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RELATIVE REARRANGEMENT ON A MEASURE SPACE APPLICATION TO THE REGULARITY OF WEIGHTED MONOTONE REARRANGEMENT PART II

J. M. RAKOTOSON AND B. SIMON

Département de Mathématiques, Faculté des Sciences de Poitiers

40 Avenue du Recteur Pineau, 86022 Poitiers Cedex, France

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Abstract—Here, we give a few examples of weights belonging to the classes Q and $\widetilde{Q}$. As an application of the first part, we also give a unified method for the proof of continuous imbeddings for weighted Sobolev spaces. The method we use allows us to estimate the imbedding constants and to get directly, and in a simple way, all the standard equivalent norms.

1. EXAMPLES

1.1. Examples of Weights Belonging to the Classes Q and $\widetilde{Q}$

Let Ω be an open bounded set of $\mathbb{R}^N$, whose boundary $\partial\Omega$ is Lipschitzian. That is, we can decompose Ω as: $\Omega = \cup_{i=0}^m \Omega_i$, where $(\Omega_i)_{i=0\ldots m}$ is a family of open sets satisfying: $\overline{\Omega}_0 \subset \Omega$, there exist two numbers $\alpha > 0$, $\beta > 0$, m systems of local coordinates $(x_i', x_{i_N})_{i=1\ldots m}$ and m Lipschitz functions $(a_i)_{i=1\ldots m}$ defined on the $N-1$-dimensional cube $Q_i = \{x_i', |x_{i_j}'| < \alpha,\ j = 1\ldots N-1\}$ such that:

- any point x of $\partial\Omega \cap \partial\Omega_i$ can be written as $x = \big(x_i', a_i(x_i')\big)$,
- if U_i indicates the open set $\{(x_i', x_{i_N}), x_i' \in Q_i, |x_{i_N} - a_i(x_i')| < \beta\}$, then $\Omega_i = U_i \cap \Omega = \{(x_i', x_{i_N}), x_i' \in Q_i, a_i(x_i') < x_{i_N} < a_i(x_i') + \beta\}$.

Then, we define $\Sigma(\Omega)$ as the class of functions σ belonging to $W^{1,\infty}(\Omega)$, strictly positive in Ω and such that in local coordinates, we have:

$$c_1\,\sigma(x) \le x_{i_N} - a_i(x_i') + b_i(x_i') \le c_2\,\sigma(x) \qquad \forall x \in \Omega_i,$$

where c_1, c_2 are two constants strictly positive and b_i a function defined on Q_i such that $0 \le b_i(x_i') \le c_3$.

We also define $\Sigma^{\#}(\Omega)$ as the sub-class of $\Sigma(\Omega)$ consisting of the functions σ such that on each connected component of $\partial\Omega$; $\partial\Omega_i$, there exists one arc $\Gamma_i \subset \partial\Omega_i$ (with $H_{N-1}(\Gamma_i) > 0$) on which the trace of σ is strictly positive. Then, we have

PROPOSITION 1. *Let Ω be an open, bounded, connected, Lipschitzian set, $\sigma \in \Sigma(\Omega)$ and $\nu \ge 0$, then $\sigma^\nu \in \widetilde{Q}\big(\Omega, \frac{N+\nu}{N+\nu-1}\big)$.*

PROPOSITION 2. *Let Ω be an open, bounded, Lipschitzian set, $\sigma \in \Sigma^{\#}(\Omega)$ and $\nu \ge 0$, then $\sigma^\nu \in Q\big(\Omega, \frac{N+\nu}{N+\nu-1}\big)$.*

The proof of these two propositions takes inspiration from [1]. We give a few examples of functions σ belonging to the classes $\Sigma(\Omega)$ and $\Sigma^{\#}(\Omega)$:

$$\sigma(x) = \mathrm{dist}(x, \partial\Omega) : \sigma \in \Sigma(\Omega),\ \sigma \notin \Sigma^{\#}(\Omega),$$
$$\sigma(x) = \mathrm{dist}(x, x_0),\ x_0 \in \partial\Omega : \sigma \in \Sigma^{\#}(\Omega),$$
$$\sigma \in W^{1,\infty}(\Omega) \text{ such that } 0 < c_1 \le \sigma : \sigma \in \Sigma^{\#}(\Omega).$$

