Structural Operators and Eigenmanifold Decomposition for Functional Differential Equations in Hilbert Spaces

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Submitted by G. F. Webb
Received January 13, 1994

This paper studies spectral properties of linear retarded functional differential equations in Hilbert spaces with the emphasis on their relations to structural operators. The equations involve unbounded operators acting on the discrete and distributed delayed terms, and the operators acting on the instantaneous term are defined through sesquilinear forms. The main concern of this paper is studying the spectral properties of the infinitesimal generators associated with the solution semigroups by means of structural operators. The characterizations of eigenmanifolds are derived and the relations between the manifolds and structural operators are shown by using the properties of structural operators.

1. INTRODUCTION

In this paper we continue the study of structural properties for functional differential equations (FDE's) in a Hilbert space, which is discussed in Jeong et al. [7]. The structural study developed by Manitius and co-workers [1, 4] for FDE's in $\mathbb{R}^n$ (see also the references in [7, 9, 10]) is extended to abstract FDE's by Nakagiri [10], Tanabe [14], Kunisch and Matišek [9], and Jeong [5, 6]. In [10] the structural study has been done extensively, but operators acting on the delayed part of equation are...
bounded. On the other hand, the works [5, 6, 9, 14] and those of Di Blasio et al. [2, 3] and Yong and Pan [17] intend to study the equation which has unbounded operators acting on the retarded part of the equation; however, the structural study seems to be insufficient. We want to fill the gap by extending the results in [10] to the equations studied in [7].

In [7], we study the basic state space theory for the following functional differential equation \((E)\) on a Hilbert space \(H\):

\[
\frac{du(t)}{dt} = A_0 u(t) + A_1 u(t - h) + \int_{-h}^{0} a(s) A_2 u(t + s) \, ds + f(t)
\]

a.e. \(t \geq 0\) \hspace{1cm} (1.1)

\[
u(0) = g^0, \quad u(s) = g^1(s) \text{ a.e. } s \in [-h,0), \hspace{1cm} (1.2)
\]

and its transposed equation \((E^T)\):

\[
\frac{dv(t)}{dt} = A_0^* v(t) + A_1^* v(t - h) + \int_{-h}^{0} a(s) A_2^* v(t + s) \, ds + h(t)
\]

a.e. \(t \geq 0\) \hspace{1cm} (1.3)

\[
v(0) = \varphi^0, \quad v(s) = \varphi^1(s) \text{ a.e. } s \in [-h,0), \hspace{1cm} (1.4)
\]

where the operators \(A_0, A_1, A_2\) and their adjoint operators \(A_0^*, A_1^*, A_2^*\) acting on instantaneous and delayed terms are all unbounded. Assuming that \(V\) is a Hilbert space, dense in \(H\), and the embeddings \(V \subset H \subset H^*\) are continuous, we study equations \((E)\) and \((E^T)\) on the product state space \(M_2 = H \times L^2(-h,0;V)\) provided that \(A_1, A_2 \in \mathcal{S}(V,V^*)\) and \(A_0\) is defined through the sesquilinear form on \(V \times V\).

The equation \((E)\) covers the following typical initial-boundary value problem for a parabolic partial functional differential equation. Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary \(\partial \Omega\). We set \(H = L^2(\Omega)\) and \(V = H^1_0(\Omega)\). Let \(a(u,v)\) be the sesquilinear form in \(H^1_0(\Omega) \times H^1_0(\Omega)\) defined by

\[
a(u,v) = \int_{\Omega} \left\{ \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) uv \right\} \, dx,
\]

\[x \in \Omega. \hspace{1cm} (1.5)
\]

Here in (1.5) we assume that the real valued coefficients \(a_{ij}, b_i, c\) satisfy

\[
a_{ij} \in C^1(\overline{\Omega}), b_i \in C^1(\overline{\Omega}), c \in L^n(\Omega),
\]

\[
a_{ij} = a_{ji} \hspace{1cm} (1 \leq i, j \leq n), \text{ and the uniform ellipticity}
\]

\[
\sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \geq \nu |y|^2, \hspace{1cm} y = (y_1, \ldots, y_n) \in \mathbb{R}^n
\]
for some positive $\nu$. As is well known (see e.g. Tanabe [12, Chap. 2]) this sesquilinear form is bounded and the operator $A_0: H^2_0(\Omega) \to H^{-1}(\Omega)$ defined through (1.5) has the following realization in $L^2(\Omega)$. Let

$$\mathcal{A}_0 = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x), \quad x \in \Omega \quad (1.6)$$

be the associated uniformly elliptic differential operator of second order. The realization of $-\mathcal{A}_0$ in $L^2(\Omega)$ under the Dirichlet boundary condition is exactly $A_0$, i.e., $\mathcal{D}(A_0) = H^2(\Omega) \cap H^1_0(\Omega)$ and $A_0 u = -\mathcal{A}_0 u$ for $u \in \mathcal{D}(A_0)$. Next, let $A_i$, $i = 1, 2$, be the restriction to $H^1_0(\Omega)$ of the second order differential operator $A_i$, $i = 1, 2$, given by

$$\mathcal{A}_i = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + e_i(x), \quad x \in \Omega, \quad (1.7)$$

where

$$a_{ij} = a_{ji} \in C^1(\overline{\Omega}), \quad b_i \in C^1(\overline{\Omega}), \quad e_i \in L^\infty(\Omega).$$

Each $A_i: H^2_0(\Omega) \to H^{-1}(\Omega)$ is bounded without the ellipticity condition. The kernel function $q(s)$ is assumed to be an element of $L^2(-\nu, 0)$. The following system of a parabolic partial functional differential equation and initial-boundary conditions is covered by (E).

$$\frac{\partial u(t, x)}{\partial t} = \mathcal{A}_0 u(t, x) + \mathcal{A}_2 u(t - h, x) + \int_{-h}^0 a(s) \mathcal{A}_2 u(t + s, x) \, ds + f(t, x), \quad 0 < t < \infty, \quad x \in \Omega, \quad (1.8)$$

$$u|_{t = 0} = 0, \quad 0 < t < \infty, \quad (1.9)$$

$$u(0, x) = g^0(x), \quad u(s, x) = g^1(s, x) \quad \text{a.e. } s \in [-h, 0), \quad x \in \Omega. \quad (1.10)$$

Here in (1.8) and (1.10) we suppose that $f \in L^2_{\text{loc}}(\mathbb{R}^+; H^{-1}(\Omega))$ and $g^0 \in L^2(\Omega)$, $g^1 \in L^2(-h, 0; H^1_0(\Omega))$. For the equations (1.8)--(1.10) in interpolation spaces we refer the reader to Jeong [6].

The objective of this paper is to develop the spectral theory for the equations (E) and (E_\tau) on the state space $M_2$ and the adjoint space.
$M_2^* = H \times L^2(-h, 0; V^*)$, which extends that of [10] for the equations with bounded operators acting on delayed terms. In our spectral theory the structural operators play an important role as shown in [4] and [10].

We briefly explain the content of this paper. Let $S(t)$ and $S_T(t)$ be the $C_0$-semigroups on $M_2$ associated with $(E)$ and $(E_T)$, respectively. We denote by $A$ and $A_T$ the infinitesimal generator of $S(t)$ and $S_T(t)$, respectively. In Section 2 we summarize some preliminary results for $S(t)$, $S_T(t)$ and structural operators $F \in \mathcal{L}(M_2^*, M_2^*)$, $G \in \mathcal{L}(M_2^*, M_2)$ given in [7] which will be used in this paper. In Section 3 we introduce various spectral operators and give the representations of resolvent operators $(\lambda - A)^{-1}, (\lambda - A_T)^{-1}, (\lambda - A^*)^{-1}, (\lambda - A_T^*)^{-1}$, where $A^*$, $A_T^*$ denote the adjoint operators of $A, A_T$, respectively. In the representations the structural operator $F$ and the characteristic operator $\Delta(\lambda) = \lambda - A_0 - e^{-\lambda h}A_1 - \int_{\mathbb{R}} e^{\lambda s}d\lambda$ appear as important cofactors. The characterizations of spectrum and resolvent sets for $A, A^*, A_T, A_T^*$ by means of $\Delta(\lambda)$ and $\Delta_T(\lambda) = \Delta(\bar{\lambda})^*$ are also given in Section 3. Using the representation formulae of these resolvents, we develop a fundamental spectral decomposition theory in Section 4 following the line of [10]. Let $\lambda$ be an eigenvalue of $A$, i.e., $\lambda \in \sigma_p(A)$. First we establish the characterization of null spaces $\ker(\lambda - A)^l$ ($l = 1, 2, \ldots$) in terms of $\Delta(\lambda)$ and its derivatives. For a pole $\lambda$ of $(\mu - A)^{-1}$ of order $k$, we show that the space $M_2$ can be decomposed as the direct sum of the generalized eigenmanifold $\mathcal{M}_\lambda = \ker(\lambda - A)^{k_\lambda}$ and its complementary space $\im(\lambda - A)^{k_\lambda}$. The other theme of Section 4 is to study the adjoint spectral decomposition theory and indicate the role of structural operators $F$ and $G$. The main results of this section are the facts that $\mathcal{M}_\lambda^* = F^*\mathcal{M}_\lambda^*$ for each $\lambda \in \sigma_p(A_T) = \sigma_p(A^*)$ and that $G^*\mathcal{M}_\lambda^* = \mathcal{M}_\lambda^*$ for each $\lambda \in \sigma_p(A_T)$ satisfying dim $\mathcal{M}_\lambda^* < \infty$, where $\mathcal{M}_\lambda^* \subset M_2^*$ and $\mathcal{M}_\lambda^* \subset M_2$ denotes the generalized eigenmanifold of $A^*$ and $A_T$, respectively. Further, we give the $M_2$-adjoint result for $\mathcal{M}_\lambda$, $\lambda$ being a pole of order $k_\lambda$, that is, we give the relation $\mathcal{M}_\lambda^* = \ker(\bar{\lambda} - A^*)^{k_\lambda} = P_{\bar{\lambda}}^* M_2^*$, where $\bar{\lambda}$ denotes the complex conjugate of $\lambda$ and $P_{\bar{\lambda}}^*$ is the spectral projection of $A^*$ for $\bar{\lambda}$. Using the relation we establish a useful expression of the spectral projection $P_\lambda$ in terms of generalized eigenvectors of $A, A_T$ and the structural operator $F$.

2. SEMIGROUPS AND STRUCTURAL OPERATORS

In this section we state the results on the semigroups associated with the functional differential equation $(E)$ and its transposed equation $(E_T)$.

