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Actions of Picard Groups on Graded Rings*

Jeremy Haefner

Department of Mathematics, University of Colorado, Colorado Springs, Colorado 80933
E-mail: haefner@math.uccs.edu

and

Angel del Río

Departamento de Matemáticas, Universidad de Murcia, 30071 Murcia, Spain
E-mail: adelrio@fcu.um.es

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1. INTRODUCTION

Throughout this paper, G denotes a group with identity e. If R is a G-graded ring, we write $R = \oplus_{g \in G} R_g$ and we refer to R_e as the *coefficient ring*. We denote the category of graded (left) R-modules (i.e., those left R-modules M with a G-grading $M = \oplus_{g \in G} M_g$ such that $R_h M_g \subseteq M_{hg}$) by R-gr.

The aim of this paper is to present a method to reduce the study of one strongly graded ring R to the study of another strongly graded ring R' that is more tractable. This reduction process has two aspects: *reducing the coefficient ring* or *reducing the grading*. Since we apply this method in our study of module-theoretic properties such as semisimplicity for graded rings, we require that this reduction process preserve the category of modules R-mod and the category of graded modules R-gr. This leads us to the notion of graded equivalence.

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In this paper, a Morita context is the usual tuple (A, B, P, Q, τ, μ) where we assume τ and μ are isomorphisms. A *G-graded context* is a Morita context (R, R', P, Q, τ, μ), where R and R' are G-graded rings, ${}_RP_{R'}$ and ${}_{R'}Q_R$ are graded bimodules (i.e., they have a grading which makes them into both graded R-modules and graded R'-modules on the corresponding side), and τ and μ are graded bimodule homomorphisms. See [14] for details about graded modules and homomorphisms. Given a functor F: R-gr $\to R'$-gr, we say that F is a *graded equivalence of categories* if any of the following equivalent conditions hold (see, for example, [8]):

1. F has an inverse and $F \circ T_g = T_g \circ F$, for every $g \in G$, where T_g denotes the gth suspension.

2. $F = \mathrm{Hom}_F(P, -)$ where P is part of a G-graded Morita context (R, R', P, Q, τ, μ).

3. $F = Q \otimes_R -$ where Q is part of a G-graded Morita context (R, R', P, Q, τ, μ).

4. F lifts to an equivalence of categories F': R-mod $\to R'$-mod; that is, there is an equivalence of categories F', as above, such that $F'(M) = F(M)$ for every $M \in R$-gr.

We say that R and R' are *graded equivalent* if there is a graded equivalence F: R-gr $\to R'$-gr.

A special type of graded equivalence arises from the graded isomorphism. Given G-graded rings R and R', we say that R and R' are *graded isomorphic* if there is a *graded isomorphism* f: $R \to R'$, that is, a ring homomorphism f such that $f(R_g) = R'_g$. In case $A = R_e = R'_e$, R is said to be *graded A-isomorphic* if there is a graded isomorphism f: $R \to R'$ that induces the identity on A. Of course, graded isomorphic rings have isomorphic categories of modules and graded modules, but this is too strong a condition for our purposes.

The reduction process mentioned above can be rephased in terms of graded equivalences. The main problem of this paper is the following:

Problem A. Given a (strongly) G-graded ring, find another G-graded ring R' that is graded equivalent to R such that either the identity component R'_e is simpler (e.g., R_e is semiperfect and R'_e is basic semiperfect) or the grading of R' is simpler (e.g., R' is a crossed product or a skew group ring).

We apply our solution of Problem A to the study of a more concrete problem:

Problem B. Characterize semisimple strongly G-graded rings.

The key to our study is the development of an action of graded Morita contexts on graded rings. For the purposes of this introduction, if C is a Morita context and R is a graded ring, then R^C denotes the action of C on R (see Section 2 for the details). We use this action to *reduce the coefficient ring*. For example, we can apply a graded Morita context on a graded ring R whose coefficient ring is of the form $M_n(A)$ to obtain a graded equivalent ring R' whose coefficient ring is A. We also use this action to *reduce the grading*. Crossed products and twisted, skew, and ordinary group rings are examples of gradings that are more tractable than strongly graded rings. The Cohen–Montgomery duality is an example of how this action can reduce the grading of a strongly graded ring to that of a skew group ring. To be more specific, we introduce some notation and present one of our main results.

Let R be G-graded and let $A = R_e$. A (graded) invertible R-module is a bimodule ${}_R P_R$ which is part of a (graded) Morita context (R, R, P, Q, μ, τ). The class of A-bimodules isomorphic to a given A-bimodule P is denoted by $[P]$. In addition, we use the following: (Here P_e denotes the identity component of the graded module P.)

$\mathrm{Aut}(A)$ = Group of automorphisms of A

$\mathrm{Inn}(A)$ = group of inner automorphisms of A

$\mathrm{Aut}_R(A)$ = Group of automorphisms of A that extends to a graded automorphism of R

$\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$

$\mathrm{Out}_R(A) = \mathrm{Aut}_R(A)/(\mathrm{Inn}(A) \cap \mathrm{Aut}_R(A))$

$\mathrm{Pic}(A)$ = Picard group of A

$\mathrm{Pic}_R(A) = \{[P_e] : P$ is a graded invertible R-bimodule$\}$

The next theorem summarizes our results: (Here given a group G and a subgroup H of G, G/H denotes the set of either right of left H-cosets of G.)

