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On Expansion Problems Involving Addition Theorems and Kapteyn Series

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The paper deals with general expansions which give as special cases new results involving the Bessel functions, Jacobi, ultraspherical, and Laguerre polynomials, where the degree of the function is incorporated in the argument. In fact, the theorems unify and extend the Neumann–Gegenbauer expansion and its generalization by Fields and Wimp, Cohen, and others, the Kapteyn expansion theory, and the Kapteyn expansion of the second kind. New expressions are given for the Neumann–type degenerate form of a Gegenbauer addition theorem, the Feldheim expansions for the Jacobi and ultraspherical polynomials, and other expressions. Also of interest is the new method of proof, involving differential and integral operators.

INTRODUCTION

The generalization of the Neumann–Gegenbauer expansion has proved to be of considerable interest to a number of workers. See Luke [9] for an excellent exposition. This interest has resulted in the unification of many expansions and has in fact also given new addition theorems and generating functions. Fields and Wimp [6] gave us a general addition theorem which was extended in different directions. See Luke [9] and Cohen [3] for lists of references, which are by no means exhaustive.

In this paper, we not only present Kapteyn-type expansions, for both the first and second kind [11], but also show that the Neumann–Gegenbauer-type expansion fits into the theory. See Watson [12, Chap. XVII] for an excellent exposition of theory and application of the Kapteyn series [8]. Special cases of our theorems generalize a number of important old results such as the degenerate Gegenbauer addition theorem and other expressions. The expansions show the degree of the functions incorporated in the argument. Previous known results of this type were recently given by Carlitz [2] and Cohen [3, 4].

The addition theorems in this paper involve the Bessel function, Jacobi, ultraspherical, Laguerre, Neumann, and allied polynomials, and extended polynomials and functions such as those considered in Luke [10]. The

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method of proof is quite different from the aproach taken by previous workers in this area.

Gegenbauer [7] (see also Watson [12, p. 368, Eq. 2]) gave an important addition theorem

$$e^{iz\cos\phi} = \frac{2^\nu \Gamma(\nu)}{z^\nu} \sum_{n=0}^{\infty} (\nu+n)\, i^n J_{\nu+n}(z)\, C_n^\nu(\cos\phi) \tag{1.1}$$

involving the Bessel function and the Gegenbauer polynomial. A special case of Theorem 1(b) is the generalization of (1.1):

$$\begin{aligned}
&\frac{z^\lambda}{2^\lambda \Gamma(\lambda)(1-\frac{1}{4}a^2z^2)} \exp[izx] \\
&\quad = \sum_{n=0}^{\infty} i^n(\lambda+n)[(\alpha+\tfrac{1}{2}an)(a\lambda-\alpha+\tfrac{1}{2}an)]^{-\frac{1}{2}\lambda/2} \\
&\quad \cdot C_n^\lambda\left[\frac{x}{[(\alpha+\frac{1}{2}an)(\lambda a-\alpha+\frac{1}{2}an)]^{1/2}}\right] \\
&\quad \times J_{\lambda+n}[z[(\alpha+\tfrac{1}{2}an)(a\lambda-\alpha+\tfrac{1}{2}an)]^{1/2}].
\end{aligned} \tag{1.2}$$

A special case of interest in (1.2) is the case $\lambda a = 2\alpha$. Putting $a=0$, $\alpha = i$, $x = \cos\phi$ reduces (1.2) to (1.1).

An important new expansion is

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n(\lambda+2n)}{[(\alpha+an)(\lambda a-\alpha+an)]^{\lambda/2}} J_{\lambda+2n}[z[(\alpha+an)(\lambda a-\alpha+an)]^{1/2}] \\
&\quad \times C_{2n}^\lambda\left[\frac{x}{[(\alpha+an)(\lambda a-\alpha+an)]^{1/2}}\right] = \frac{z^\lambda}{2^\lambda\Gamma(\lambda)(1-\frac{1}{4}a^2z^2)}\cos[xz].
\end{aligned} \tag{1.3}$$

Putting $a=0$, $\alpha=i$, $x=\cos\phi$ gives the known Gegenbauer result [12, p. 369, Eq. 5].