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REMARK. It is shown in [2] that if σ belongs to $\Sigma(\Omega)$, then $W^{1,p}(\Omega,\sigma^\nu) = V^{1,p}(\Omega,\sigma^\nu)$ $\forall p \in [1,+\infty], \forall \nu \geq 0$.

1.2. Estimates of the Constant Q_a

THEOREM 1. *Let Ω be an open set of $\mathbb{R}^N$ and $\nu > 0$.*

(i) *Let $x_0 \in \mathbb{R}^N$, we define the weight function $\sigma(x) = \big(\operatorname{dist}(x,x_0)\big)^\nu$. We assume that $|\Omega|_\sigma < +\infty$. Then, σ belongs to the class $Q\big(\Omega,\frac{N+\nu}{N+\nu-1}\big)$ and we have the estimate:*

$$Q_\sigma\left(\Omega,\frac{N+\nu}{N+\nu-1}\right) \leq N^{1+(\nu/2)}(2\nu^{-\nu}N^{-\nu/2})^{1/(N+\nu)}.$$

(ii) *Let H be a hyperplane of $\mathbb{R}^N$, we define $\sigma(x) = \big(\operatorname{dist}(x,H)\big)^\nu$. We assume that $|\Omega|_\sigma < +\infty$. Then, σ belongs to the class $Q\big(\Omega,\frac{N+\nu}{N+\nu-1}\big)$ and we have the estimate:*

$$Q_\sigma\left(\Omega,\frac{N+\nu}{N+\nu-1}\right) \leq (2\nu^{-\nu})^{1/(N+\nu)}.$$

Notice that the two previous estimates are independant of Ω, and if $\sigma = 1$ (that is $\nu = 0$), the isoperimetric constant $Q_1\big(\Omega,\frac{N}{N-1}\big)$ is equal to $\big(N\alpha_N^{1/N}\big)^{-1}$.

2. WEIGHTED SOBOLEV IMBEDDINGS

Using Theorems 4 and 5 of the first part and basic properties linked to rearrangement, we obtain continuous imbeddings for weighted Sobolev spaces. This method generalizes and unifies such of the results already known [1–5] and allows one to estimate the imbeddings constants.

THEOREM 2. *Let Ω be an open set of $\mathbb{R}^N$, $a \in Q(\Omega,q)$ $(q>1)$ and $p > 1$. q' (resp p') will denote the conjuguate of q (resp p). Then, we have the following continuous imbeddings:*

If $p > q'$, then $W_0^{1,p}(\Omega,a) \hookrightarrow L^\infty(\Omega)$ and $\forall u \in W_0^{1,p}(\Omega,a)$,

$$|u|_\infty \leq |\Omega|_a^{(1/p')-(1/q)}\left(\frac{q}{q-p'}\right)^{1/p'} Q_a(\Omega,q)\|\nabla u\|_{p,a}.$$

If $p = q'$ then $W_0^{1,q'}(\Omega,a) \hookrightarrow L^r(\Omega,a)\,\forall r \in [1,+\infty[$ and $\forall u \in W_0^{1,q'}(\Omega,a)$,

$$|u|_{r,a} \leq |\Omega|_a^{1/r}\left[\int_0^\infty \sigma^{r/q}e^{-\sigma}\,d\sigma\right]^{1/r} Q_a(\Omega,q)\|\nabla u\|_{q',a}.$$