THEOREM A. *Let R be a graded ring with coefficient ring A.*

1. *If R and R' are strongly G-graded rings, then R and R' are graded equivalent if and only if there exists a Morita context $\tilde{C}$ between A and the coefficient ring of R', such that $\tilde{C}$ induces a graded Morita context C between R and R' so that R' and R^C are graded isomorphic.*

2. *If R is strongly graded,* $\mathrm{Pic}(A)/\mathrm{Pic}_R(A)$ *parametrizes the strongly graded rings R', with coefficient ring A, that are graded equivalent to R but not graded A-isomorphic to R.*

3. *If R is strongly graded,* $\mathrm{Pic}(A)/\mathrm{Out}(A)$ *parametrizes the strongly graded rings, with coefficient ring A, that are graded equivalent to R but not graded isomorphic to R.*

4. $\mathrm{Out}(A)/\mathrm{Out}_R(A) \simeq \mathrm{Aut}(A)/\mathrm{Aut}_R(A)$ *parametrizes the graded rings that are graded isomorphic to R, but not graded A-isomorphic to R.*

See Theorem 3.2 and Propositions 4.4 and 4.6. As a consequence of (1), the graded equivalence class of R is completely determined by the Morita equivalence class of its coefficient ring. Statements (2) through (4) give parametrizations of the equivalence classes of (strongly) graded rings with the same coefficient ring, under graded equivalence, graded isomorphism, or graded A-isomorphism. Finally, synthesizing some results from [14, 5], we reduce the distinction between graded equivalence and graded isomorphism to a cohomology problem (see Section 5) and we obtain the following:

THEOREM B. *There are graded equivalent, strongly G-graded rings R and R', sharing coefficient ring A, and, for every $g \in G$, $R_g \simeq R'_g$ as A-bimodules but R and R' are not graded isomorphic.*

See Theorem 5.8

Theorem A implies that reducing the coefficient ring A is limited by the Morita equivalence class of A. As an example we obtain a result of [9]: If R is a strongly graded ring so that R_e is semiperfect, then R is graded equivalent to a strongly graded ring R' such that R'_e is the basic ring of R_e. In particular, it follows that R' is a crossed product. As a result, in the semiperfect situation, we can reduce not only the coefficient ring, but the grading as well. Skew group rings and twisted group rings are examples of gradings that are more tractable than crossed products, and so we ask whether the process can go further to one of these two cases. However, reducing to a twisted group ring is impossible if the original graded ring is not already twisted (Corollary 3.4). On the other hand, reducing to a skew group ring is possible via the Cohen−Montgomery duality but at the cost of complicating the coefficient ring. Nonetheless, this is the best we can expect because reducing the grading to a skew group ring and keeping a tractable coefficient ring are somehow incompatible. Specifically, if $R_e \simeq R'_e$ is basic semiperfect and R is a skew group ring, then R' is a skew group ring as well (Proposition 6.5).

This result is helpful to the solution of Problem B. In particular, if R is semisimple, so is R_e and hence, to solve Problem B, we may assume that R_e is semisimple. By Theorem A, we may reduce R to a crossed product over a finite product of division rings. Moreover, since Theorem A determines how a graded equivalence results from a Morita context of the coefficient rings, we are very specific on the resulting crossed product (see

Section 7). However we cannot expect to go further (e.g., to unskew or to untwist) via graded equivalence. Nonetheless, using the particularities of these crossed products, we can reduce the semisimplicity of the original ring to the semisimplicity of a finite set of concrete crossed products over division rings. We prove:

THEOREM C. *Let R be a strongly graded ring with semisimple coefficient ring A. Let B be the semisimple basic ring that is Morita equivalent to A (so B is a direct sum of division rings). Then R is graded equivalent to a crossed product R' with coefficient ring B. Moreover, there exists a finite collection of crossed products over division rings, $\{D_i * G_i : i = 1, \ldots, n\}$, such that R is semisimple if and only if each $D_i * G_i$ is semisimple.*

See Theorem 7.5. Moreover, if the original grading is already "untwisted" (see Section 7), then we can reduce to skew group rings over division rings (Corollary 7.7). Finally, we mention that, when the above mentioned division rings are fields, we can apply the results of [2, 3] to characterize all the semisimple strongly graded rings of this form (Corollary 7.8).

Notation. We denote ring automorphisms exponentially; that is, the action of $\alpha \in \mathrm{Aut}(R)$ on $r \in R$ is denoted by r^{α}. Accordingly $\alpha\beta$ means, first α, then β.

If u is a unit, then ι_u denotes the inner automorphism $(r^{\iota_u} = u^{-1}ru)$ induced by u.

Let R be a G-graded ring. For every $g \in G$, R_g denotes the gth *homogeneous component* of R. The notation r_g is normally used to emphasize that $r_g \in R_g$. R-gr denotes the category of left graded R-modules.

R is said to be *strongly graded* if $R_g R_h = R_{gh}$ for ever $g, h \in G$. R is said to be a *crossed product* if R_g contains a unit for every $g \in G$. Crossed products are determined by parameter sets: A *parameter set* of a group G over a ring A is a pair of maps $(\alpha\colon G \to \mathrm{Aut}(A),\ t\colon G \times G \to \mathrm{U}(A))$ satisfying the following conditions:

$$\alpha_g \alpha_h = t(g,h)\,\iota_{t(g,h)} \qquad \text{and} \qquad t(gh,k)\cdot t(g,h)^{\alpha_k} = t(g,hk)t(h,k).$$

The crossed product $A *_t^{\alpha} G$ defined by the parameter set (α, t) is the free right A-module with basis $\{\bar{g} : g \in G\}$ with multiplication given by

$$r\bar{g} = \bar{g}r^{\alpha_g}, \qquad \bar{g}\bar{h} = \overline{gh}\,t(g,h).$$

A *skew group ring* is a crossed product $A *_t^{\alpha} G = A *^{\alpha} G$, such that $t(g,h) = 1$ for every $g, h \in G$. A *twisted group ring* is a crossed product $A *_t^{\alpha} G = A *_t G$, such that $\alpha_g = 1$ for every $g \in G$.

When a map $X \otimes Y \to Z$ is denoted by M, we mean that the map is given by $M(x \otimes y) = xy$, where the multiplication will be clear from the context. Finally, we have frequently abused the notation and identified $X \otimes_A A$, $A \otimes_A X$, and X without explicit mention.