First we give exact descriptions of the equation $(E)$. Let $H$ and $V$ be complex Hilbert spaces such that $V$ is dense in $H$ and the inclusion map $i: V \to H$ is continuous. The norms of $H, V$ and the inner product of $H$
are denoted by $|\cdot|$, $\|\cdot\|$, and $\langle \cdot, \cdot \rangle$, respectively. By identifying the antidual of $H$ with $H$ we may consider $V \subset H \subset V^*$. Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding’s inequality

$$\Re a(u, v) \geq c_0 \|u\|^2 - c_1 |u|^2,$$  \hfill (2.1)

where $c_0 > 0$ and $c_1 \geq 0$ are real constants. Let $A_0$ be the operator associated with this sesquilinear form

$$\langle v, A_0 u \rangle = -a(u, v), \quad u, v \in V,$$  \hfill (2.2)

where $\langle \cdot, \cdot \rangle$ denotes also the duality pairing between $V$ and $V^*$. The operator $A_0$ is a bounded linear form from $V$ into $V^*$. The realization of $A_0$ in $H$, which is the restriction of $A_0$ to the domain $\mathcal{D}(A_0) = \{ u \in V: A_0 u \in H \}$, is also denoted by $A_0$. It is known (cf. Tanabe [12, Chap. 3]) that $A_0$ generates an analytic semigroup $e^{tA_0}$ both in $H$ and $V^*$ and that $T(t): V^* \to V$ for each $t > 0$. It is assumed that each $A_i$ ($i = 1, 2$) is bounded and linear from $V$ to $V^*$ (i.e., $A_i \in \mathcal{L}(V, V^*)$) such that $A_i$ maps $\mathcal{D}(A_0)$ endowed with the graph norm of $A_0$ to $H$ continuously. The real valued scalar function $a(s)$ is assumed to be $L^2$-integrable on $[-h, 0]$, that is $a(s) \in L^2(-h, 0)$.

The operator valued Stieltjes measure $\eta$ is defined by

$$\eta(s) = -\chi_{(-\infty, -h]}(s) A_1 - \int_{s}^{0} a(\xi) \, d\xi A_2: V \to V^*, \quad s \in [-h, 0],$$  \hfill (2.3)

where $\chi_{(-\infty, -h]}$ denotes the characteristic function of $(-\infty, -h]$. Then the delayed terms in (1.1) are written simply by $\int_{s}^{a} d\eta(s) u(t + s)$. A function $u \in L^2_{\text{loc}}(\mathbb{R}^+; V) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^+; V^*)$ is said to be a solution of $(E)$ if $u(t)$ satisfies (1.1) and (1.2).

The existence and uniqueness result and a variation of constants formula of solutions in terms of the fundamental solution are established in Jeong et al. [7].

**Theorem 1.** Let $f \in L^2_{\text{loc}}(\mathbb{R}^+; V^*)$ and $g = (g^0, g^1) \in H \times L^2(-h, 0; V)$. Then there exists a unique solution $u(t) = u(t; f, g)$ of $(E)$ satisfying

$$u \in L^2_{\text{loc}}(-h, \infty; V) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^+; V^*) \subset C([0, \infty); H).$$  \hfill (2.4)

Under the additional condition that $a(s)$ is Hölder continuous on $[-h, 0]$,

$$a(s)$$
Tanabe [13, 15] has constructed the fundamental solution $W(t)$ of (E) as the solution of the following integral equation with delay

$$
W(t) = \begin{cases} 
T(t) + \int_0^t T(t-s) \int_{-h}^0 d\eta(\xi) W(\xi + s) \, ds, & t \geq 0 \\
0 & t < 0,
\end{cases} \tag{2.6}
$$

where $O$ denotes the null operator.

This fundamental solution $W(t)$ is strongly continuous both in $H$ and $V^*$ and $W(t): V^* \to V$ for each $t > 0$.

Let $M_2 = H \times L^2(-h, 0; V^*)$ be the state space of the equation (E). $M_2$ is a product Hilbert space with the norm

$$
\|g\|_{M_2} = \left( |g^0|^2 + \int_{-h}^0 \|g^1(s)\|^2 \, ds \right)^{1/2}, \quad g = (g^0, g^1) \in M_2.
$$

Let $g \in M_2$ and $u(t; g)$ be the solution of (E) with $f = 0$. The segment $u_t$ is given by $u_t(s; g) = u(t + s; g), s \in [-h, 0]$. The solution semigroup $S(t)$ associated with (E) is defined by

$$
S(t) g = (u(t; g), u_t(\cdot; g)), \quad t \geq 0, \quad g \in M_2. \tag{2.7}
$$

Then $S(t)$ is a $C_0$-semigroup on $M_2$. Let $A$ be the infinitesimal generator of $S(t)$. The characterization of $A$ is given by the following theorem.

**Theorem 2.** The operator $A$ is given by

$$
\mathcal{D}(A) = \left\{ g = (g^0, g^1) : g^1 \in W^{1,2}(-h, 0; V), \right. \\
\left. g^1(0) = g^0 \in V, \quad A_0 g^0 + \int_{-h}^0 d\eta(s) g^1(s) \, ds \in H \right\}, \tag{2.8}
$$

$$
Ag = \left( A_0 g^0 + \int_{-h}^0 d\eta(s) g^1(s) \, ds, \dot{g}^1 \right) \quad \text{for } g = (g^0, g^1) \in \mathcal{D}(A). \tag{2.9}
$$

Next we consider the adjoint semigroup $S^a(t)$ on the adjoint space of $M_2$. The adjoint space $M_2^a$ of $M_2$ can be identified with the product space $H \times L^2(-h, 0; V^*)$ via the duality pairing (we use the same bracket as that of inner product)

$$
\langle g, f \rangle_{M_2} = \langle g^0, f^0 \rangle + \int_{-h}^0 \langle g^1(s), f^1(s) \rangle \, ds,
$$

$$
g = (g^0, g^1) \in M_2, \quad f = (f^0, f^1) \in M_2^a. \tag{2.10}
$$
By identifying the second dual $V^{**}$ of $V$ with itself, the second adjoint space $M_2^*$ is identified with $M_2$. That is, $M_2$ is reflexive in this identification. Hence, as is well known, the adjoint semigroup $S^*(t)$ is strongly continuous in $M_2^*$. The infinitesimal generator of $S^*(t)$ is given by the adjoint $A^*$ and is characterized precisely by the following theorem. For relevant results for abstract functional differential equations we refer the reader to Webb [16], Kunisch and Mastinsek [9], and Nakagiri [10].

**Theorem 3.** The infinitesimal generator $A^*$ of $S^*(t)$ is given by

$$\mathcal{D}(A^*) = \{ f = (f^0, f^1) : f^0 \in V, f^1 \in W^{1,2}(-h, 0; V^*) \},$$

$$A^* f^0 + f^1(0) \in H, f^1(-h) = A^*_0 f^0, \quad (2.11)$$

$$A^* f = (A^*_0 f^0 + f^1(0), a(\cdot) A^*_2 f^0 - \hat{f}^1(\cdot)) \quad \text{for } f = (f^0, f^1) \in \mathcal{D}(A^*). \quad (2.12)$$

Next we study the transposed equation $E_\tau$ and their associated semigroups. In (1.3) and (1.4) it is assumed that $A^*_j$ denote the adjoint operator of $A_j (j = 0, 1, 2)$, $(\varphi^0, \varphi^1) \in M_2$, and $h \in L^2_{loc}(\mathbb{R}^+; V^*)$. The adjoint operator $A^*_0$ generates an analytic semigroup $T^*(t)$, which is the adjoint of $T(t)$, both in $H$ and $V^*$, and that $T^*(t) : V^* \to V$ for each $t > 0$. It is evident that $A^*_0 \in \mathcal{D}(V, V^*) (\tau = 0, 1, 2)$. Then the solution $v(t)$ of $E_\tau$ exists uniquely by Theorem 1. Consequently we can construct the fundamental solution $W_\tau(t)$ of $E_\tau$, which is strongly continuous both in $H$ and $V^*$ and $W_\tau(t) : V^* \to V$, $t > 0$.

We denote the adjoint of $W(t)$ by $W^*(t)$. The following useful relation $W_\tau(t) = W^*(t)$, $t \in \mathbb{R}$, is proved in [7, Lemma 4].

Let us denote by $(S^*_\tau(t))_{t \geq 0}$ the $C_0$-semigroup on $M_2$ corresponding to the transposed equation $E_\tau$. We denote by $(S^*_{\tau}^*(t))_{t \geq 0}$ the adjoint $C_0$-semigroup on the adjoint space $M_2^*$. Then we have the following result.

**Theorem 4.** (i) The infinitesimal generator $A_T$ of $S_T(t)$ is given by

$$\mathcal{D}(A_T) = \{ g = (g^0, g^1) : g^1 \in W^{1,2}(-h, 0; V),$$

$$g^1(0) = g^0 \in V, \quad A^*_0 g^0 + \int_{-h}^0 d\eta^*(s) g^1(s) ds \in H \}, \quad (2.13)$$

$$A_T g = \left( A^*_0 g^0 + \int_{-h}^0 d\eta^*(s) g^1(s) ds, g^1 \right) \quad \text{for } g = (g^0, g^1) \in \mathcal{D}(A_T), \quad (2.14)$$

where $\eta^*(s) = -\chi_{(-h, -\alpha)}(s) A^*_1 - \int_{-h}^0 a(\xi) d\xi A^*_2$, $s \in [-h, 0]$. 

By identifying the second dual $V^{***}$ of $V$ with itself, the second adjoint space $M_2^*$ is identified with $M_2$. That is, $M_2$ is reflexive in this identification. Hence, as is well known, the adjoint semigroup $S^*(t)$ is strongly continuous in $M_2^*$. The infinitesimal generator of $S^*(t)$ is given by the adjoint $A^*$ and is characterized precisely by the following theorem. For relevant results for abstract functional differential equations we refer the reader to Webb [16], Kunisch and Mastinsek [9], and Nakagiri [10].

