(t) = 0. \]

In fact, by (3.28), we know that for any given \( 0 < \varepsilon < 1 \), there exists \( \varepsilon_1 > 0 \) and \( \varepsilon_0 > 0 \) such that
\[ A_{\omega} \left[ -d(t) + e(t)(x_1^*(t) + \varepsilon_1) - f(t)\varepsilon \right] \leq \varepsilon_1 A_{\omega} e(t) - \varepsilon A_{\omega} f(t) \leq -\varepsilon_0. \] (3.29)

Since
\[
\begin{align*}
\dot{x}_1 & \leq x_1 \left[ b_1(t) - a_1(t)x_1 \right] + D_{12}(t)x_2 - D_{21}(t)x_1, \\
\dot{x}_2 & = x_2 \left[ b_2(t) - a_2(t)x_2 \right] + D_{21}(t)x_1 - D_{12}(t)x_1,
\end{align*}
\]

we know that for the given \( \varepsilon_1 \) there exists \( T^{(1)} > 0 \) such that
\[ x_1(t) < x_1^*(t) + \varepsilon_1, \quad t > T^{(1)}. \]

By (3.29), we have
\[ A_{\omega} \left[ -d(t) + e(t)x_1(t) - f(t)\varepsilon \right] \leq -\varepsilon_0, \] (3.30)
for \( t \geq T^{(1)} \). First, we show that there must exist \( T^{(2)} \) such that \( y(T^{(2)}) < \varepsilon \). Otherwise, we have
\[ \varepsilon \leq y(t) \leq \varepsilon \exp \left( T^{(1)} \right) \exp \int_{T^{(1)}}^{t} \left[ -d(s) + e(s)x_1(s) - f(s)\varepsilon \right] ds \to 0, \quad \text{as} \ t \to \infty. \]

This implies \( \varepsilon \leq 0 \), which is a contradiction. Let \( M(\varepsilon) = \max_{0 \leq t \leq \omega} \{ d(t) + e(t)x_1(t) + f(t)\varepsilon \} \). By Proposition 3.1, we know that \( x_1(t) \) is bounded. So \( M(\varepsilon) \) is also bounded for \( \varepsilon \in [0, 1] \).

Second, we will show that
\[ y(t) \leq \varepsilon \exp \left( M(\varepsilon) \omega \right), \quad \text{for} \ t \geq T^{(2)}. \] (3.31)

Otherwise, there exists \( T^{(3)} > T^{(2)} \) such that
\[ y \left( T^{(3)} \right) > \varepsilon \exp \left( M(\varepsilon) \omega \right). \]

By the continuity of \( y(t) \), there must exist \( T^{(4)} \subset (T^{(2)}, T^{(3)}) \) such that \( y(T^{(4)}) = \varepsilon \) and \( y(t) > \varepsilon \) for \( t \in (T^{(4)}, T^{(3)}) \). Let \( P_1 \) be the nonnegative integer such that \( T^{(3)} \in (T^{(4)} + P_1\omega, T^{(4)} + (P_1 + 1)\omega) \), by (3.30), we have
\[ \varepsilon \exp \left( M(\varepsilon) \omega \right) < y \left( T^{(3)} \right) \]
\[ = \varepsilon \exp \left\{ \int_{T^{(4)}}^{T^{(3)}} \left[ -d(t) + e(t)x_1(t) - f(t)\varepsilon \right] dt \right\} \]
\[ < \varepsilon \exp \left( M(\varepsilon) \omega \right), \]
which is a contradiction. This implies (3.31) holds. Further, by the arbitrariness of \( \varepsilon \), we know that \( y(t) \to 0 \) as \( t \to \infty \). This completes the proof.
REFERENCES