Watson [12, p. 140, Eq. 3] gives the expansion of a Bessel function in terms of Bessel functions and modified Jacobi polynomials.

$$(\tfrac{1}{2}kz)^{\mu-\nu} J_\nu(kz) = k^\mu \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n)(\mu+2n)}{n!\,\Gamma(\nu+1)}\, {}_2F_1\left[\begin{matrix} -n, \mu+n; \\ \nu+1; \end{matrix}\ k^2\right] J_{\mu+2n}(z). \tag{1.4}$$

A particular example of Theorem 1(a) extends (1.4):

$$\sum_{n=0}^{\infty} \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{n!\,[(\alpha+an)(\lambda a-\alpha+an)]^{\lambda/2}} J_{\lambda+2n}[z[(\alpha+an)(\lambda a-\alpha+an)]^{1/2}]$$
$$\cdot {}_2F_1\left[\begin{matrix}-n, \lambda+n; \\ 1+c;\end{matrix} \quad \frac{x^2}{(\alpha+an)(\lambda a-\alpha+an)}\right] = \frac{\Gamma(c+1)(\frac{1}{2}z)^{\lambda-c}}{x^c(1-\frac{1}{4}a^2z^2} J_c[xz]. \tag{1.5}$$

Note $\lambda a = 2\alpha$ is a case of interest.

Putting $a=0$, $\alpha=i$ in (1.5) gives essentially (1.4). Letting $x=0$, $\alpha=0$, $\lambda=0$, $a=2$ in (1.5) gives a Kapteyn-type result in Watson [12, p. 566]. Letting $x=0$, $\alpha=1$, $\lambda=1$, $a=2$ gives a similar result in Watson [12, p. 567].

Watson [12, p. 283, Eq. 1] gives

$$\frac{z^{\nu}}{t-z} = \sum_{n=0}^{\infty} A_{n,\nu}(t)\, J_{\nu+n}(z), \tag{1.6}$$

where $A_{n,\nu}(t)$ is the Gegenbauer polynomial generalization of Neumann's polynomial. A particular example of Theorem 1(d) is

$$\frac{z^{\lambda}}{(x^{2/s}-z^{2/s})(1-\frac{1}{4}a^2z^2)} = \sum_{n=0}^{\infty} A_{n,\lambda}^{a,\alpha,s}(x)$$
$$\times J_{\lambda+(2n/s)}\left| z \left| \left(\alpha+\frac{an}{s}\right)\left(a\lambda-\alpha+\frac{an}{s}\right) \right|^{1/2} \right|, \tag{1.7}$$

where

$$A_{n,\lambda}^{a,\alpha,s}(x) = \sum_{k=0}^{[n/s]} \frac{\left[\begin{matrix}2^{\lambda}(\lambda+2n/s)\,\Gamma(\lambda+2n/s-k) \\ \times\, [\frac{1}{2}x[(\alpha+an/s)(a\lambda-\alpha+an/s)]^{1/2}]^{2k-(2n/s)}\end{matrix}\right]}{x^{2/s}k!\,[(\alpha+an/s)(a\lambda-\alpha+an/s)]^{\lambda/2}}. \tag{1.8}$$

Putting $a=0$, $s=2$, $\alpha=i$ gives (1.6).

Another special case of interest is

$$\frac{z^{\lambda}}{x^{2/s}-z^{2/s}} = \sum_{n=0}^{\infty} B_{n,\lambda}^{a,\alpha,s}(x)\, J_{\lambda+(2n/s)}\left[z\left[\left(\alpha+\frac{an}{s}\right)\left(a\lambda-\alpha+\frac{an}{s}\right)\right]^{1/2}\right], \tag{1.9}$$

where

$$B_{n,\lambda}^{a,\alpha,s}(x)=\sum_{k=0}^{[n/s]}\frac{\begin{bmatrix}2^{\lambda}(\lambda+2n/s)(\alpha+an/s-ak)\\ \times(a\lambda-\alpha+an/s-ak)\,\Gamma(\lambda+2n/s-k)\end{bmatrix}}{x^{2/s}k!\,[(\alpha+an/s)(a\lambda-\alpha+an/s)]^{\lambda/2+1}}$$

$$\cdot\left[\frac{1}{2}x\left[\left(a\lambda-\alpha+\frac{an}{s}\right)\left(\alpha+\frac{an}{s}\right)\right]^{1/2}\right]^{2k-(2n/s)}.$$

Letting $\alpha=i$, $a=0$, $s=2$ gives (1.6), while putting $s=2$, $a\lambda=2\alpha$ and algebra gives [12, p. 571, Eq. 2].