2. MORITA CONTEXTS ACTING ON GRADED RINGS

We begin this section by defining a general action by Morita contexts. Let A and B be arbitrary unital rings. By "Morita context," we mean the usual Morita context 6-tuple (A, B, P, Q, τ, μ) where ${}_AP_B$ and ${}_BQ_A$ are bimodules and $\tau\colon P \otimes_B Q \to A$ and $\mu\colon Q \otimes_A P \to B$ are bimodule isomorphisms. In the literature, the condition that τ and μ are bijective is usually not required and a Morita context satisfying this condition is called a strict Morita context. But all the Morita contexts used in this paper are strict, so we just say "Morita context" to mean "strict Morita context." In particular, a Morita context shall always induce an equivalence of categories between A-mod and B-mod.

For shorthand notation, all Morita contexts will be denoted by C with some modifier. For example, C' denotes the Morita context $(A', B', P', Q', \tau', \mu')$.

DEFINITION 2.1. Let C and C' be two Morita contexts. A morphism of Morita contexts from C to C' is a 4-tuple of maps $\phi = (\alpha, \beta, \pi, \varrho)$ where:

1. $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ are ring homomorphisms,
2. $\pi\colon P \to P'$ is an $\alpha-\beta$ semilinear map,
3. $\varrho\colon Q \to Q'$ is a $\beta-\alpha$ semilinear map, and
4. the following diagrams are commutative:

$$\begin{array}{ccc} P \otimes_B Q & \xrightarrow{\tau} & A \\ {\scriptstyle \pi\otimes\varrho}\downarrow & & \downarrow{\scriptstyle \alpha} \\ P' \otimes_{B'} Q' & \xrightarrow{\tau'} & A' \end{array} \qquad \begin{array}{ccc} Q \otimes_A P & \xrightarrow{\mu} & B \\ {\scriptstyle \varrho\otimes\pi}\downarrow & & \downarrow{\scriptstyle \beta} \\ Q' \otimes_{A'} P' & \xrightarrow{\mu'} & B' \end{array}$$

As with our notation convention for Morita contexts, we shall use similar rules to denote morphisms of Morita contexts; e.g., the morphism ϕ_1 is formed by the maps $\alpha_1, \beta_1, \pi_1, \varrho_1$.

Morita context and the morphisms between Morita contexts define a category in a natural way.

DEFINITION 2.2. Given two Morita contexts C and C', with $B = A'$, we define the multiplication

$$\begin{aligned} C \cdot C' = \big(A, B', P \otimes_B P', Q' \otimes_B Q,&\\ \tau(1 \otimes \tau' \otimes 1)\colon (P \otimes_B P') \otimes_{B'} (Q' \otimes_B Q) &\to A,\\ \mu'(1 \otimes \mu \otimes 1)\colon (Q' \otimes_B Q) \otimes_A (P \otimes_B P') &\to B'\big). \end{aligned}$$

It is straightforward to check that this product is well defined and associative. The next lemma shows that this product can be translated to the isomorphism classes of Morita contexts.

LEMMA 2.3. *The product of Morita contexts is compatible with the isomorphism of Morita contexts; i.e., if $C \simeq C_1$ and $C' \simeq C'_1$ are Morita contexts so that $B = A'$ and $B_1 = A'_1$, then $C \cdot C' \simeq C_1 \cdot C'_1$.*

Proof. Let $\phi: C \to C_1$ and $\phi: C' \to C'_1$ be isomorphisms of Morita contexts. Then one checks by straightforward computations that $(\alpha, \beta', \pi \otimes \pi', \varrho' \otimes \varrho)$ is an isomorphism of Morita contexts $C \cdot C' \simeq C_1 \cdot C'_1$. ∎

We now turn our attention to graded rings. Let R be a G-graded ring. As we want a context $C = (A, B, P, Q, \tau, \mu)$ to act on R, we need to make sure that R can be viewed as an A-bimodule. Thus, for technical reasons, it is convenient to assume that the coefficient ring R_e is isomorphic to A. This leads us to the following definition of an (A, G)-graded ring and the appropriate context action.

DEFINITION 2.4. Let A be a ring and G a group. An (A, G)-*graded ring* is a pair (R, f) where R is a G-graded ring and $f: A \to R_e$ is a ring isomorphism.

Let $\alpha: A \to A'$ be a ring homomorphism, (R, f) an (A, G)-graded ring, and (R', f') an (A', G)-graded ring. A graded *α-homomorphism* from (R, f) to (R', f') is a graded ring homomorphism $\Phi: R \to R'$ which makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & R \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\Phi} \\ A' & \xrightarrow{f'} & R' \end{array}$$

commutative.

(A, G)-graded rings and the 1_A-morphisms of (A, G) define a category in a natural way. The graded 1_A-isomorphism class of an (A, G)-graded ring (R, f) is denoted by $[R, f]$. Now we define an action of Morita contexts on graded rings.

DEFINITION 2.5. Given a Morita context C and an (A, G)-graded ring (R, f), we define $(R, f)^C = (R^C, f^C)$, where

1. $R^C = Q \otimes_A R \otimes_A P = \bigoplus_{g \in G} Q \otimes_A R_g \otimes_A P$ with the product defined by

$$(q \otimes r \otimes p)(q' \otimes r' \otimes p') = (q \otimes r\tau(p \otimes q')r' \otimes p').$$

2. $f^C = (1 \otimes f \otimes 1)\mu^{-1}$.

The next lemma shows that this action induces an action of the isomorphism classes of Morita contexts on the class of (graded) isomorphism classes of graded rings.

LEMMA 2.6. *Let* (R, f) *be an* (A, G)*-graded ring,* (R', f') *an* (A', G)*-graded ring,* $\phi = (\alpha, \beta, \pi, \varrho)$: $C \to C'$ *a morphism of Morita contexts, and* Φ: $(R, f) \to (R', f')$ *a graded* α*-morphism of rings. Then* $\Phi^{\phi} = \varrho \otimes \Phi \otimes \pi$: $R^C \to R'^{C'}$ *is a graded* β*-morphism. In particular, if* ϕ: $C \simeq C'$ *and* (R, f) *is* α*-isomorphic to* (R', f')*, then* $(R, f)^C$ *is* β*-isomorphic to* $(R', f')^{C'}$.