**Theorem 3.** The infinitesimal generator $A^*$ of $S^*(t)$ is given by

$$\mathcal{D}(A^*) = \{ f = (f^0, f^1) : f^0 \in V, f^1 \in W^{1,2}(-h, 0; V^*) ,$$

$$A^*_0 f^0 + f^1(0) \in H, f^1(-h) = A^*_0 f^0, \quad (2.11)$$

$$A^* f = (A^*_0 f^0 + f^1(0), a(\cdot) A^*_2 f^0 - \hat{f}^1(\cdot)) \quad \text{for } f = (f^0, f^1) \in \mathcal{D}(A^*). \quad (2.12)$$

Next we study the transposed equation $E_\tau$ and their associated semigroups. In (1.3) and (1.4) it is assumed that $A^*_j$ denote the adjoint operator of $A_j (j = 0, 1, 2)$, $(\varphi^0, \varphi^1) \in M_2$, and $h \in L^2_{loc}(\mathbb{R}^+; V^*)$. The adjoint operator $A^*_0$ generates an analytic semigroup $T^*(t)$, which is the adjoint of $T(t)$, both in $H$ and $V^*$, and that $T^*(t) : V^* \to V$ for each $t > 0$. It is evident that $A^*_0 \in \mathcal{D}(V, V^*) (\tau = 0, 1, 2)$. Then the solution $v(t)$ of $E_\tau$ exists uniquely by Theorem 1. Consequently we can construct the fundamental solution $W_\tau(t)$ of $E_\tau$, which is strongly continuous both in $H$ and $V^*$ and $W_\tau(t) : V^* \to V$, $t > 0$.

We denote the adjoint of $W(t)$ by $W^*(t)$. The following useful relation $W_\tau(t) = W^*(t)$, $t \in \mathbb{R}$, is proved in [7, Lemma 4].

Let us denote by $(S_\tau^*(t))_{t \geq 0}$ the $C_0$-semigroup on $M_2$ corresponding to the transposed equation $E_\tau$. We denote by $(S_\tau^*_{\tau}^*(t))_{t \geq 0}$ the adjoint $C_0$-semigroup on the adjoint space $M_2^*$. Then we have the following result.

**Theorem 4.** (i) The infinitesimal generator $A_T$ of $S_T(t)$ is given by

$$\mathcal{D}(A_T) = \{ g = (g^0, g^1) : g^1 \in W^{1,2}(-h, 0; V),$$

$$g^1(0) = g^0 \in V, \quad A^*_0 g^0 + \int_{-h}^0 d\eta^*(s) g^1(s) ds \in H \}, \quad (2.13)$$

$$A_T g = \left( A^*_0 g^0 + \int_{-h}^0 d\eta^*(s) g^1(s) ds, g^1 \right) \quad \text{for } g = (g^0, g^1) \in \mathcal{D}(A_T), \quad (2.14)$$

where $\eta^*(s) = -\chi_{(-h, -\alpha)}(s) A^*_1 - \int_{-h}^0 a(\xi) d\xi A^*_2$, $s \in [-h, 0]$. 

(ii) The infinitesimal generator $A_t^\#$ of $S_t^\#(t)$ is given by

$$\mathcal{D}(A_t^\#) = \{ f = (f^0, f^1) : f^0 \in V, f^1 \in W^{1,2}(-h, 0; V^*) \},$$

$$A_t^\#f^0 + f^1(0) \in H, \quad f^1(-h) = A_1f^0 \}.$$  \hspace{1cm} (2.15)

\[
A_t^\#f = (A_0f^0 + f^1(0), a(\cdot)A_2f^0 - f^1(\cdot)) \quad \text{for } f = (f^0, f^1) \in \mathcal{D}(A_t^\#). \hspace{1cm} (2.16)
\]

The first structural operator $F : M_2 \to M_2^\#$ is defined by $F = \left( \begin{array}{c} f^0 \\ v^1 \end{array} \right)$ i.e.,

$$[Fg]^0 = g^0, \quad [Fg]^1 = F_1g^1, \quad \text{for } g = (g^0, g^1) \in M_2,$$  \hspace{1cm} (2.17)

where $F_1 : L_2(-h, 0; V) \to L_2(-h, 0; V^*)$ is given by

$$[F_1g^1](s) = \int_{-h}^{s} d\eta(\xi) g^1(\xi - s) \quad \text{a.e. } s \in [-h, 0].$$  \hspace{1cm} (2.18)

The operator $F$ is into and bounded.

The second structural operator $G : M_2^\# \to M_2$ is defined by

$$\left\{ \begin{array}{c}
[Gf]^1(s) = W(h + s) f^0 + \int_{-h}^{0} W(h + s + \xi) f^1(\xi) d\xi \\
[Gf]^0 = [Gf]^1(0), \quad f = (f^0, f^1) \in M_2^\#.
\end{array} \right.$$  \hspace{1cm} (2.19)

$G$ is also into and bounded.

**Proposition 1.** (i) The adjoint $F^* : M_2 \to M_2^\#$ of $F$ is given by

$$[F^*g]^0 = g^0, \quad [F^*g]^1 = F_1^*g^1, \quad \text{for } g = (g^0, g^1) \in M_2,$$  \hspace{1cm} (2.20)

where $F_1^* : L_2(-h, 0; V) \to L_2(-h, 0; V^*)$ denotes the adjoint of $F_1$ and is represented by

$$[F_1^*g^1](s) = \int_{-h}^{s} d\eta^*(\xi) g^1(\xi - s) \quad \text{a.e. } s \in [-h, 0].$$  \hspace{1cm} (2.21)
The adjoint $G^*$: $M_2^* \to M_2$ of $G$ is given by
\[
\begin{align*}
[G^*f]^1(s) &= W_T(h + s)f^0 + \int_{-h}^{0} W_T(h + s + \xi)f^1(\xi)\,d\xi \\
[G^*f]^0 &= [G^*f]^1(0),
\end{align*}
\]
for $s \in [-h, 0]$, \hspace{1cm} (2.22)

These $G$ and $G^*$ satisfy
\[
\begin{align*}
\ker G &= \ker G^* = \{0\}; \quad \text{Cl}(\text{Im} G) = \text{Cl}(\text{Im} G^*) = M_2. \hspace{1cm} (2.23)
\end{align*}
\]

Now we can state the crucial interconnected properties between the structural operators $F, G$ and the semigroups given above.

**Theorem 5.** (i) The following decompositions hold true:
\[
\begin{align*}
S(h) &= GF, \quad S_T(h) = G^*F^*; \\
S^*(h) &= F^*G^*, \quad S_T^*(h) = FG.
\end{align*}
\]

(ii) The following relations on $F$ hold true:
\[
\begin{align*}
FS(t) &= S_T^*(t)F, \quad S^*(t)F^* = F^*S_T(t), \quad t \geq 0. \hspace{1cm} (2.26) \\
F(\mathcal{D}(A)) &= \mathcal{D}(A^*_T) \text{ and } FA = A^*_T F \quad \text{on} \quad \mathcal{D}(A). \hspace{1cm} (2.27) \\
F^*(\mathcal{D}(A_T)) &= \mathcal{D}(A^*_T) \text{ and } A^*F^* = F^*A_T \quad \text{on} \quad \mathcal{D}(A_T). \hspace{1cm} (2.28)
\end{align*}
\]

(iii) The following relations on $G$ hold true:
\[
\begin{align*}
S(t)G &= GS^*_T(t), \quad G^*S^*(t) = S_T(t)G^*, \quad t \geq 0. \hspace{1cm} (2.29) \\
G(\mathcal{D}(A^*_T)) &= \mathcal{D}(A) \text{ and } AG = GA^*_T \quad \text{on} \quad \mathcal{D}(A^*_T). \hspace{1cm} (2.30) \\
G^*(\mathcal{D}(A^*_T)) &= \mathcal{D}(A_T) \text{ and } G^*A^* = A_TG^* \quad \text{on} \quad \mathcal{D}(A^*_T). \hspace{1cm} (2.31)
\end{align*}
\]

3. RESOLVENT AND SPECTRUM

This section is devoted to give exact forms of the resolvent operators of generators associated with FDE's and to study their spectrum by means of the characteristic operators. Following Delfour and Manitius [4] and Nakagiri [10], for each $\lambda \in \mathbb{C}$, we introduce the following linear operators

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\[ E_\lambda: V \to M_2, \quad T_\lambda: M_2 \to M_2, \quad K_\lambda: M_2^* \to M_2^*, \quad \text{and} \quad H_\lambda: M_2^* \to V^* \] by

\[ E_\lambda u = (u, e^{\lambda s}u) \quad \text{for} \quad u \in V, \quad (3.1) \]

\[ T_\lambda g = \left( 0, \int_s^0 e^{\lambda(s-\xi)}g^1(\xi) \, d\xi \right) \quad \text{for} \quad g = (g^0, g^1) \in M_2, \quad (3.2) \]

\[ H_\lambda f = f^0 + \int_{-h}^0 e^{\lambda s}f^1(s) \, ds \quad \text{for} \quad f = (f^0, f^1) \in M_2^*, \quad (3.3) \]

\[ K_\lambda f = \left( 0, \int_{-h}^s e^{\lambda(s-\xi)}f^1(\xi) \, d\xi \right) \quad \text{for} \quad f = (f^0, f^1) \in M_2^*, \quad (3.4) \]

respectively. We shall call these operators the spectral operators.

Also we introduce the characteristic operator \( D_\lambda \) to study the relations between this and \( \lambda - A \). For each \( \lambda \in \mathbb{C} \), we define the operator \( \Delta(\lambda) \) by

\[ \Delta(\lambda) = \lambda - A_0 - \int_{-h}^0 e^{\lambda s} \eta(s) : V \to V^*. \quad (3.5) \]

By considering \( \Delta(\lambda) \) as an operator in \( \mathcal{L}(V, V^*) \), we define the following resolvent set and spectrum for \( \Delta(\lambda) \).

\[ \rho(\Delta) = \{ \lambda: \Delta(\lambda) \text{ is bijective} \}, \quad \sigma(\Delta) = \mathbb{C} \setminus \rho(\Delta). \quad (3.6) \]

For the spectrum \( \sigma(\Delta) \), we divide it into the following three disjoint subsets:

\[ \sigma_c(\Delta) = \{ \lambda: \Delta(\lambda) \text{ is injective, } \text{Im} \Delta(\lambda) \text{ is dense in } V^*, \]
\[ \quad \text{and } \Delta(\lambda)^{-1} \text{ is unbounded on } V^* \}, \quad (3.7) \]

\[ \sigma_p(\Delta) = \{ \lambda: \Delta(\lambda) \text{ is injective and } \text{Im} \Delta(\lambda) \text{ is not dense in } V^* \}, \quad (3.8) \]

\[ \sigma_f(\Delta) = \{ \lambda: \Delta(\lambda) \text{ is not injective} \}. \quad (3.9) \]

Compare our definitions with those given in [3] and [17], where \( \Delta(\lambda) \) is considered as an unbounded operator on \( H \).

It is easy to see, by open mapping theorem, that for each \( \lambda \in \rho(\Delta) \), \( \Delta(\lambda)^{-1} \) exists and belongs to \( \mathcal{L}(V^*, V) \).