For the Laguerre polynomial, we extend a result due to [1, p. 538, Eq. (7.4)].

$$\frac{(\frac{1}{2}z)^{\lambda}}{\Gamma(\lambda)(1-\frac{1}{4}a^{2}z^{2})}\exp\left[\frac{z^{2}}{4x}\right]=\sum_{n=0}^{\infty}\frac{(\lambda+2n)(\lambda)_{n}}{[(\alpha+an)(-\alpha+a\lambda+an)]^{\lambda/2}(-x)^{n}}$$
$$\cdot L_{n}^{-\lambda-2n}[x(\alpha+an)(a\lambda-\alpha+an)]\,J_{\lambda+2n}[z[\alpha+an)(-\alpha+a\lambda+an)]^{1/2}]. \tag{1.10}$$

Letting $\alpha=i$, $a=0$ gives the known result. If required, one may also derive the generalization of [1, p. 541, Eq. (7.15)].

Special cases of Theorem 1 also give extensions of the important Feldheim connection coefficients and expansions, both for the Jacobi and ultraspherical polynomials.

For the Gegenbauer polynomial, we have

$$z^{l-sm}C_{m}^{\mu}(xz^{s})=\sum_{n=0}^{[l/2]}f_{n}(x)\,C_{l-2n}^{\nu}[z[(\alpha+an)(-a\nu-al-\alpha+an)]^{-1/2}], \tag{1.11}$$

where

$$f_{n}(x)=\frac{(\nu+l-2n)\,2^{m-l}l!\,(\mu)_{m}}{(\nu)\,n!\,(1+\nu)_{l-n}m!}$$
$$\cdot\sum_{k=0}^{[n/s]}\frac{\begin{bmatrix}x^{m-2k}(\alpha+ask)(a\nu-al-\alpha+ask)(-m)_{2k}(-n)_{sk}\\ \times(\alpha+an)^{l/2-sk-1}(-a\nu-al-\alpha+an)^{l/2-sk-1}\end{bmatrix}}{(-l)_{2sk}(1-\mu-m)_{k}\,k!}$$
$$\cdot\,2^{2sk-2k}(-\nu-l+n)_{sk}$$

and $m\leqslant[l/s]$.

Letting $a=0$, $\alpha=i$, $s=1$, $l=m$ gives [5, p. 474, Eq. (4.12)], and letting $a=0$, $\alpha=i$, $s=1$, $m=0$ gives [5, p. 480, Eq. (5.17), third equation].

For the Jacobi polynomial, we have

$$y^{l-ms}R_m^{(c,d)}(xy^s)=\sum_{n=0}^{l} g_n(x)\, R_{l-n}^{(\mu,\nu)}\left[\frac{y}{(\alpha+an)(a\lambda-\alpha+an)}\right], \tag{1.12}$$

where $R_m^{(c,d)}(x)$ is the shifted Jacobi polynomial [10, p. 434], and

$$g_n(x)=\frac{\begin{bmatrix}(1+c+d)_{2m}(\alpha+an)^{l-1}(a\lambda-\alpha+an)^{l-1}\, l! \\ \times(1+\nu+\mu)_{l-n}(\lambda)_n(-\nu-l)_n(a\alpha\lambda-\alpha^2)\end{bmatrix}}{m!\,(1+c+d)_m\, n!\,(1+\nu+\mu)_{2l-n}(\lambda)_{2n}}$$