Proof. We leave it to the reader to check that Φ^{ϕ} is well defined. We check that Φ^{ϕ} is a ring homomorphism. Given $q, q' \in Q$, $r, r' \in R$, and $p, p' \in P$, then

$$\begin{aligned}
&\Phi^{\phi}((q \otimes r \otimes p)(q' \otimes r' \otimes p')) \\
&\quad = \Phi^{\phi}(q \otimes r\tau(p \otimes q')r' \otimes p') \\
&\quad = \varrho(q) \otimes \Phi(r\tau(p \otimes q')r') \otimes \pi(p') \\
&\quad = \varrho(q) \otimes \Phi(r)(\Phi f \tau)(p \otimes q')\Phi(r') \otimes \pi(p') \\
&\quad = \varrho(q) \otimes \Phi(r)(f'\alpha\tau)(p \otimes q')\Phi(r') \otimes \pi(p') \\
&\quad = \varrho(q) \otimes \Phi(r)(f'\tau'(\pi \otimes \varrho))(p \otimes q')\Phi(r') \otimes \pi(p') \\
&\quad = \varrho(q) \otimes \Phi(r)\tau'(\pi(p) \otimes \varrho(q'))\Phi(r') \otimes \pi(p') \\
&\quad = (\varrho(q) \otimes \Phi(r) \otimes \pi(p))(\varrho(q') \otimes \Phi(r') \otimes \pi(p')) \\
&\quad = \Phi^{\phi}(q \otimes r \otimes p)\Phi^{\phi}(q' \otimes r' \otimes p').
\end{aligned}$$

It is straightforward to see that Φ^{ϕ} is graded. Finally we prove that Φ^{ϕ} is a β-homomorphism. Since the diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & R \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\Phi} \\ A' & \xrightarrow{f'} & R' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Q \otimes_A P & \xrightarrow{\mu} & B \\ {\scriptstyle\varrho \otimes \pi}\downarrow & & \downarrow{\scriptstyle\beta} \\ Q' \otimes_{A'} P' & \xrightarrow{\mu'} & B' \end{array}$$

are commutative, the diagram

$$\begin{array}{ccc} B & \xrightarrow{f^C} & Q \otimes_A R \otimes_A P \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\Phi^{\phi}} \\ B' & \xrightarrow{f'^{C'}} & Q' \otimes_{A'} R' \otimes_{A'} P' \end{array}$$

is also commutative. ∎

We close this section by showing that the action of (isomorphism classes) of Morita contexts on the (isomorphism classes) of graded rings is multiplicative.

LEMMA 2.7. *Let (R, f) be an (A, G)-graded ring and C and C' Morita contexts so that $B = A'$. Then $(R, F)^{C \cdot C'}$ is isomorphic to $((R, f)^C)^{C'}$.*

Proof. This follows by showing that the classical isomorphism

$$(Q' \otimes_{A'} Q) \otimes_A R \otimes_A (P \otimes_{A'} P') \to Q' \otimes_{A'} (Q \otimes_A R \otimes_A P) \otimes_{A'} P'$$

is an isomorphism of (B', G)-graded rings. ∎

Notation 2.8. We denote the action of the isomorphism class of the Morita context C on the isomorphism class $[R, f]$ of the (A, G)-graded ring (R, f) by $[R, f]^C$.

3. GRADED EQUIVALENCES

The moral of the previous section is that we may consider the equivalence classes of Morita contexts (partially) acting multiplicatively on graded rings. In this section, we show that this action completely characterizes graded equivalences for strongly graded rings.

The key to graded contexts is that they induce Morita contexts for the coefficient ring when the ring R is strongly graded. This is the essence of the following lemma.

LEMMA 3.1. *Let (R, f) be a strongly (A, G)-graded ring and (R', f') a strongly (A', G)-graded ring. If $\mathscr{C} = (R, R', X, Y, \tau', \mu')$ is a graded context, then $C = (A, A', P = X_e, Q = Y_e, \tau = f^{-1}\tau'_e, \mu = f'^{-1}\mu'_e)$ is a Morita context, where τ'_e: $P \otimes_{A'} Q \to A$ and μ'_e: $Q \otimes_A P \to A'$ are given by $\tau'_e(p \otimes q) = \tau'(p \times q)$ and $\mu'_e(q \otimes q) = \mu'(q \times p)$.*

Proof. Since τ' is surjective, there are $x_i \in X$ and $y_i \in Y$ $(i = 1, 2, \ldots, n)$ so that $\tau'(\sum_{i=1}^n x_i \otimes y_i) = 1$. We may assume that every x_i is homogeneous, say of degree g_i, and that every y_i is homogeneous of degree g_i^{-1}. For every $i = 1, 2, \ldots, n$, choose $r_{ij} \in R'_{g_i^{-1}}$ and $s_{ij} \in R'_{g_i}$ $(j = 1, 2, \ldots, k_i)$ so that $\sum_j r_{ij} s_{ij} = 1$. Then $\tau(\sum_i \sum_j x_i r_{ij} \otimes s_{ij} y_i) = f^{-1}\tau'(\sum_i \sum_j x_i r_{ij} \otimes s_{ij} y_i) = f^{-1}\tau'(\sum_i (x_i \otimes \sum_j (r_{ij} s_{ij}) y_i) = f^{-1}(\tau'(\sum_{i=1}^n x_i \otimes y_i)) = f^{-1}(1) = 1$. This shows that τ is surjective. By symmetry, μ is also surjective. ∎

We can now present our characterization of graded equivalences using our context action.