If $1 < p < q'$, then $W_0^{1,p}(\Omega,a) \hookrightarrow L^r(\Omega,a)\,\forall r \in [1,p^*[$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{q'}$ and $\forall u \in W_0^{1,p}(\Omega,a)$,

$$|u|_{r,a} \leq |\Omega|_a^{(1/p')-(1/q)+(1/r)}\left(\frac{q}{p'-q}\right)^{1/p'}\left[\int_0^1 \left(t^{1-(p'/q)}-1\right)^{r/p'} dt\right]^{1/r} Q_a(\Omega,q)\|\nabla u\|_{p,a}.$$

REMARKS. If $p = 1$, we know by definition of $Q(\Omega,q)$ that $W_0^{1,1}(\Omega,a) \hookrightarrow L^q(\Omega,a)$ and $|u|_{q,a} \leq Q_a(\Omega,q)\,\|\nabla u\|_{1,a} \forall u \in W_0^{1,1}(\Omega,a)$.

This theorem generalizes perfectly the classical Sobolev inequalities. Indeed, by Poincaré inequality, we know that the weight function $a = 1$ belongs to the class $Q(\Omega,\frac{N}{N-1})$. Namely, $q = \frac{N}{N-1}$ and $q' = N$.

Of course, we can apply this theorem to the weights that we have introduced in the preceding paragraph (then, the exponent q' is $N+\nu$) and take back the estimates of the constants Q_a.

We also deduce a weighted Trudinger inequality:

THEOREM 3. *Let Ω be an open set of $\mathbb{R}^N$ and $a \in Q(\Omega, q)$ $(q > 1)$. Then for all $u \in W_0^{1,q'}(\Omega, a)$ and for all $\lambda > Q_a(\Omega, q)$, we have:*

$$\int_\Omega \exp\left[\left(\frac{|u(x)|}{\lambda\|\nabla u\|_{q',a}}\right)^q\right] a(x)\,dx \le \frac{|\Omega|_a}{1-\left(\frac{Q_a(\Omega,q)}{\lambda}\right)^q}.$$

Namely, $\exp(c|u|^q) \in L^1(\Omega, a)$.

THEOREM 4. *Let Ω be a (connected) open set of $\mathbb{R}^N$, $a \in \widetilde{Q}(\Omega, q)$ $(q > 1)$ and $p > 1$. Then, we have the following continuous imbeddings:*

If $p > q'$, then $V^{1,p}(\Omega, a) \hookrightarrow L^\infty(\Omega)$ and $\forall u \in V^{1,p}(\Omega, a)$,

$$|u_{*,a}(s) - u_{*,a}(s')| \le 2^{1-(1/q)}\left|\int_s^{s'} k^{-p'}(\sigma)\,d\sigma\right|^{1/p'} \widetilde{Q}_a(\Omega, q)\|\nabla u\|_{p,a} \qquad \forall s, s' \in \Omega^*,$$

where

$$k(\sigma) = \min\left(\sigma^{1/q}, (|\Omega|_a - \sigma)^{1/q}\right),$$

in particular,

$$|u_{*,a}(s) - \bar{u}_{*,a}| \le C\|\nabla u\|_{p,a} \qquad \forall s \in \Omega^*,$$

where

$$\bar{u}_{*,a} = \frac{1}{|\Omega|_a}\int_{\Omega^*} u_{*,a}(\sigma)\,d\sigma$$

and

$$C = 2|\Omega|_a^{(1/p')-(1/q)}\left(\frac{q}{q-p'}\right)^{1/p'} \widetilde{Q}_a(\Omega, q),$$

moreover,

$$|u|_\infty \le C\|\nabla u\|_{p,a} + |\Omega|_a^{-1}|u|_{1,a}.$$

If $p = q'$, then $V^{1,q'}(\Omega, a) \hookrightarrow L^r(\Omega, a)\,\forall r \in [1, +\infty[$ and $\forall u \in V^{1,q'}(\Omega, a)$,

$$\left|u_{*,a}(.) - u_{*,a}\left(\frac{|\Omega|_a}{2}\right)\right|_r \le C\|\nabla u\|_{q',a},$$

where

$$C = 2^{1/q'}|\Omega|_a^{1/r}\left[\int_0^\infty \sigma^{r/q}e^{-\sigma}\,d\sigma\right]^{1/r} \widetilde{Q}_a(\Omega, q),$$