$$\cdot\, x^m\, {}_{2s+4}F_{2s+3}\left[\begin{matrix}-m,-d-m,\dfrac{\alpha}{as}+1,\dfrac{a\lambda-\alpha}{as}+1,\Delta(-n,s),\Delta(\lambda+n,s);\\ -c-d-2m,\dfrac{\alpha}{as},\dfrac{a\lambda-\alpha}{as},\Delta(-l,s),\Delta(-\nu-l,s);\end{matrix}\ \frac{1}{x(\alpha+an)^s(a\lambda-\alpha+an)^s}\right],$$

$$\lambda=-\nu-\mu-2l-1,\qquad\text{and}\qquad m\leqslant[l/s],\qquad\text{and}$$

$$\Delta(m,s)=m/s,(m+1)/s,\ldots,(m+s-1)/s.$$

Putting $a=0$, $\alpha=i$, $s=1$, $l=m$ gives [5, p. 472, Eq. (4.6)], and $m=0$, $a=0$, $\alpha=i$, and $s=1$ gives [5, p. 480, Eq. (5.16), third equation].

Note that the expansions given in this section are of special interest for $a\lambda=2\alpha$.

The theorem and its constituent parts are enunciated and proved in the next section.

The method of proof developed in the next section involves differential and integral operators of a particular kind. Other types of operators, which give expansions of a different character, will be presented elsewhere.

II

LEMMA 1. *For b,β real or complex numbers, then for p a non-negative integer*

$$\text{(a)}\quad \sum_{i=0}^{p}\frac{(-1)^{p-i}(b+\beta+i)(b+i)^p(b+2\beta+i)^p}{i!\,(p-i)!\,(2\beta+2b+i)_{p+1}}=\frac{1}{2}, \tag{2.1}$$

$$\text{(b)}\quad \sum_{i=0}^{p} \frac{(-1)^{p-i}(b+\beta+i)(b+i)^{p-1}(b+2\beta+i)^{p-1}}{i!\,(p-i)!\,(2\beta+2b+i)_{p+1}} = \begin{cases} 0 & p \neq 0 \\ \dfrac{1}{(2b)(b+2\beta)} & p = 0. \end{cases} \tag{2.2}$$

Proof of Lemma 1(a).

$$\int_0^1 D[x^\beta (xD)^p [x^b (1-x)^p]]^2\, dx \tag{2.3}$$

$$= 2\int_0^1 x^\beta (xD)^p [x^b(1-x)^p]$$

$$\times \sum_{i=0}^{p} \frac{(-p)_i (b+i)^p (b+\beta+i)\, x^{b+\beta+i-1}}{i!}\, dx \tag{2.4}$$

$$= 2\sum_{i=0}^{p} \frac{(-1)^p(-p)_i(b+\beta+i)(b+i)^p(b+2\beta+i)^p}{i!}$$

$$\times \int_0^1 x^{2b+2\beta+i-1}(1-x)^p\, dx \tag{2.5}$$

$$= 2\sum_{i=0}^{p} \frac{(-1)^p(-p)_i(b+i)^p(b+2\beta+i)^p(b+\beta+i)\,p!\,\Gamma(2b+2\beta+i)}{i!\,\Gamma(2b+2\beta+i+p+1)}. \tag{2.6}$$

Going from (2.3) to (2.4) involves expanding $(1-x)^p$ and operating. Integrating (2.4) p times gives (2.5). The integral in (2.5) is now the beta function, reducing to (2.6). Now integrating (2.3) directly gives

$$p!\,p!. \tag{2.7}$$

Equating (2.6) and (2.7) and simplifying completes the proof.

Proof of Lemma 1(b). The proof involves the consideration of

$$\int_0^1 D[x^\beta (xD)^{p-1}[x^b(1-x)^p]]^2\, dx. \tag{2.8}$$

Proceeding as in Lemma 1(a) and simplifying results in Lemma 1(b).