**Lemma 1.** The set \( \rho(\Delta) \) contains a right half plane and is open in \( \mathbb{C} \). The inverse \( \Delta(\lambda)^{-1} \) is analytic on \( \rho(\Delta) \).
Proof. Since
\[
\Delta(\lambda) = (\lambda - A_0) \left( I - (\lambda - A_0)^{-1} \int_{-h}^{0} e^{\lambda s} \, d\eta(s) \right)
\]  
(3.10)
and, by Tanabe [12, Chap. 3], there exists a \( k_0 > 0 \) such that
\[
\left\| (\lambda - A_0)^{-1} \int_{-h}^{0} e^{\lambda s} \, d\eta(s) \right\|_{\mathcal{L}(V)} \\
\leq \left\| (\lambda - A_0)^{-1} \right\|_{\mathcal{L}(V)} \\
\times \left( e^{(\Re \lambda)h} \| A_1 \|_{\mathcal{L}(V', V)} + \int_{-h}^{0} e^{(-\Re \lambda)s} \| a(s) \|_{\mathcal{L}(V', V')} \right) \\
< 1 \quad \text{for } \Re \lambda \geq k_0,
\]  
(3.11)
then \( \Delta(\lambda) \) has a bounded inverse \( \Delta(\lambda)^{-1} : V^* \to V \) on \( \{ \lambda : \Re \lambda \geq k_0 \} \). Let \( \lambda \in \rho(\Delta) \) and \( \mu \in \mathbb{C} \). The openness of \( \rho(\Delta) \) and the analyticity of \( \Delta(\lambda)^{-1} \) on \( \rho(\Delta) \) follows from the equation
\[
\Delta(\mu) = \Delta(\lambda) \left( I - \Delta(\lambda)^{-1} \left( (\lambda - \mu) I - \int_{-h}^{0} (e^{\lambda s} - e^{\mu s}) \, d\eta(s) \right) \right)
\]  
(3.12)
and that
\[
\left\| \Delta(\lambda)^{-1} \right\|_{\mathcal{L}(V', V')} < +\infty, \lim_{\mu \to \lambda} \left\| \int_{-h}^{0} (e^{\lambda s} - e^{\mu s}) \, d\eta(s) \right\|_{\mathcal{L}(V', V')} = 0.
\]

Theorem 6. If \( \lambda \in \rho(\Delta) \), then \( \lambda \in \rho(A) \) and the resolvent \( (\lambda - A)^{-1} \) of \( A \) is represented by
\[
(\lambda - A)^{-1} = E_{\lambda} \Delta(\lambda)^{-1} H_F + T_{\lambda}.
\]  
(3.13)

Proof. Let \( \varphi = (\varphi^0, \varphi^1) \in M_2 \) and \( g = (g(0), g(\cdot)) \in \mathcal{D}(A) \). We consider the resolvent equation \( (\lambda - A)g = \varphi \). In view of Theorem 2, this
equation is equivalent to
\[
\begin{align*}
\lambda g(0) - A_0 g(0) - \int_{-h}^0 d\eta(s) g(s) &= \varphi^0 \in H, \quad g(0) \in V \\
\lambda g(s) - \frac{d}{ds} g(s) &= \varphi^1(s) \in V, \quad \text{a.e. } s \in [-h, 0].
\end{align*}
\] (3.14)

By solving the differential equation and using definitions of spectral operators, as shown in [10, Theorem 6.1], we see that (3.14) is equivalent to
\[
\begin{align*}
\Delta(\lambda) g(0) &= H_\lambda F \varphi \in H, \quad g(0) \in V \\
( g(0), g(\cdot) ) &= E_\lambda g(0) + T_\lambda \varphi \in M_2.
\end{align*}
\] (3.15)

Since \( \lambda \in \rho(\Delta) \), there exists a bounded inverse \( \Delta(\lambda)^{-1} : V^* \to V \). Then by (3.15), we have
\[
( g(0), g ) = E_\lambda \Delta(\lambda)^{-1} H_\lambda F \varphi + T_\lambda \varphi. \] (3.16)

The operators \( F : M_2 \to M_2^* \), \( H_\lambda : M_2^* \to V^* \), \( \Delta(\lambda)^{-1} : V^* \to V \), \( E_\lambda : V \to M_2 \), and \( T_\lambda : M_2 \to M_2 \) are all bounded, and so the operator defined by the right-hand side of (3.16) is a bounded operator in \( M_2 \). This proves that \( \lambda \) belongs to the resolvent set \( \rho(A) \) and the resolvent \( (\lambda - A)^{-1} \) is given by (3.13).

Since \( S(t) \) is a \( C_0 \)-semigroup on \( M_2 \), there exist \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that
\[
\| S(t) g \|_{M_2} \leq Me^{\omega t} \| g \|_{M_2}, \quad g \in M_2. \] (3.17)

Let \( \Pi_0 \) be the projection of \( M_2 \) onto \( H \). Then by (2.7) and (3.17),
\[
|W(t)g^0| = |\Pi_0 S(t)(g^0, 0)| \leq \| S(t)(g^0, 0) \|_{M_2} \leq Me^{\omega t} |g^0|, \quad g^0 \in H. \] (3.18)

This shows that \( W(t) \) is Laplace transformable in \( \mathcal{L}(H) \). The Laplace transform of \( W(t) \) is given by the inverse \( \Delta(\lambda)^{-1} \). That is, we have the following

**Corollary 1.** If \( \lambda \in \rho(\Delta) \) satisfies \( \Re \lambda \geq \omega \), then the inverse \( \Delta(\lambda)^{-1} \) is given by
\[
\Delta(\lambda)^{-1} = \int_0^\infty e^{-\lambda t} W(t) \, dt. \] (3.19)
Proof. Since $\text{Re} \lambda \geq \omega$, we see $\lambda \in \rho(A)$ and for $g^0 \in H$

$$\int_{0}^{\infty} e^{-\lambda t}W(t)g^0 \, dt = \Pi \int_{0}^{\infty} e^{-\lambda t}S(t)(g^0, 0) \, dt = \Pi \delta(\lambda - A)^{-1}(g^0, 0).$$

(3.20)

By virtue of Theorem 6, it is easily verified that for $\lambda \in \rho(\Delta)$,

$$(\lambda - A)^{-1}(g^0, 0) = (\Delta(\lambda)^{-1}g^0, e^{\lambda t}\Delta(\lambda)^{-1}g^0).$$

(3.21)

Combining (3.20) and (3.21) we obtain the assertion (3.19).

Next we shall give the representation of the resolvent $(\lambda - A^*)^{-1}$. To give the convenient form we introduce the transposed characteristic operator $\Delta_T(\lambda)$. Let $A_0^*, A_1^*, A_2^*; V \rightarrow V^*$ be adjoint operators of $A_0, A_1, A_2$, respectively. We assume also that each $A_i^*$ $(i = 1, 2)$ maps $\mathcal{D}(A_0^*)$ endowed with the graph norm of $A_0^*$ to $H$ continuously. The operator $\Delta_T(\lambda)$ is defined by

$$\Delta_T(\lambda) = \lambda - A_0^* - \int_{-h}^{0} e^{\lambda s} d\eta^*(s); V \rightarrow V^*, \quad \lambda \in \mathbb{C},$$

(3.22)

where

$$\eta^*(s) = -\chi(-s, -h)A_1^* - \int_{s}^{0} a(\xi) \, d\xi A_2^*; V \rightarrow V^*, \quad s \in [-h, 0].$$

(3.23)

The resolvent set $\rho(\Delta_T)$ and the three types of spectrum $\sigma_c(\Delta_T), \sigma_p(\Delta_T), \sigma_p(\Delta_T)$ for the operator $\Delta_T(\lambda)$ are defined similarly as for $\Delta(\lambda)$. By Corollary 1 and Kato [8, p. 169] we have the following result.

**Corollary 2.** (i) $\lambda \in \rho(\Delta)$ if and only if $\lambda$ (complex conjugate) $\in \rho(\Delta_T)$ and

$$(\Delta(\lambda)^{-1})^* = \Delta_T(\lambda)^{-1}.\quad (3.24)$$

(ii) The set $\rho(\Delta_T)$ contains a right half plane and

$$\Delta_T(\lambda)^{-1} = \int_{0}^{\infty} e^{-\lambda t}W_T(t) \, dt \quad (3.25)$$

for sufficiently large $\text{Re} \lambda$. 
Let $\phi = (\phi^0, \phi^1) \in M_2^g$ and $f = (f^0, f^1) \in \mathcal{D}(A^g)$. Similarly to [10, Lemma 6.1], in view of Theorem 3 we can verify that $(\lambda - A^g)f = \phi$ is equivalent to

$$
\begin{align*}
\Delta_T(\lambda)f^0 &= H_\lambda \phi \in H, \\
 f &= K_\lambda \phi + F^* E_\lambda f^0 \in M_2^g.
\end{align*}
$$

(3.26)

Hence the following theorem follows immediately from (3.26).

**Theorem 7.** If $\lambda \in \rho(\Delta_T)$, then $\lambda \in \rho(A^g)$ and the resolvent $(\lambda - A^g)^{-1}$ is represented by

$$
(\lambda - A^g)^{-1} = F^* E_\lambda \Delta_T(\lambda)^{-1} H_\lambda + K_\lambda.
$$

(3.27)

For the resolvents $(\lambda - A_T^g)^{-1}$ and $(\lambda - A_T^g)^{-1}$, we have similar representations.

**Theorem 8.** (i) If $\lambda \in \rho(\Delta_T)$, then $\lambda \in \rho(A_T)$ and $(\lambda - A_T)^{-1}$ is expressed by

$$
(\lambda - A_T)^{-1} = E_\lambda \Delta_T(\lambda)^{-1} H_\lambda F^* + T_\lambda.
$$

(3.28)

(ii) If $\lambda \in \rho(\Delta)$, then $\lambda \in \rho(A_T^g)$ and $(\lambda - A_T^g)^{-1}$ is expressed by

$$
(\lambda - A_T^g)^{-1} = FE_\lambda \Delta(\lambda)^{-1} H_\lambda + K_\lambda.
$$

(3.29)

For the operator valued entire functions $E_\lambda$, $\Delta(\lambda)$, and others, we denote by $E_\lambda^{(k)}$, $\Delta^{(k)}(\lambda)$, and so on the $k$th derivatives in $\lambda$. By repeating the similar calculations in [10, Prop. 6.1], we can establish the following equalities.