moreover,

$$|u|_{r,a} \le C\|\nabla u\|_{q',a} + 2|\Omega|_a^{(1/r)-1}|u|_{1,a}.$$

If $1 < p < q'$, then $V^{1,p}(\Omega, a) \hookrightarrow L^r(\Omega, a)\,\forall r \in [1, p^*[$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{q'}$ and $\forall u \in V^{1,p}(\Omega, a)$,

$$\left|u_{*,a}(.) - u_{*,a}\left(\frac{|\Omega|_a}{2}\right)\right|_r \le C\|\nabla u\|_{p,a},$$

where

$$C = 2^{1/p}|\Omega|_a^{(1/p')-(1/q)+(1/r)}\left(\frac{q}{p'-q}\right)^{1/p'}\left[\int_0^1 \left(t^{1-(p'/q)} - 1\right)^{r/p'} dt\right]^{1/r} \widetilde{Q}_a(\Omega, q),$$

moreover,

$$|u|_{r,a} \leq C\big|\big|\nabla u\big|\big|_{p,a} + 2\big|\Omega\big|_a^{(1/r)-1}|u|_{1,a}.$$

REMARK. We get directly the following equivalent norms; for instance, if $1 < p < q'$:

$$\big|u\big|_{V^{1,p}(\Omega,a)} \sim \big|\big|\nabla u\big|\big|_{L^p(\Omega,a)} + \big|u\big|_{L^s(\Omega,a)} \qquad 1 \leq s < p^*.$$

If $a = 1$, $\widetilde{Q}_1(\Omega, \frac{N}{N-1}) \leq U(\Omega, \frac{N}{N-1})$: the relative isoperimetric constant [6]. It is known for some domains [7], for instance:

- If Ω is a ball of $\mathbb{R}^N$, $U\left(\Omega, \frac{N}{N-1}\right) = \frac{1}{\alpha_{N-1}}\left(\frac{1}{2}\alpha_N\right)^{1-(1/N)}$ (where α_m is the measure of the unit ball in $\mathbb{R}^m$).
- If Ω is a rectangle (in $\mathbb{R}^2$) whose sides have lengths a and b, $a \geq b$, $U(\Omega, 2) = a^{1/2}(2b)^{-1/2}$.
- If Ω is a triangle (in $\mathbb{R}^2$) whose the smallest of its angles is ω, $U(\Omega, 2) = (2\omega)^{-1/2}$.

3. REGULARITY

THEOREM 5. *Let Ω be an open set of $\mathbb{R}^N$ and $a \in Q(\Omega, q)$ $(q > 1)$. Let $u \in W_0^{1,p}(\Omega, a)$ solution of $-\operatorname{div}\big(a|\nabla u|^{p-2}\nabla u\big) = f$. We assume that $h = \frac{|f|}{a} \in L^r(\Omega, a)$ $(r \geq 1)$. Then,*

$$(|u|)_{*,a}(s) \leq \big(Q_a(\Omega, q)\big)^{p'} \int_s^{|\Omega|_a} \sigma^{-\frac{p'}{q}} \left(\int_0^\sigma h_{*,a}(\tau)\, d\tau\right)^{p'/p} d\sigma \qquad \forall s \in \Omega^*,$$

and if $r > \frac{q'}{p}$, then $u \in L^\infty(\Omega)$ and we have the estimate:

$$|u|_\infty \leq \frac{1}{\gamma}\big|\Omega\big|_a^\gamma \big(Q_a(\Omega, q)\big)^{p'} \big|h\big|_{r,a}^{p'/q} \qquad \textit{with } \gamma = 1 - \frac{p'}{q} + \frac{p'}{pr'}.$$

Details of the proof will be given in [8,9] (these proofs are very easy). Other estimates of $\widetilde{Q}_a$ will be also given. The proof of Theorem 5 follows the ideas of [10] (see also [11]).

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