THEOREM 1(a). *For α, λ, a real or complex numbers, c_k, d_k arbitrary functions of k, then for s a positive integer*

$$\frac{1}{1-a^2 z} \sum_{k=0}^{\infty} (xz^s)^k c_k = \sum_{n=0}^{\infty} (\lambda+2n)\, R_n(x)\, S_n(z), \tag{2.9}$$

where

$$R_n(x) = \sum_{k=0}^{[n/s]} \frac{x^k c_k(\alpha + an)^{n-sk}(\lambda a - \alpha + an)^{n-sk}\, \Gamma(\lambda + n + sk)}{(n-sk)!};$$

$$S_n(z) = \sum_{p=0}^{n} \frac{(-1)^p\, z^{n+p}(\alpha + an)^p(\lambda a - \alpha + an)^p}{p!\, \Gamma(\lambda + 2n + p + 1)}.$$

THEOREM 1(b)

$$\frac{1}{1 - a^2 z^s} \sum_{n=0}^{\infty} \frac{c_n(xz)^n}{n!} = \sum_{n=0}^{\infty} \left(\lambda + \frac{2n}{s}\right) U_n(x)\, V_n(z), \tag{2.10}$$

where

$$U_n(x) = \sum_{k=0}^{[n/s]} \frac{c_{n-sk} x^{n-sk}(\alpha + an/s)^k(a\lambda - \alpha + an/s)^k\, \Gamma(\lambda + 2n/s - k)}{(n-sk)!\, k!}$$

$$V_n(z) = \sum_{p=0}^{\infty} \frac{(-1)^p\, z^{n+ps}(\alpha + an/s)^p(a\lambda - \alpha + an/s)^p}{p!\, \Gamma(\lambda + 2n/s + p + 1)}.$$

THEOREM 1(c)

$$\sum_{k=0}^{\infty} \frac{(xz^s)^k\, c_k d_{sk}}{k!\, (\alpha + ask)(a\lambda - \alpha + ask)} = \sum_{n=0}^{\infty} (\lambda + 2n)\, F_n(x)\, G_n(z), \tag{2.11}$$

where

$$F_n(x) = \sum_{k=0}^{[n/s]} \frac{(-1)^{n-sk} c_k(\alpha + an)^{n-sk-1}(a\lambda - \alpha + an)^{n-sk-1}\, \Gamma(\lambda + n + sk)\, x^k}{(n-sk)!\, k!},$$

$$G_n(z) = \sum_{p=0}^{\infty} \frac{(\alpha + an)^p(a\lambda - \alpha + an)^p\, d_{n+p} z^{n+p}}{p!\, \Gamma(\lambda + 2n + p + 1)}.$$

THEOREM 1(d).

$$\sum_{n=0}^{\infty} \frac{(xz)^n\, c_n d_n}{n!\, (\alpha + an/s)(a\lambda - \alpha + an/s)} = \sum_{n=0}^{\infty} \left(\lambda + \frac{2n}{s}\right) A_n(x)\, B_n(z), \tag{2.12}$$

where

$$A_n(x) = \sum_{k=0}^{[n/s]} \frac{\left[\begin{array}{l}(-1)^k\, c_{n-sk}(\alpha + an/s)^{k-1} \\ \qquad \times\, (a\lambda - \alpha + an/s)^{k-1}\, \Gamma(\lambda + 2n/s - k)\, x^{n-sk}\end{array}\right]}{(n-sk)!\, k!}$$

$$B_n(z) = \sum_{p=0}^{\infty} \frac{d_{n+sp}(\alpha + an/s)^p(a\lambda - \alpha + an/s)^p\, z^{n+sp}}{p!\, \Gamma(\lambda + 2n/s + p + 1)}.$$

THEOREM 1(e).

$$x^p = \frac{p!\,(\alpha + asp)(a\lambda - \alpha + asp)}{cp} \times \sum_{n=0}^{sp} \frac{(\lambda + 2n)(\alpha + an)^{sp-n}(a\lambda - \alpha + an)^{sp-n}}{(sp-n)!\,\Gamma(\lambda + sp + n + 1)} F_n(x) \tag{2.13}$$

THEOREM 1(f).

$$x^n = \frac{n!\,(\alpha + an/s)(a\lambda - \alpha + an/s)}{c_n} \times \sum_{p=0}^{[n/s]} \frac{(\lambda + 2n/s - 2p)(\alpha + an/s - ap)^p(a\lambda - \alpha + an/s - ap)^p}{p!\,\Gamma(\lambda + 2n/s - p + 1)} \times A_{n-sp}(x), \tag{2.14}$$

where $|z\exp(1-z^2)^{1/2}[1 + (1-z^2)^{1/2}]^{-1}| < 1$, *and the left-hand sides of* (2.9) *through* (2.12) *are assumed to be the convergent. The condition given above is sufficient for convergence, but not necessary.*