**Lemma 2.** For each $\lambda \in \mathbb{C}$ and $k = 0, 1, 2, \ldots,$

$$
\begin{align*}
E_\lambda^g &= H_\lambda \text{ in } \mathcal{L}(M_2^g, V^g), & H_\lambda^g &= E_\lambda \text{ in } \mathcal{L}(V, M_2), \\
T_\lambda^g &= K_\lambda \text{ in } \mathcal{L}(M_2^g), & K_\lambda^* &= T_\lambda \text{ in } \mathcal{L}(M_2), \\
F T_\lambda^k &= K_\lambda^k F, & F^* T_\lambda^k &= K_\lambda^k F^* \text{ in } \mathcal{L}(M_2, M_2^g), \\
(F E_\lambda)^{(k)} &= F E_\lambda^{(k)}, & (F^* E_\lambda)^{(k)} &= F^* E_\lambda^{(k)} \text{ in } \mathcal{L}(V, M_2^g).
\end{align*}
$$

(3.30)

(3.31)

(3.32)

(3.33)
Next we study the spectra of $A, A^*, A_T, A_T^*$ by means of the characteristic operators $\Delta(\lambda)$ and $\Delta_T(\lambda)$.

**Lemma 3.** For each $\lambda \in \mathbb{C}$,

\[
\ker(\lambda - A) = \{0\} \iff \ker \Delta(\lambda) = \{0\};
\]

\[
\ker(\lambda - A^*) = \{0\} \iff \ker \Delta_T(\lambda) = \{0\};
\]

\[
\ker(\lambda - A_T) = \{0\} \iff \ker \Delta_T(\lambda) = \{0\};
\]

\[
\ker(\lambda - A_T^*) = \{0\} \iff \ker \Delta(\lambda) = \{0\}.
\]

**Proof.** By substituting $\varphi = 0$ in (3.15), we see that $(\lambda - A)g = 0$ if and only if $\Delta(\lambda)g^0 = 0$, $g^0 \in V$, and $g = E_s g^0$. As is easily checked this implies the equivalence (3.36). Similarly, by (3.26), we have that $(\lambda - A^*)f = 0$ if and only if $\Delta_T(\lambda)f^0 = 0$, $f^0 \in V$, and $f = F^* E_s f^0$. From this the second equivalence (3.37) follows readily. Now the rest, (3.38) and (3.39), are obvious.

As usual we denote by $\sigma(A)$ the spectrum of $A$, and by $\sigma_p(A), \sigma_c(A), \sigma_r(A)$ the point, continuous, and residual spectrum of $A$, respectively. We can now state the following result on the relationship between three kinds of spectrum for $A, A^*, A_T, A_T^*$ and those corresponding sets defined for $\Delta(\lambda)$ and $\Delta_T(\lambda)$, which gives a slightly sharpened version of [3, Theorem 3.9] on the part of residual spectrum.

**Theorem 9.** The following relations hold:

\[
\rho(\Delta) \subset \left\{ \rho(A), \rho(A^*), \rho(A_T^*) \right\} \subset \rho(\Delta_T) \cup \sigma_c(\Delta);
\]

\[
\rho(\Delta_T) \subset \left\{ \rho(A^*), \rho(A_T^*) \right\} \subset \rho(\Delta_T) \cup \sigma_c(\Delta_T);
\]

\[
\sigma_p(\Delta) = \sigma_p(A) = \sigma_p(A_T^*), \quad \sigma_p(\Delta_T) = \sigma_p(A^*) = \sigma_p(A_T);
\]
\( \sigma_R(\Delta) = \sigma_R(A) = \sigma_R(A^*_r), \quad \sigma_R(\Delta_T) = \sigma_R(A^*_r) = \sigma_R(A_T); \quad (3.43) \)

\[
\begin{align*}
&\left\{ \sigma_c(A) \right\} \subseteq \sigma_c(\Delta) \subseteq \left\{ \rho(A) \cup \sigma_c(A) \right\}; \\
&\left\{ \sigma_c(A^*_r) \right\} \subseteq \sigma_c(\Delta_T) \subseteq \left\{ \rho(A^*_r) \cup \sigma_c(A^*_r) \right\}.
\end{align*}
\]  

(3.44)

(3.45)

**Proof.** The inclusions \( \rho(\Delta) \subset \rho(A), \rho(\Delta_T) \subset \rho(A^*_r), \rho(\Delta) \subset \rho(A^*_r), \rho(\Delta_T) \subset \rho(A_T) \) follow from Theorems 6–8. Let \( \lambda \in \rho(A) \). Then \( (\lambda - A) \mathcal{D}(A) = M_2 \) and this implies, by substituting \( \varphi = (\varphi^0, 0) \) in (3.15), that for any \( \varphi^0 \in H \) there exists a \( g^0 \in V \) such that \( \Delta(\lambda)g^0 = \varphi^0 \). This shows \( \text{Im} \Delta(\lambda) \supset H \), and then \( \text{Im} \Delta(\lambda) \) is dense in \( V^* \). Since \( \Delta(\lambda) \) is injective by Lemma 3, \( \Delta(\lambda)^{-1} \) exists. If \( \Delta(\lambda)^{-1} \) is bounded on \( V^* \), we have \( \text{Im} \Delta(\lambda) = V^* \), so that \( \lambda \in \rho(\Delta) \). If \( \Delta(\lambda)^{-1} \) is unbounded, then \( \lambda \in \sigma_c(\Delta) \) by (3.7).

Hence \( \rho(A) \subset \rho(\Delta) \cup \sigma_c(\Delta) \) is proved. The rest inclusions in (3.40) and (3.41) are shown similarly. The equalities in (3.42) are direct consequences from Lemma 3. Since \( M_2 \) is reflexive, we can apply the duality theorem and use (3.37) to obtain that

\[
\text{Cl}(\text{Im}(\lambda - A)) = M_2 \iff \ker(\lambda - A^*_r) = \{0\}
\]

\[
\iff \ker \Delta_T(\lambda) = \ker \Delta(\lambda)^* = \{0\}
\]

\[
\iff \text{Cl}(\text{Im} \Delta(\lambda)) = V^*.
\]

Therefore we have, by contradiction, that the image \( \text{Im}(\lambda - A) \) is not dense in \( M_2 \) if and only if \( \text{Im} \Delta(\lambda) \) is not dense in \( V^* \). Combining this and (3.36) we have by (3.8) \( \sigma_R(\Delta) = \sigma_R(A) \). Similarly, by the equivalences

\[
\text{Cl}(\text{Im}(\lambda - A_T^*_r)) = M_2^* \iff \ker(\lambda - A_T) = \{0\}
\]

\[
\iff \ker \Delta_T(\lambda) = \{0\}
\]

\[
\iff \text{Cl}(\text{Im} \Delta(\lambda)) = V^*.
\]

we have \( \sigma_R(\Delta) = \sigma_R(A_T^*_r) \). This shows the first part of (3.43). The second part of (3.43) is shown analogously. The inclusions (3.44), (3.45) are consequences from (3.40)–(3.43).

To illustrate Theorem 9 we consider the following special case of practical interest.

\[ A_1 = \gamma A_0, \quad A_2 = A_0 \quad \text{and} \quad \gamma \text{ is a real constant.} \quad (3.46) \]

In this case we have

\[
\Delta(\lambda) = \lambda - m(\lambda) A_0, \quad \Delta_T(\lambda) = \lambda - m(\lambda) A_0^*,
\]

where \( m(\lambda) = 1 + \gamma e^{-\lambda h} + \int_0^h e^{\lambda s} a(s) \, ds \).
Now let
\[ \Xi = \{ \lambda : m(\lambda) \neq 0, \lambda/m(\lambda) \in \rho(A_0) \}, \]
\[ \Sigma_C = \{ \lambda : m(\lambda) \neq 0, \lambda/m(\lambda) \in \sigma_C(A_0) \}, \]
\[ \Sigma_R = \{ \lambda : m(\lambda) \neq 0, \lambda/m(\lambda) \in \sigma_R(A_0) \}, \]
\[ \Sigma_p = \{ \lambda : m(\lambda) \neq 0, \lambda/m(\lambda) \in \sigma_p(A_0) \}, \]
\[ \Sigma_0 = \{ \lambda : \lambda \neq 0, m(\lambda) = 0 \}. \]

We denote by \( \Xi^*, \Sigma_C^*, \Sigma_R^*, \Sigma_p^*, \Sigma_0^* \) the corresponding sets given above in which \( A_0 \) is replaced by \( A_U^* \). Then we have the following proposition.

**Proposition 2.** Let \( A \) and \( A_U \) satisfy (3.46). Then the following relations hold:

\[ \Xi \subset \rho(A) \subset \Xi \cup \Sigma_C, \quad \Xi \subset \rho(A_U^*) \subset \Xi \cup \Sigma_C; \quad (3.47) \]
\[ \Xi^* \subset \rho(A^*) \subset \Xi^* \cup \Sigma_C^*, \quad \Xi^* \subset \rho(A_T^*) \subset \Xi^* \cup \Sigma_C^*; \quad (3.48) \]
\[ \sigma_p(A) = \sigma_p(A_T^*) = \begin{cases} \Sigma_p & \text{if } 1 + \gamma + \int_{-h}^{0} a(s) ds \neq 0 \\ \Sigma_p \cup \{0\} & \text{if } 1 + \gamma + \int_{-h}^{0} a(s) ds = 0 \end{cases}; \quad (3.49) \]
\[ \sigma_p(A^*) = \sigma_p(A_T^*) = \begin{cases} \Sigma_p^* & \text{if } 1 + \gamma + \int_{-h}^{0} a(s) ds \neq 0 \\ \Sigma_p^* \cup \{0\} & \text{if } 1 + \gamma + \int_{-h}^{0} a(s) ds = 0 \end{cases}; \quad (3.50) \]
\[ \sigma_R(A) = \sigma_R(A_T^*) = \Sigma_R, \quad \sigma_R(A^*) = \sigma_R(A_T^*) = \Sigma_R^*; \quad (3.51) \]
\[ \Sigma_0 \subset \sigma_C(A) \subset \Sigma_0 \cup \Sigma_C, \quad \Sigma_0 \subset \sigma_C(A_T^*) \subset \Sigma_0 \cup \Sigma_C; \quad (3.52) \]
\[ \Sigma_0^* \subset \sigma_C(A^*) \subset \Sigma_0^* \cup \Sigma_C^*, \quad \Sigma_0^* \subset \sigma_C(A_T^*) \subset \Sigma_0^* \cup \Sigma_C^*. \quad (3.53) \]

Further, if the inclusion map \( i : V \to H \) is compact, then
\[ \rho(A) = \rho(A_T^*) = \Xi, \quad \rho(A^*) = \rho(A_T^*) = \Xi^*; \quad (3.54) \]
\[ \sigma_p(A) = \sigma_p(A_T^*) = \begin{cases} \Sigma_p & \text{if } 1 + \gamma + \int_{-h}^{0} a(s) ds \neq 0 \\ \Sigma_p \cup \{0\} & \text{if } 1 + \gamma + \int_{-h}^{0} a(s) ds = 0 \end{cases}; \quad (3.55) \]
\[ \sigma_p(A^*) = \sigma_p(A_T) = \begin{cases} \Sigma^+ & \text{if } 1 + \gamma + \int_{-h}^{0} a(s) \, ds \neq 0 \\ \Sigma^+ \cup \{0\} & \text{if } 1 + \gamma + \int_{-h}^{0} a(s) \, ds = 0 \end{cases}; \quad (3.56) \]

\[ \sigma_R(A) = \sigma_R(A_T^*) = \sigma_R(A^*) = \sigma_R(A_T) = \emptyset; \quad (3.57) \]

\[ \sigma_C(A) = \sigma_C(A_T^*) = \Sigma_0, \quad \sigma_C(A^*) = \sigma_C(A_T) = \Sigma_0^* \quad (3.58) \]

**Proof.** Using Theorem 9 and repeating the same argument in Di Blasio et al. [3, Lemma 4.1], we have the first part (3.47)–(3.53) of this proposition. By Riesz–Schauder theory, the compactness of the imbedding of \( V \) to \( H \) implies that \( \sigma(A_0) \) consists entirely of point spectra and \( \sigma(A) \) is given by \( (\lambda; m(\lambda) = 0, \lambda/m(\lambda) \in \sigma(A_0)) \). Hence the remainder part of proposition follows from the first part.