THEOREM 3.2. *Let (R, f) be an (A, G)-graded ring and let (R', f') be an (A', G)-graded ring.*

1. *If there exists a Morita context C such that $[R', f'] = [R, f]^C$, then R and R' are graded equivalent.*

2. *If R and R' are strongly graded, then the converse of (1) holds; that is, if R and R' are strongly graded and graded equivalent, then there exists a Morita context C such that $[R', f'] = [R, f]^C$.*

Proof. 1. It is enough to show that R and R^C are graded equivalent. Let $Q \otimes_A -$: R-gr $\to R^C$-gr be the functor defined as follows: If $M \in R$-gr then $Q \otimes_A -$ maps M to $Q \otimes_A M$ with the grading $(Q \otimes_A M)_g = Q \otimes_A M_g$ and the left multiplication by elements of R^C is defined by

$$(q \otimes r \otimes p)(q' \otimes m) = qr\tau(p \otimes q') \otimes m.$$

The action of $Q \otimes_A -$ on morphisms is the natural one. It is straightforward to see that $Q \otimes_A -$ commutes with the suspension functor T_g and that $Q \otimes_A -$ is a category equivalence. It follows that $Q \otimes_A -$ is a graded equivalence from our discussion in the Introduction. For further details, see [8].

2. Let $\mathscr{C} = (R, R', X, Y, \tau', \mu')$ be a graded context and assume that R and R' are strongly graded. By Lemma 3.1 $C = (A, A', P = X_e, Q = Y_e, \tau = f^{-1}\tau'_e, \mu = f'^{-1}\mu'_e)$ is a Morita context. Let Φ: $R^C = Q \otimes_A R \otimes_A P \to R'$ be defined by $\Phi(q \otimes r \otimes p) = \mu'(qr \otimes p)$.

We first show that Φ is a ring homomorphism

$$\begin{aligned}\Phi((q \otimes r \otimes p)(q' \otimes r' \otimes p')) &= \Phi(q \otimes r\tau(q \otimes q')r' \otimes p')\\ &= \mu'(qr\tau'(p \otimes q')r' \otimes p')\\ &= \mu'(\mu'(qr \otimes p)q'r' \otimes p')\\ &= \mu'(qr \otimes p)\mu'(q'r' \otimes p')\\ &= \Phi(q \otimes r \otimes p)\Phi(q' \otimes r' \otimes p').\end{aligned}$$

It is clear that Φ is graded.

Let $r'_g \in R'_g$. Since μ' is surjective, there are $x_i \in X$ and $y_i \in Y$ ($i = 1, \dots, n$) so that $r'_g = \mu'(\sum_i x_i \otimes y_i)$. We may assume that every x_i is homogeneous (of degree gh_i) and y_i is homogeneous (of degree h_i^{-1}). Then $r'_g = \Phi(\sum_{ij} x_i a'_{ij} \otimes b'_{ij} a_{ik} \otimes b_{ik} y_i)$, where $a_{ik} \in R_{h_i^{-1}}$, $b_{ik} \in R_{h_i}$, $a'_{ij} \in$

$R_{(gh_i)^{-1}}$, $b'_{ij} \in R_{gh_i}$, and $\sum_k a_{ik} b_{ik} = 1 = \sum_j a'_{ij} b'_{ij}$, for every i. This shows that Φ is surjective.

To prove that Φ is injective, we first prove μ' restricts to a bijection μ_g: $Y_g \otimes_A P \to R'_g$, for every $g \in G$. Indeed, since $\mu = f^{-1}\mu_e$, μ_e is a bijection. On the other hand the following diagram is commutative

$$\begin{array}{ccc} R'_{g^{-1}} \otimes_A Y_g \otimes_A P & \xrightarrow{1 \otimes \mu_g} & R'_{g^{-1}} \otimes_A R'_g \\ \Big\downarrow{\scriptstyle M \otimes 1} & & \Big\downarrow{\scriptstyle M} \\ Q \otimes_A P & \xrightarrow{\quad \mu_e \quad} & R'_e \end{array}$$

and so $1 \otimes \mu_g$ is a bijection. Since $R_{g^{-1}}$ is faithfully flat as a right A-module, μ_g is a bijection.

Now assume that $\Phi(\sum_i q_i \otimes r_i \otimes p_i) = 0$. Then $\mu'(\sum_i q_i r_i \otimes p_i) = 0$. We may assume that every r_i is homogeneous and that they all have the same degree, say g. Then $\mu_g(\sum_i q_i r_i \otimes p_i) = 0$ and hence $\sum_i q_i r_i \otimes p_i = 0$ in $X_g \otimes_A P$. Therefore, $\sum_i q_i \otimes r_i \otimes p_i = 0$ in $Q \otimes_A R_g \otimes_A P$.

It only remains to show that the following diagram is commutative

$$\begin{array}{ccc} A' & \xrightarrow{f^C} & R^C \\ \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \Phi} \\ A' & \xrightarrow{f'} & R' \end{array}$$

However $\Phi f^C \mu(q \otimes p) = \Phi(1 \otimes f \otimes 1)(q \otimes p) = \Phi(q \otimes 1 \otimes p) = \mu'(q \otimes p) = f'\mu(q \otimes p)$. ▮

The following corollary is a direct consequence of Theorem 3.2

COROLLARY 3.3. *If R is a strongly graded ring and A is Morita equivalent to R_e, then R is graded equivalent to a strongly graded ring R' such that $R'_e = A$.*

Since a strongly graded ring R is a twisted group ring if and only if the homogeneous components of R are isomorphic to R_e as R_e-bimodules, we have the following corollary.

COROLLARY 3.4. *If R is a twisted group ring, then every ring graded equivalent to R is also a twisted group ring.*

Remark 3.5. We note that the results of [12, 13] are similar to some of the results found in this and the previous section. For example, Theorem 1.1 of [13] compares to Lemmas 2.3 and 2.7, which show that Morita contexts, as well as their action on graded rings, are multiplicative. Moreover, Theorem 3.4 of [12] is similar in flavor to Theorem 3.2, which shows that graded equivalences arise from Morita contexts of the coefficient ring.