Proof of Theorem 1(a).

$$\sum_{n=0}^{\infty} (\lambda + 2n) \sum_{k=0}^{[n/s]} \frac{c_k x^k(\alpha + an)^{n-sk}(\lambda a - \alpha + an)^{n-sk}\,\Gamma(\lambda + n + sk)}{(n-sk)!} \cdot \sum_{p=0}^{\infty} \frac{(-1)^p z^{n+p}(\alpha + an)^p(\lambda a - \alpha + an)^p}{p!\,\Gamma(\lambda + 2n + p + 1)} \tag{2.15}$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left[\begin{array}{l}(\lambda + 2sk + 2n)\,c_k x^k(\alpha + ask + an)^{n+p} \\ \quad\times (\lambda a - \alpha + ask + an)^{n+p}\,\Gamma(\lambda + n + 2sk)\end{array}\right]}{n!} \cdot \frac{(-1)^p z^{n+p+sk}}{p!\,\Gamma(\lambda + 2sk + 2n + p + 1)} \tag{2.16}$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{n=0}^{p} \frac{\left[\begin{array}{l}z^{p+sk}(\lambda + 2sk + 2n)\,c_k x^k(\alpha + ask + an)^p \\ \quad\times (\lambda a - \alpha + ask + an)^p\,\Gamma(\lambda + n + 2sk)(-1)^{p-n}\end{array}\right]}{n!\,(p-n)!\,\Gamma(\lambda + 2sk + p + n + 1)} \tag{2.17}$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} (za^2)^p z^{sk} c_k x^k \tag{2.18}$$

$$= \frac{1}{1 - za^2} \sum_{k=0}^{\infty} (xz^s)^k c_k \tag{2.19}$$

which completes the proof. Equation (2.17) reduces to (2.18) by Lemma 1(a).

Proof of Theorem 1(b).

$$\sum_{n=0}^{\infty}\left(\lambda+\frac{2n}{s}\right)\sum_{k=0}^{[n/s]}\frac{c_{n-sk}x^{n-sk}(\alpha+an/s)^k(a\lambda-\alpha+an/s)^k\,\Gamma(\lambda+2n/s-k)}{(n-sk)!\,k!}$$

$$\cdot\sum_{p=0}^{\infty}\frac{(-1)^p z^{n+ps}(\alpha+an/s)^p(a\lambda-\alpha+an/s)^p}{p!\,\Gamma(\lambda+2n/s+p+1)} \tag{2.20}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left[\begin{array}{c}(\lambda+2k+2n/s)\,c_nx^n(\alpha+ak+an/s)^{k+p}\\ \times(a\lambda-\alpha+ak+an/s)^{k+p}\end{array}\right]}{n!\,k!}$$

$$\cdot\frac{\Gamma(\lambda+k+2n/s)(-1)^p z^{n+ks+ps}}{p!\,\Gamma(\lambda+2n/s+2k+p+1)} \tag{2.21}$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{p}$$

$$\times\frac{\left[\begin{array}{c}(\lambda+2k+2n/s)\,c_nx^n(\alpha+ak+an/s)^p\\ \times(a\lambda-\alpha+ak+an/s)^p\,\Gamma(\lambda+k+2n/s)((-1)^{p-k}z^{n+ps}\end{array}\right]}{n!\,k!\,(p-k)!\,\Gamma(\lambda+k+2n/s+p+1)} \tag{2.22}$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{c_nx^nz^{n+ps}a^{2p}}{n!} \tag{2.23}$$

$$=\frac{1}{1-a^2z^s}\sum_{n=0}^{\infty}\frac{c_n(xz)^n}{n!} \tag{2.24}$$

and Theorem 1(b) is proved. Lemma 1(a) is employed in going from (2.22) to (2.23).