### 4. EIGENMANIFOLD DECOMPOSITION

In this section we apply the results of previous sections to develop the spectral theory for the functional differential equations (E) and (E_T) with the emphasis of structural operators.

Let \( \lambda \) be an eigenvalue of \( A \). The generalized eigenmanifold \( \mathcal{M}_\lambda \) corresponding to \( \lambda \in \sigma_p(A) \) is defined by

\[ \mathcal{M}_\lambda = \bigcup_{l=1}^{\infty} \ker(\lambda - A)^l. \quad (4.1) \]

In order to characterize the structure of \( \ker(\lambda - A)^l \) in terms of \( \Delta(\lambda) \) we introduce the operator valued matrix \( \mathcal{A}_l \); \( (V \times \cdots \times V)^l = V^l \rightarrow (V^* \times \cdots \times V^*)^l = V^{*l} \) by

\[ \mathcal{A}_l = \begin{bmatrix} \Delta(\lambda) & \Delta'(\lambda) & \cdots & 1 & \frac{1}{(l-1)!} \Delta^{(l-1)}(\lambda) \\ 0 & \Delta(\lambda) & \cdots & \frac{1}{(l-2)!} \Delta^{(l-2)}(\lambda) \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \Delta(\lambda) \\ 0 & 0 & \cdots & \Delta(\lambda) \end{bmatrix}. \quad (4.2) \]

Using Lemma 2 we can derive the following important characterization of \( \ker(\lambda - A)^l \), which extends that of Nakagiri [10, Prop. 7.2] to our
equations involving unbounded operators acting on the delayed terms. For corresponding result in Euclidean spaces we refer the reader to Delfour and Manitius [4, Prop. 4.3].

**Theorem 10.** Let \( \lambda \in \sigma_p(A) \). Then for each \( l = 1, 2, \ldots \),

\[
\ker(\lambda - A)^l = \left\{ \varphi \in M_2 : \varphi = \sum_{j=0}^{l-1} \frac{1}{j!} (E_{\lambda} \gamma_{l+1})^{(j)}, \gamma_1, \ldots, \gamma_l' = 0 \text{ in } V^{*l} \right\},
\]

\[\text{dim } \ker(\lambda - A)^l = \text{dim } \ker \mathcal{A}_l. \quad (4.3)\]

**Proof.** Let \( \phi \in \ker(\lambda - A)^l \). We set \( \phi_0 = \phi \) and \( \phi_j = (\lambda - A)\phi_{j-1} \) \((j = 1, 2, \ldots, l)\). It is easy to see that \( \phi \in \ker(\lambda - A)^l \) if and only if \( \phi_0 = 0 \). The equation \( \phi_j = (\lambda - A)\phi_{j-1} \) is, in view of Theorem 2, equivalent to

\[
\begin{cases}
\Delta(\lambda) \phi_{j-1}(0) = H_e F \phi_j \in H, & \phi_{j-1}(0) \in V \\
\phi_{j-1} = E_{\lambda} \phi_{j-2}(0) + T_{\lambda} \phi_j \in M_2.
\end{cases} \quad (4.5)
\]

Since \( \phi_l = 0 \), it follows from (4.5) and (3.34) that

\[
\begin{align*}
\phi_{l-1} &= E_{\lambda} \phi_{l-2}(0), \\
\phi_{l-2} &= E_{\lambda} \phi_{l-3}(0) + T_{\lambda} \phi_{l-1} \\
&= E_{\lambda} \phi_{l-2}(0) + T_{\lambda} E_{\lambda} \phi_{l-1}(0), \ldots, \\
\phi_l &= E_{\lambda} \phi_0, \\
\phi &= \phi_0 = \sum_{j=0}^{l-1} T_{\lambda}^j E_{\lambda} \phi_0 \phi_0 = \sum_{j=0}^{l-1} \frac{1}{j!} (E_{\lambda} (-1)^j \phi_0)^{(j)}.
\end{align*} \quad (4.6, 4.7)
\]
Putting $\gamma_{j+1} = (-1)^j \phi_j(0) \in V, j = 0, 1, \ldots, l - 1$, we have by (4.5), (4.6), and (3.35) that

$$\Delta(\lambda) \gamma_j = (-1)^{l-1} H \gamma_j = (-1) \left( \sum_{r=j}^{l-1} (-1)^{r-j} H_r FT^{r-j}_\lambda E_{\lambda} \gamma_{r+1} \right)$$

$$= (-1) \left( \sum_{r=j}^{l-1} \frac{1}{(r-j+1)!} \Delta^{r-j+1}(\lambda) \gamma_{r+1} \right),$$

so that

$$\sum_{r=j}^{l-1} \frac{1}{(r-j)!} \Delta^{r-j}(\lambda) \gamma_r = 0, \quad j = 1, \ldots, l$$

by changing $r \to r + 1$. This proves that $(\gamma_1, \ldots, \gamma_l)$ satisfies $\mathcal{A}(\gamma_1, \ldots, \gamma_l) = 0$ in $V$, and $\phi = \sum_{j=0}^{l-1} (1/j!) (E_j \gamma_j)^{(j)}$ by (4.7). Hence the proof of (4.3) is completed. The equality (4.4) is now obvious.

**Corollary 3.** For $\lambda \in \sigma_p(A)$,

$$\ker(\lambda - A) = \{ E_j \varphi^0 : \Delta(\lambda) \varphi^0 = 0 \}.$$  \hspace{1cm} (4.8)

Let $\lambda$ be an isolated singular point of $(\mu - A)^{-1}$. The spectral projection $P_\lambda$ and the operator $Q_\lambda$ for $\lambda$ are defined respectively by

$$P_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - A)^{-1} d\mu$$  \hspace{1cm} (4.9)

$$Q_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - \lambda)(\mu - A)^{-1} d\mu,$$  \hspace{1cm} (4.10)

where $\Gamma_\lambda$ is a small circle with center $\lambda$ such that its interior and $\Gamma_\lambda$ contains no points of $\sigma(A)$ except $\lambda$. By Yosida [18, p. 229] we have the following direct sum decomposition of the space $M_2$.

**Theorem 11.** If $\lambda$ is a pole of $(\mu - A)^{-1}$ of order $k_\lambda$, then $\lambda \in \sigma_p(A)$ and the following decomposition holds:

$$\mathcal{M}_\lambda = \ker(\lambda - A)^{k_\lambda}, \quad M_2 = \mathcal{M}_\lambda \oplus \text{im}(\lambda - A)^{k_\lambda}. \hspace{1cm} (4.11)$$

Both $\mathcal{M}_\lambda$ and $\text{im}(\lambda - A)^{k_\lambda} = \ker P_\lambda$ are closed and invariant under the semigroup $S(t)$. Further, the resolvent $(\mu - A)^{-1}$ has the Laurent series expansion

$$(\mu - A)^{-1} = \sum_{n=-k_\lambda}^{\infty} (\mu - \lambda)^n Q_n \hspace{1cm} (4.12)$$
in a neighborhood of \( \lambda \), where \( Q_n \) is given by

\[
Q_n = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{(\mu - \lambda)^{n-1}(\mu - A)^{-1}} \, d\mu
\]  

(4.13)

and

\[
P_\lambda = Q_{-1}, \quad Q_\lambda = Q_{-2}, \quad Q_{-n} = Q_{-n+1}^{-1}, \quad n = 2, \ldots, k, \quad Q^{-k}_\lambda = 0,
\]

(4.14)

\[
P_\lambda Q_\lambda = Q_\lambda P_\lambda = Q_\lambda, \quad AP_\lambda = \lambda P_\lambda + Q_\lambda.
\]

(4.15)

In Theorem 11 we remark that the dimension of \( \text{Ker}(\lambda - A)^l \) may be infinite even if \( \lambda \) is a pole of \( (\mu - A)^{-1} \).

Next we consider the transposed operator \( A^T \). Let \( A_\lambda^T \) denote the generalized eigenspace of \( A^T \) corresponding to \( \lambda \in \sigma_p(A^T); \) let the matrices \( \mathcal{A}_\lambda^T = \mathcal{A}_\lambda^T(\lambda) \) \((l = 1, 2, \ldots)\) be defined by (4.2) in which \( \Delta(\lambda) \) is replaced by \( \Delta_T(\lambda) \); and let \( P_\lambda^T, Q_\lambda^T \), denote the spectral projection and the operator corresponding to an isolated singular point \( \lambda \) of \( (\mu - A_T)^{-1} \).