4. THE PICARD GROUP ACTING ON GRADED RINGS

The philosophy of this section is that, using Corollary 3.3, we have reduced the coefficient ring (via graded equivalence) as much as possible and, consequently, we now fix the coefficient ring A. That is, we are interested in all the G-graded rings R, with $R_e = A$, and our goal is to identify the graded equivalence classes. In this case, the action of Morita contexts on graded rings reduces to an action of $\mathrm{Pic}(A)$ on graded rings and the graded equivalent classes turn out to be the orbits under this action. For technical reasons, it is more convenient to consider all the G-graded rings R such that R_e is isomorphic to A. This does not change our study but rather simplifies the proofs.

Notation 4.1. $\mathrm{Pic}(A)$ denotes the Picard group of A. If P is an invertible A-bimodule, $[P] \in \mathrm{Pic}(A)$ denotes the class of invertible A-bimodules isomorphic to P.

Given an automorphism σ of A, A^σ is the A-bimodule defined as follows: $A^\sigma = A$ as a left A-module and right multiplication is given by $p \cdot a = p\sigma(a)$ $(a \in A,\ p \in A^\sigma)$. Similarly, we define ${}^\sigma A$. It is well known that A^σ is invertible and that the map $\alpha \mapsto [A^\sigma]$ defines a group homomorphism $\mathrm{Aut}(A) \to \mathrm{Pic}(A)$ whose kernel is $\mathrm{Inn}(A)$, the set of inner automorphisms of A. Accordingly we identify $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ with the image of this map.

Now let R be a G-graded ring. Then $\mathrm{Pic}^{\mathrm{gr}}(R)$ denotes the group of graded isomorphism classes of invertible R-bimodules P that occur in a graded context (R, R, P, Q, τ, μ).

Let (R, f) be an (A, G)-graded ring. If σ is a graded automorphism of R, then $R^\sigma \in \mathrm{Pic}^{\mathrm{gr}}(R)$. Moreover, σ induces an automorphism $\sigma_e = f^{-1}\sigma f$ in A. We denote the set of graded automorphisms of R by $\mathrm{Aut}^{\mathrm{gr}}(R)$ and the set of inner automorphisms of R induced by an invertible element of A by $\mathrm{Inn}_A(R)$. Set

$$\mathrm{Pic}_R(A) = \{[P] \in \mathrm{Pic}(A) : [P] = [Q_e] \text{ for some } Q \in \mathrm{Pic}^{\mathrm{gr}}(R)\},$$

$$\mathrm{Aut}_R(A) = \{\sigma \in \mathrm{Aut}(A) : f^{-1}\sigma f \text{ extends to a graded automorphism of } R\},$$

and

$$\mathrm{Out}_R(A) = \mathrm{Aut}_R(A)/\mathrm{Inn}(A).$$

If (R, f) is a strongly (A, G)-graded ring, then G acts on the center of A via the Miyashita action $\sigma\colon G \to \mathrm{Aut}(Z(A))$. That is, if $g \in G$ and $a \in Z(A)$, then a^{σ_g} is defined by

$$ar_g = r_g a^{\sigma_g}, \qquad r_g \in R_g.$$

In this paper, the coboundary, cocycle, and cohomology groups $B^n(G, U(Z(A)))$, $Z^n(G, U(Z(A)))$, and $H^n(G, U(Z(A)))$ are considered with respect to this action.

PROPOSITION 4.2. *If (R, f) is a strongly G-graded ring, there is a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & B^1(G, U(Z(A))) & \xrightarrow{\alpha_1} & \mathrm{Inn}_A(R) & \xrightarrow{\beta_1} & \mathrm{Inn}(A) & \longrightarrow & 1 \\
 & & j\downarrow & & j\downarrow & & j\downarrow & & \\
1 & \longrightarrow & Z^1(G, U(Z(A))) & \xrightarrow{\alpha_2} & \mathrm{Aut}^{\mathrm{gr}}(R) & \xrightarrow{\beta_2} & \mathrm{Aut}_R(A) & \longrightarrow & 1 \\
 & & T\downarrow & & T\downarrow & & T\downarrow & & \\
1 & \longrightarrow & H^1(G, U(Z(A))) & \xrightarrow{\alpha_3} & \mathrm{Pic}^{\mathrm{gr}}(R) & \xrightarrow{\beta_3} & \mathrm{Pic}_R(A) & \longrightarrow & 1 \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

with exact rows and columns of group homomorphisms, and all the vertical homomorphisms are canonical.

Proof. We first define the morphisms α and β. If $c \in Z^1(G, U(Z(A)))$, then $\alpha_2(c)\colon R \to R$ is given by $\alpha_2(c)(r_g) = r_g c(g)$ ($g \in G$ and $r_g \in R_g$). α_1 is the restriction of α_2 to $B^1(G, U(Z(A)))$. So the upper left square is commutative and then α_3 is the only group homomorphism making the lower left square commutative.

β_1 and β_2 are the restriction maps and β_3 is the map given by $\beta_3([P]) = [P_e]$. Note that, by Lemma 3.1, β_3 maps $\mathrm{Pic}^{\mathrm{gr}}(R)$ into $\mathrm{Pic}(A)$.

Commutativity of the diagram is straightforward to check. We only have to prove that β_3 is a group homomorphism and that the rows are exact.

For the remainder of the proof g denotes an arbitrary element of G and $x_i \in R_{g^{-1}}$ and $y_i \in R_g$ are such that $\sum_i x_i y_i = 1$.

Let $P, P' \in \mathrm{Pic}^{\mathrm{gr}}(R)$ and consider the maps $\Phi\colon P_e \otimes_A P'_e \to (P \otimes_R P')_e$ the inclusion map and $\Psi\colon (P \otimes_R P')_e \to P_e \otimes_A P'_e$ given by $\Psi(p \otimes p') = \sum_i px_i \otimes y_i p'$, for $p \otimes p' \in P_g \otimes P'_g$.