Proof of Theorem 1(c).

$$\sum_{n=0}^{\infty}(\lambda+2n)\sum_{k=0}^{[n/s]}\frac{\left[\begin{array}{c}c_k(-1)^{n-sk}(\alpha+an)^{n-sk-1}\\ \times(a\lambda-\alpha+an)^{n-sk-1}\,\Gamma(\lambda+n+sk)\,x^k\end{array}\right]}{(n-sk)!\,k!}$$

$$\cdot\sum_{p=0}^{\infty}\frac{z^{n+p}(\alpha+an)^p(a\lambda-\alpha+an)^p\,d_{n+p}}{p!\,\Gamma(\lambda+2n+p+1)} \tag{2.25}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left[\begin{array}{c}(\lambda+2sk+2n)\,c_k(-1)^n(\alpha+ask+an)^{n+p-1}\\ \times(a\lambda-\alpha+ask+an)^{n+p-1}\end{array}\right]}{n!\,k!}$$

$$\cdot \frac{\Gamma(\lambda + n + 2sk)\, x^k z^{n+sk+p} d_{n+p+sk}}{p!\, \Gamma(\lambda + 2n + 2sk + p + 1)} \tag{2.26}$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{z^{p+sk} c_k x^k d_{p+sk}}{k!}$$

$$\times \sum_{n=0}^{p} \frac{\begin{bmatrix} (\lambda + 2sk + 2n)(-1)^n \\ \times (\alpha + ask + an)^{p-1} (a\lambda - \alpha + ask + an)^{p-1} \end{bmatrix}}{n!\, (p-n)!}$$

$$\cdot \frac{\Gamma(\lambda + n + 2sk)}{\Gamma(\lambda + n + 2sk + p + 1)}. \tag{2.27}$$

The summations over n and p reduce with the aid of Lemma 1(b) to give Theorem 1(c).

Proof of Theorem 1(d).

$$\sum_{n=0}^{\infty} \left(\lambda + \frac{2n}{s}\right) \sum_{k=0}^{[n/s]} \frac{\begin{bmatrix} (-1)^k x^{n-sk} c_{n-sk} (\alpha + an/s)^{k-1} \\ \times (a\lambda - \alpha + an/s)^{k-1}\, \Gamma(\lambda + 2n/s - k) \end{bmatrix}}{(n-sk)!\, k!}$$

$$\cdot \sum_{p=0}^{\infty} \frac{z^{n+sp} d_{n+sp} (\alpha + an/s)^p (a\lambda - \alpha + an/s)^p}{p!\, \Gamma(\lambda + 2n/s + p + 1)} \tag{2.28}$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{\begin{bmatrix} (\lambda + 2n/s + 2k)(-1)^k x^n c_n (\alpha + ak + an/s)^{p+k-1} \\ \times (a\lambda - \alpha + ak + an/s)^{p+k-1}\, \Gamma(\lambda + 2n/s + k) \end{bmatrix}}{n!\, k!}$$

$$\cdot \frac{z^{n+sk+sp} d_{n+sk+sp}}{p!\, \Gamma(\lambda + 2n/s + p + 2k + 1)} \tag{2.29}$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{\begin{bmatrix} ((\lambda + 2n/s + 2k)(-1)^k x^n c_n (\alpha + ak + an/s)^{p-1} \\ \times (a\lambda - \alpha + ak + an/s)^{p-1}\, \Gamma(\lambda + 2n/s + k) \end{bmatrix}}{n!\, k!}$$

$$\cdot \frac{z^{n+sp} d_{n+sp}}{(p-k)!\, \Gamma(\lambda + 2n/s + p + k + 1)} \tag{2.30}$$

Equation (2.30) reduces to Theorem 1(d) using lemma 1(b).

Proof of Theorem 1(e). Consider Theorem 1(c), and compare coefficients of powers of z.

Proof of Theorem 1(f). Consider Theorem 1(d), and compare coefficients of powers of z.

Putting $a=0$, $\alpha=i$, $s=l=1$ in Theorem 1(c, d) gives [6]. See also [10, Sect. 11.3.6.2, p. 446; 3, Theorem 5].

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