Then we have:

**Theorem 12.** (i) For \( \lambda \in \sigma_p(A^T) \) and each \( l = 1, 2, \ldots, \)

\[
\text{Ker}(\lambda - A^T)^l = \left\{ \varphi \in M_2 : \varphi = \sum_{j=0}^{l-1} \frac{1}{j!} (E_{\lambda} \gamma_{l+1}^T)^j \right\},
\]

where \( \mathcal{A}_\lambda^T(\gamma_1^T, \ldots, \gamma_l^T)^T = 0 \) in \( V^{l+1} \),

\[
\dim \text{Ker}(\lambda - A^T)^l = \dim \text{Ker} \mathcal{A}_\lambda^T.
\]

(4.16)

(4.17)

(ii) If \( \lambda \) is a pole of \( (\mu - A_T)^{-1} \) of order \( m_\lambda \), then \( \lambda \in \sigma_p(A^T) \) and the following decomposition holds:

\[
\mathcal{A}_\lambda^T = \text{Ker}(\lambda - A^T)^{m_\lambda}, \quad M_2 = \mathcal{A}_\lambda^T \oplus \text{Im}(\lambda - A^T)^{m_\lambda}.
\]

(4.18)

Both \( \mathcal{A}_\lambda^T \) and \( \text{Im}(\lambda - A^T)^{m_\lambda} = \text{Ker} P_\lambda^T \) are closed and invariant under \( S_T(t) \). Further, the resolvent \( (\mu - A_T)^{-1} \) has the Laurent series expansion

\[
(\mu - A_T)^{-1} = \sum_{n=-m_\lambda}^{\infty} (\mu - \lambda)^n Q_n^T
\]

(4.19)

in a neighborhood of \( \lambda \), where \( Q_n^T \) is given by (4.13) in which \( (\mu - A)^{-1} \) is replaced by \( (\mu - A_T)^{-1} \).
The following proposition is useful in applications to control theory involving FDE's (cf. Jeong [5, 6], Nakagiri and Yamamoto [11]).

**Proposition 3.** (i) For $\lambda \in \sigma_p(A)$, $F$ is injective on $\mathcal{M}$.

(ii) For $\lambda \in \sigma_p(A_T)$, $F^*$ is injective on $\mathcal{M}^T$.

**Proof.** It is sufficient to prove (i). Assume first that $\varphi \in \text{Ker}(\lambda - A)$ and $F\varphi = 0$. Then by (2.24), $0 = GF\varphi = S(h)\varphi = e^{kh}\varphi$ and hence $\varphi = 0$. This proves $F$ is injective on $\text{Ker}(\lambda - A)$. Next let $\varphi \in \mathcal{M}$ and $F\varphi = 0$. Then there exists an $l$ such that $\varphi \in \text{Ker}(\lambda - A)^l$. If we set $\varphi_1 = (\lambda - A)^{l-1}\varphi$, then $\varphi_1 \in \text{Ker}(\lambda - A)$ and

$$S(h)\varphi_1 = (\lambda - A)^{l-1}S(h)\varphi = (\lambda - A)^{l-1}GF\varphi = 0.$$ 

Hence $\varphi_1 = 0$. Continuing this procedure $l$ times we reach $\varphi = 0$, i.e., $F$ is injective on $\mathcal{M}$.

Next we investigate the structure of generalized eigenspaces of the adjoint operators $A^*$ and $A_T^*$. Let us denote the generalized eigenspace of $T \in A$ (resp. $A_T$) corresponding to $\lambda \in \sigma_p(A^*)$ (resp. $\lambda \in \sigma_p(A_T^*)$) by $\mathcal{M}^*$ (resp. $\mathcal{M}^T$).

**Theorem 13.** (i) For each $\lambda \in \sigma_p(\Delta_T) = \sigma_p(A^*) = \sigma_p(A_T^*)$, $\text{Ker}(\lambda - A^*)^l = F^* \text{Ker}(\lambda - A_T)^l$, $l = 1, 2, \ldots$  \hspace{1cm} (4.20)

$$\text{dim Ker}(\lambda - A^*)^l = \text{dim Ker}(\lambda - A_T)^l = \text{dim Ker} \mathcal{M}^T.$$ \hspace{1cm} (4.21)

In particular

$$\mathcal{M}^* = \bigcup_{l=0}^{\infty} \text{Ker}(\lambda - A^*)^l = F^* \mathcal{M}^T. \hspace{1cm} (4.22)$$

(ii) For each $\lambda \in \sigma_p(\Delta) = \sigma_p(A_T^*) = \sigma_p(A)$, $\text{Ker}(\lambda - A_T^*)^l = F \text{Ker}(\lambda - A)^l$, $l = 1, 2, \ldots$  \hspace{1cm} (4.23)

$$\text{dim Ker}(\lambda - A_T^*)^l = \text{dim Ker}(\lambda - A)^l = \text{dim Ker} \mathcal{M}_T.$$ \hspace{1cm} (4.24)

In particular,

$$\mathcal{M}_T^* = \bigcup_{l=0}^{\infty} \text{Ker}(\lambda - A_T^*)^l = F \mathcal{M}_T. \hspace{1cm} (4.25)$$

**Proof.** Since the proof of part (ii) is quite similar, we shall prove only part (i). By (2.28) and using induction, we have

$$(\lambda - A^*)^lF^* = F^*(\lambda - A_T)^l \text{ on } \mathcal{D}(A_T^l)$$ \hspace{1cm} (4.26)
for each \( l = 1, 2, \ldots \). This implies
\[
F^* \ker(\lambda - A_T)^{l} \subset \ker(\lambda - A^*)^{l}. \tag{4.27}
\]

Now we shall show the reverse inclusion of (4.27). Let \( \psi \in \ker(\lambda - A^*)^{l} \) and set \( \psi_0 = \psi, \psi_j = (\lambda - A^*)\psi_{j-1} \) \((j = 1, 2, \ldots, l)\). Then \( \psi \in \ker(\lambda - A^*)^{l} \) if and only if \( \psi_l = 0 \). It is verified, by (3.26), that the equality \( \psi_j = (\lambda - A^*)\psi_{j-1} \) is equivalent to
\[
\Delta_T(\lambda)\psi_{j-1}^0 = H_A\psi_j, \psi_j^0 \in V \quad \text{and} \quad \psi_{j-1} = K_A\psi_j + F^*E_A\psi_{j-1}^0. \tag{4.28}
\]

Since \( \psi_j = 0 \), by substituting this into the last equality in (4.28) with \( j = l \) we have \( \psi_{l-1} = F^*E_A\psi_{l-1}^0 \), so that again by (4.28) and (3.32)
\[
\psi_{l-2} = F^*E_A\psi_{l-2}^0 + K_AF^*E_A\psi_{l-2}^0 = F^*(E_A\psi_{l-2}^0 + T_AE_A\psi_{l-1}^0).
\]

Continuing this procedure, we reach \( \psi = \psi_0 = F^*(\sum_{j=0}^{l-1}T_AE_A\psi_j^0) \). If we set \( \gamma_{j+1}^l = (-1)^j\psi_j^0 \in V, j = 0, 1, \ldots, l - 1 \), then by (3.34) \( \psi \) is written as \( \psi = F^*(\sum_{j=0}^{l-1}(1/j!)(E_A\gamma_{j+1}^l)^0) \) and \( (\gamma_1^l, \ldots, \gamma_l^l) \) satisfies \( s_A^T(\gamma_1^l, \ldots, \gamma_l^l)^0 = 0 \) in \( V_{\#l} \) as shown in the proof of Theorem 10. Thus by Theorem 12(i), \( \psi \in F^* \ker(\lambda - A_T)^{l} \). This proves the reverse inclusion of (4.27).

The following inclusions involving the operator \( G \) are verified similarly as in Theorem 13 by using the relations (2.30), (2.31).

**Lemma 4.** (i) For each \( \lambda \in \sigma_p(\Delta_T) = \sigma_p(A^*) = \sigma_p(A_T) \),
\[
G^* \ker(\lambda - A^*)^{l} \subset \ker(\lambda - A_T)^{l}, \quad l = 1, 2, \ldots. \tag{4.29}
\]

In particular
\[
G^* \mathcal{M}_A \subset \mathcal{M}_T. \tag{4.30}
\]

(ii) For each \( \lambda \in \sigma_p(\Delta) = \sigma_p(A^*_T) = \sigma_p(A) \),
\[
G \ker(\lambda - A^*_T)^{l} \subset \ker(\lambda - A)^{l}, \quad l = 1, 2, \ldots. \tag{4.31}
\]

In particular
\[
G^* \mathcal{M}_A \subset \mathcal{M}_T. \tag{4.32}
\]
To study the structure of finite dimensional generalized eigenmanifolds in detail, we introduce the following sets:

\[ \sigma_0(\Delta) = \{ \lambda : \lambda \text{ is a pole of } \Delta(\mu)^{-1} \}, \]
\[ \sigma_0(A) = \{ \lambda : \lambda \text{ is a pole of } (\mu - A)^{-1} \}, \]
\[ \sigma_\delta(A) = \{ \lambda : \lambda \text{ is isolated and } \dim(\text{Im } P_\lambda) < +\infty \}. \]

By Kato [8, p. 181] we know that \( \sigma_\delta(A) \subseteq \sigma_0(A) \subseteq \sigma_p(A) \). The sets \( \sigma_0(\Delta_T), \sigma_0(A_T), \sigma_\delta(A_T) \) and others for \( A^\circ, A_T^\circ \) are defined similarly.

**Theorem 14.** (i) Assume that \( \sigma_0(\Delta_T) = \sigma_0(A^\circ) = \sigma_0(A_T) \). Then

\[ \sigma_\delta(A^\circ) = \sigma_\delta(A_T), \] (4.33)

and for each \( \lambda \in \sigma_\delta(A^\circ) = \sigma_\delta(A_T) \) and each \( l = 1, 2, \ldots \),

\[ G^\circ \operatorname{Ker}(\lambda - A^\circ)^l = \operatorname{Ker}(\lambda - A_T)^l, \] (4.34)
\[ \dim \operatorname{Ker}(\lambda - A^\circ)^l = \dim \operatorname{Ker}(\lambda - A_T)^l = \dim \operatorname{Ker} \mathcal{A}_l^T < \infty. \] (4.35)

In particular,

\[ G^\circ \mathcal{M}_A^\circ = \mathcal{M}_A^T. \] (4.36)

(ii) Assume that \( \sigma_0(\Delta_T) = \sigma_0(A_T^\circ) = \sigma_0(A) \). Then

\[ \sigma_\delta(A_T^\circ) = \sigma_\delta(A), \] (4.37)

and for each \( \lambda \in \sigma_\delta(A_T^\circ) = \sigma_\delta(A) \) and each \( l = 1, 2, \ldots \),

\[ G \operatorname{Ker}(\lambda - A_T^\circ)^l = \operatorname{Ker}(\lambda - A)^l, \] (4.38)
\[ \dim \operatorname{Ker}(\lambda - A_T^\circ)^l = \dim \operatorname{Ker}(\lambda - A)^l = \dim \operatorname{Ker} \mathcal{A}_l < \infty. \] (4.39)

In particular,

\[ G \mathcal{M}_A^{T\circ} = \mathcal{M}_A. \] (4.40)