We have to check that Ψ is well defined. Let $a_i \in R_{g^{-1}}$ and $b_i \in R_g$ such that $\sum_i a_i b_i = 1$. Let $p \times p' \in P_g \otimes P'_g$. Then

$$\sum_i px_i \otimes y_i p' = \sum_i p \sum_j a_j b_j x_i \otimes y_i p'$$
$$= \sum_i \sum_j pa_j \otimes b_j x_i y_i p'$$
$$= \sum_j pa_j \otimes b_j p'.$$

By straightforward computations, one shows that Φ and Ψ are inverse A-bimodule isomorphisms. This shows that β_3 is a group homomorphism.

We leave the details of checking that the first row is exact. For the second, let $c \in \mathrm{Ker}\, \alpha_2$ and note that $r_g c(g) = r_g$, for every $r_g \in R_g$. Since R_g is faithful as a right A-module, $c = 1$. Thus α_2 is injective. It is clear that $\mathrm{Im}\, \alpha_2 \subseteq \mathrm{Ker}\, \beta_2$. Let $\sigma \in \mathrm{Ker}\, \beta_2$. We claim that, for every $g \in G$, there is a unit $c(g) \in A$, such that $\sigma(r_g) = r_g c(g)$, for every $r_g \in R_g$. Let $c(g) = f^{-1}(\sum_i x_i \sigma(y_i))$. Then $\sigma(r_g) = \sigma(r_g \sum_i x_i y_i) = r_g c(g)$. If $a \in A$, then $r_g ac(g) = \sigma(r_g a) = \sigma(r_g)a = r_g c(g)a$. Therefore $c(g)$ is central and, since $r_g \mapsto r_g c(g)$ is an automorphism of ${}_A R_g$, $c(g)$ is invertible in A. Moreover, for every $r_g \in R_g$ and $s_h \in R_h$, $r_g s_h c(gh) = r_g c(g) s_h c(h) = r_g s_h c(g)^{\sigma_h} c(h)$ and hence $c \in Z^1(G, U(Z(A)))$. Thus $\sigma \in \mathrm{Im}\, \alpha_2$.

For the third row, let $c \in Z^1(G, U(Z(A)))$. If $c + B^1(G, U(Z(A))) \in \mathrm{Ker}\, \alpha_3$, then $\alpha_2(c) \in \mathrm{Inn}_A(R)$. Thus, there exists u a unit in $Z(A)$, such that $r_g c(g) = ur_g u^{-1} = r_g u^{\sigma_g} u^{-1}$, for every $g \in G$ and every $r_g \in R_g$. Consequently, $c(g) = u^{\sigma_g} u^{-1}$ and hence $c \in B^1(G, U(Z(A)))$. This shows that α_3 is injective. Since $\mathrm{Im}\, \alpha_2 = \mathrm{Ker}\, \beta_2$ and $T\colon Z^1(G, U(Z(A))) \to H^1(G, U(Z(A)))$ is surjective, $\mathrm{Im}\, \alpha_3 \subseteq \mathrm{Ker}\, \beta_3$. Assume that $[P] \in \mathrm{Ker}\, \beta_3$; this means that P_e is isomorphic to A as an A-bimodule and hence $P \simeq R \otimes_A P_e$ is isomorphic to R as a graded left R-module. It follows that $[P] = [A^\sigma]$ for some $\sigma \in \mathrm{Ker}\, \beta_2$. Using that the middle exact sequence is exact, it is now easy to prove that $[P] \in \mathrm{Im}\, \alpha_3$. ∎

Notation 4.3. Set

$$\mathrm{Gr}(A, G) = \{[R, F] : (R, f) \text{ is an } (A, G)\text{-graded ring}\}$$

and

$$\mathrm{StGr}(A,G) = \{[R,f] \in \mathrm{Gr}(A,f) : R \text{ is strongly graded}\}.$$

Every element $[P] \in \mathrm{Pic}(A)$ canonically defines a Morita context

$$C(P) = \left(A, A, P, P^{-1} = \mathrm{Hom}({}_A P, {}_A A), \tau, \mu\right),$$

where $\tau(p \otimes f) = (p)f$ and $p'\mu(f \otimes p) = (p')fp$. Moreover if P and P' are two invertible A-modules then $[P] \simeq [P']$ if and only if there is an isomorphism ϕ: $C(P) \simeq C(P')$ of Morita contexts so that $\alpha = \beta = 1$.

By the results from the previous section, we define the following action of $\mathrm{Pic}(A)$ on $\mathrm{Gr}(A,G)$:

$$[R,f]^{[P]} = [R,f]^{C(P)}.$$

It is clear that $\mathrm{StGr}(A,G)$ is invariant under this action.

We can use Theorem 3.2 to compute the orbit of an $[R,f] \in \mathrm{StGr}(A,G)$.

PROPOSITION 4.4. *Let* $[R,f] \in \mathrm{StGr}(A,G)$.

1. *The orbit of* $[R,f]$ *in* $\mathrm{StGr}(A,G)$ *by* $\mathrm{Pic}(A)$ *is*

$$\mathrm{Orb}_{\mathrm{Pic}(A)}([R,f]) = \{[R',f'] \in \mathrm{StGr}(A,G) : R' \textit{ is graded equivalent to } R\}.$$

2. *The stabilizer of* $[R,f]$ *by* $\mathrm{Pic}(A)$ *is* $\mathrm{Pic}_R(A)$.
3. *There is a one-to-one correspondence between the set of isomorphism classes of strongly* (A,G)*-graded rings, graded Morita equivalent to* R, *and the set of cosets* $\mathrm{Pic}(A)/\mathrm{Pic}_R(A)$.

Proof. (1) This is a consequence of Theorem 3.2.

(2) Theorem 3.2 shows that $\mathrm{Pic}_R(A)$ is contained in the stabilizer of (R,f) in $\mathrm{Pic}(A)$.