**Proof.** It is sufficient to prove only part (i). First we shall show (4.33). Let \( \lambda \in \sigma_0(A^\circ) \). Since \( \sigma_\delta(A^\circ) \subseteq \sigma_0(A^\circ) \) is obvious, we have by the assumption that \( \lambda \in \sigma_0(A_T^\circ) \). Let \( k_\lambda \) be the order as a pole of \( (\mu - A^\circ)^{-1} \), which is identical with that of \( (\mu - A_T)^{-1} \). Then by Yosida [18,
We have
\[
\text{Ker}(\lambda - A_T) \subset \text{Ker}(\lambda - A_T)^2 \subset \cdots \subset \text{Ker}(\lambda - A_T)^{k_\lambda} = P_\lambda^T M_2 = \mathcal{M}_\lambda^T,
\]
(4.41)
\[
\text{Ker}(\lambda - A_T)^{l} = \mathcal{M}_\lambda^T \quad \text{for } l \geq k_\lambda + 1.
\]
(4.42)

On the other hand, since \(\lambda \in \sigma_p(A^*)\), we have similar inclusions
\[
\text{Ker}(\lambda - A^*) \subset \text{Ker}(\lambda - A^*)^2 \subset \cdots \subset \text{Ker}(\lambda - A^*)^{k_\lambda} = P_\lambda^* M_2^* = \mathcal{M}_\lambda^*,
\]
(4.43)
\[
\text{Ker}(\lambda - A^*)^{l} = \mathcal{M}_\lambda^* \quad \text{for } l \geq k_\lambda + 1.
\]
(4.44)

Further, since \(\lambda \in \sigma_d(A^*)\), then by (4.43) and (4.44) it follows that
\[
\dim \text{Ker}(\lambda - A^*)^{l} \leq \dim \mathcal{M}_\lambda^* < \infty, \quad \text{for } l = 1, 2, \ldots.
\]
(4.45)

Since \(\sigma_d(A_T) \subset \sigma_p(A_T)\) is clear, Theorem 13(i) implies
\[
F^* \text{Ker}(\lambda - A_T)^{l} = \text{Ker}(\lambda - A^*)^{l}, \quad \text{for } l = 1, 2, \ldots.
\]
(4.46)

Noting that \(F^*\) is injective on each \(\text{Ker}(\lambda - A_T)^{l}\) \((l = 1, 2, \ldots)\) by Proposition 3, we have by (4.45) and (4.46)
\[
\dim \text{Ker}(\lambda - A_T)^{l} \leq \dim \mathcal{M}_\lambda^T = \dim \mathcal{M}_\lambda^* < \infty, \quad \text{for } l = 1, 2, \ldots.
\]
(4.47)

Hence the dimension of \(\text{Im} P_\lambda^T\) is finite, i.e., \(\lambda \in \sigma_d(A_T)\). Next we show the reverse inclusion \(\sigma_d(A_T) \subset \sigma_p(A_T)\). Let \(\lambda \in \sigma_d(A_T)\). Then from the assumption it follows by an argument similar to that above that \(\lambda \in \sigma_p(A^*)\) and (4.43), (4.44) hold true. Since
\[
G^* \text{Ker}(\lambda - A^*)^{l} \subset \text{Ker}(\lambda - A_T)^{l}, \quad l = 1, 2, \ldots,
\]
(4.48)

by Lemma 4, we see by the injectivity of \(G^*\) and \(\lambda \in \sigma_d(A_T)\) that for each \(l = 1, 2, \ldots,\)
\[
\dim \text{Ker}(\lambda - A^*)^{l} \leq \dim \text{Ker}(\lambda - A_T)^{l} \leq \dim \mathcal{M}_\lambda^T < \infty.
\]
(4.49)

Hence by (4.44) \(\dim \text{Im} P_\lambda^* < \infty\), that is, \(\lambda \in \sigma_p(A^*)\). Therefore the equality (4.33) is proved. Next we shall show (4.34). For each \(\lambda \in \sigma_d(A^*) = \sigma_p(A_T)\), the nullspaces \(\text{Ker}(\lambda - A_T)^{l}\) \((l = 1, 2, \ldots)\) are all finite dimensional and are invariant under the semigroup \(S_T(t)\). Then the operator
$S_T(h) = G^*F^*$ is bijective on Ker$(\lambda - A_T)^T$. Thus by Theorem 13(i),

$$\text{Ker}(\lambda - A_T)^T = G^*F^*\text{Ker}(\lambda - A_T)^T = G^*\text{Ker}(\lambda - A^T)^T.$$  \hspace{1cm} (4.50)

This shows (4.34). Now (4.35) is clear.

For the operators $A$ and $A_T$, we have the following adjoint result which is a direct consequence from Kato [8, p. 184].

**Proposition 4.** (i) Let $\lambda$ be a pole of $(\mu - A)^{-1}$ of order $k_\lambda$ at $\mu = \lambda$. Then $\overline{\lambda}$ is a pole of $(\mu - A^T)^{-1}$ of same order $k_\lambda$ at $\mu = \overline{\lambda}$. Further we have

$$\mathcal{M}_\lambda^* = \text{Ker}(\lambda - A^*)^{k_\lambda} = (P_\lambda)^*M_2^\perp$$

$$\dim \mathcal{M}_\lambda = \dim \mathcal{M}_\lambda^* \leq \infty,$$ \hspace{1cm} (4.51)

where $(P_\lambda)^*$ is the adjoint of $P_\lambda$ and is given by

$$(P_\lambda)^* = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - A^*)^{-1} d\mu = P_\lambda^*$$ \hspace{1cm} (4.52)

and $\Gamma_\lambda$ denotes the Miller image of $\Gamma_\lambda$.

(ii) Let $\lambda$ be a pole of $(\mu - A_T)^{-1}$ of order $k_\lambda^T$ at $\mu = \lambda$. Then $\overline{\lambda}$ is a pole of $(\mu - A^T)^{-1}$ of same order $k_\lambda^T$ at $\mu = \overline{\lambda}$. Further we have

$$\mathcal{M}_{\lambda}^{T*} = \text{Ker}(\overline{\lambda} - A_T^*)^{k_{\lambda}^T} = (P_{\lambda}^{T*})^*M_2^\parallel$$

$$\dim \mathcal{M}_{\lambda}^{T*} = \dim \mathcal{M}_{\lambda}^{T*} \leq \infty,$$ \hspace{1cm} (4.53)

where $(P_{\lambda}^{T*})^*$ is the adjoint of $P_{\lambda}^{T*}$ and is given by

$$(P_{\lambda}^{T*})^* = \frac{1}{2\pi i} \int_{\Gamma_{\lambda}^{T*}} (\mu - A_T^*)^{-1} d\mu = P_{\lambda}^{T*}.$$ \hspace{1cm} (4.54)

Finally we consider the characterizations of spectral projections in terms of generalized eigenvectors. Under the assumption $\sigma_D(\Delta_T) = \sigma_D(A^*) = \sigma_D(A_T)$, it was shown in Proposition 4 and Theorem 14 that if $\lambda \in \sigma_D(A)$, then $\overline{\lambda} \in \sigma_D(A^T)$ and $\lambda \in \sigma_D(A^*) = \sigma_D(A_T)$ and

$$d_{\lambda} = \dim \mathcal{M}_{\lambda} = \dim \mathcal{M}_{\lambda}^* = \dim \mathcal{M}_{\lambda}^{T*} < \infty.$$ \hspace{1cm} (4.55)

Let $\Phi = \{\phi_1, \ldots, \phi_d\}$ and $\Psi = \{\psi_1, \ldots, \psi_d\}$ be the bases of $\mathcal{M}_{\lambda}$ and $\mathcal{M}_{\lambda}^{T*}$, respectively. Since $\mathcal{M}_{\lambda}^* = F^*\mathcal{M}_{\lambda}^{T*}$, by (4.57) the $d_{\lambda} \times d_{\lambda}$ matrix $M$ defined
by \( M = (m_{ij}) = (\phi_i, F^*\psi_j)_{M_2} \) is nonsingular. Hence we can suppose
\[
(\phi_i, F^*\psi_j)_{M_2} = \delta_{ij}, \quad i, j = 1, \ldots, d_x,
\]  
where \( \delta_{ij} \) denotes the Kronecker’s delta. Now we introduce the continuous projection \( \tilde{P}_\lambda \) defined by
\[
\tilde{P}_\lambda g = \sum_{i=1}^{d_x} \langle g, F^*\psi_i \rangle_{M_2} \phi_i, \quad g \in M_2.
\]  
It is easily verified that \( \text{Im} \tilde{P}_\lambda = \mathcal{M} \) and \( \text{Ker} \tilde{P}_\lambda = \text{Im}(\lambda - A)^{k_\lambda} \), so that \( P_\lambda = \tilde{P}_\lambda \). Thus we obtain the following result.

**Theorem 15.** (i) Assume that \( \sigma_0(\Delta_T) = \sigma_0(A^+) = \sigma_0(A_T) \). Then for \( \lambda \in \sigma_d(A) \) the spectral projection \( P_\lambda \) has the following equivalent representation
\[
\tilde{P}_\lambda g = \sum_{i=1}^{d_x} \langle g, F^*\psi_i \rangle_{M_2} \phi_i, \quad g \in M_2,
\]  
where \( \{\phi_1, \ldots, \phi_{d_x}\} \) is the basis of \( \mathcal{M} \) and \( \{\psi_1, \ldots, \psi_{d_x}\} \) is the basis of \( \mathcal{M}^T \) satisfying (4.58).

(ii) Assume that \( \sigma_0(\Delta) = \sigma_0(A^+_T) = \sigma_0(A) \). Then for \( \lambda \in \sigma_d(A_T) \) the spectral projection \( P_\lambda^T \) has the following equivalent representation
\[
\tilde{P}_\lambda^T g = \sum_{i=1}^{d_x^T} \langle g, F\psi_i^T \rangle_{M_2} \phi_i^T, \quad g \in M_2,
\]  
where \( d_x^T = \dim \mathcal{M}^T = \dim \mathcal{M}_x \), \( \{\phi_1^T, \ldots, \phi_{d_x^T}\} \) is the basis of \( \mathcal{M}^T \) and \( \{\psi_1, \ldots, \psi_{d_x^T}\} \) is the basis of \( \mathcal{M}_x \) satisfying \( \langle \phi_i^T, F\psi_i^T \rangle = \delta_{ij}, i, j = 1, \ldots, d_x^T \).

We remark that the assumptions \( \sigma_0(\Delta) = \sigma_0(A^+_T) = \sigma_0(A) \) and \( \sigma_0(\Delta_T) = \sigma_0(A^+) = \sigma_0(A_T) \) are satisfied provided that \( A_1 \) and \( A_2 \) satisfy (3.46) and the inclusion map \( i: V \to H \) is compact.

**References**