Assume now that $[P]$ belongs to the stabilizer of (R,f) in $\mathrm{Pic}(A)$. Set $C = C(P)$ and Φ: $R \to R^C$ a graded isomorphism making the diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & R \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \Phi} \\ A & \xrightarrow{f^C} & R^C \end{array}$$

Let Ψ be the composition $P \otimes_A R \xrightarrow{1 \otimes \Phi} P \otimes_A Q \otimes_A R \otimes_A P \xrightarrow{\tau \otimes 1} R \otimes_A P$. Using this isomorphism, we can endow $X = R \otimes_A P$ with an R-bimodule structure by defining the left multiplication in the canonical way and the right multiplication by $xr = \Psi(\Psi^{-1}(x)r)$. We check that these two multiplications make $R \otimes_A P$ into an R-bimodule. Assume that $\tau^{-1}(1) = \sum_i p_i \otimes q_i$ and that $\Phi(r') = \sum_j x_j \otimes s_j \otimes y_j$. Then

$$\begin{aligned}
(r \otimes p)r' &= \Psi\big(\Psi^{-1}(r \otimes p)r'\big) \\
&= \Psi\left((1 \otimes \Phi^{-1})\left(\sum_i p_i \otimes q_i \otimes r \otimes p\right)r'\right) \\
&= \Psi\left(\sum_i p_i \otimes \Phi^{-1}(q_i \otimes r \otimes p)r'\right) \\
&= (\tau \otimes 1)\left(\sum_i p_i \otimes (q_i \otimes r \otimes p)\Phi(r')\right) \\
&= (\tau \otimes 1)\left(\sum_i p_i \otimes (q_i \otimes r \otimes p)\sum_j x_j \otimes s_j \otimes y_j\right) \\
&= (\tau \otimes 1)\left(\sum_i p_i \otimes \sum_j q_i \otimes r\tau(p \otimes x_j)s_j \otimes y_j\right) \\
&= \sum_j \left(\sum_i \tau(p_i \otimes q_i)r\tau(p \otimes x_j)s_j \otimes y_j\right) \\
&= \sum_j \big(r\tau(p \otimes x_j)s_j \otimes y_j\big) \\
&= r\big((1 \otimes p)r'\big).
\end{aligned}$$

Therefore $r((r_1 \otimes p)r') = r(r_1((1 \otimes p)r')) = (rr_1)((1 \otimes p)r') = (rr_1 \otimes p)r' = (rr_1 \otimes p)r' = (r(r_1 \otimes p))r'$. We make X into a graded left R-module by using the grading of R. In a similar way, we make $P \otimes_A R$ into a graded right R-module. Since Ψ is a graded isomorphism of abelian groups, X and $P \otimes_A R$ are graded R-bimodules, so that Ψ is a graded isomorphism of graded R-bimodules. Similarly, $Y = Q \otimes_A R$ and $R \otimes_A Q$ are isomorphic R-bimodules.

Now $X \otimes_R Y \simeq P \otimes_A R \otimes_R Q \otimes_A R \simeq P \otimes_A R \otimes_R R \otimes_A Q \simeq P \otimes_A R \otimes_A Q \simeq P \otimes_A Q \otimes_A R \simeq A \otimes_A R \simeq R$. Similarly, $Y \otimes_R X \simeq R$ and these isomorphisms are graded. Since $[P] \in \mathrm{Pic}(A)$ and $P \otimes_A R \in \mathrm{Pic}^{\mathrm{gr}}(R)$, we have shown that $[P] \in \mathrm{Pic}_R(A)$. Moreover $P \simeq X_e$ as A-bimodules.

(3) This follows from (1) and (2). ∎

The following fact is elementary: Every element of $\mathrm{Gr}(A, G)$ has a representative of the form $[R, 1]$, i.e., a representative, where $R_e = A$ and $f = 1$. Thus we have the following corollary.

COROLLARY 4.5. *Given a ring A, there is a one-to-one correspondence between the graded equivalence classes of strongly G-graded rings R such that $R_e = A$ and the orbits of* $\mathrm{Pic}(A)$ *in* $\mathrm{StGr}(A, G)$.

The above descriptions of the orbit require strongly graded rings. However, when we restrict our attention to the action induced by $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$, we can describe the orbits even for non-strongly graded rings. The key observation is that the homomorphisms β_1 and β_2 can be defined even if R is not strongly graded. In that case, we set $\mathrm{Out}^{\mathrm{gr}}(R) = \mathrm{Aut}^{\mathrm{gr}}(R)/\mathrm{Inn}_A(R)$ and so there is a homomorphism β_3': $\mathrm{Out}^{\mathrm{gr}}(R) \to \mathrm{Out}_R(A)$ making the diagram (with exact rows and columns)

$$\begin{array}{ccccc}
1 & & 1 & & \\
\downarrow & & \downarrow & & \\
\mathrm{Inn}_A(R) & \xrightarrow{\beta_1} & \mathrm{Inn}(A) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\mathrm{Aut}^{\mathrm{gr}}(R) & \xrightarrow{\beta_2} & \mathrm{Aut}_R(A) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\mathrm{Out}^{\mathrm{gr}}(R) & \xrightarrow{\beta_3'} & \mathrm{Out}_R(A) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
1 & & 1 & &
\end{array}$$

commutative.

PROPOSITION 4.6. *Let* $[R, f] \in \mathrm{Gr}(A, G)$.

1. *The orbit of* $[R, f]$ *by* $\mathrm{Out}(A)$ *is*

$$\mathrm{Orb}_{\mathrm{Out}(A)}([R, f]) = \{[R', f'] \in \mathrm{Gr}(A, G) : R' \textit{ is graded isomorphic to } R\}.$$

2. *The stabilizer of* $[R, f]$ *by* $\mathrm{Out}(A)$ *is* $\mathrm{Aut}_R(A)$.

3. *There is a one-to-one correspondence between the set of isomorphism classes of* (A, G)*-graded rings, graded A-isomorphic to R, and the set of cosets* $\mathrm{Out}(A)/\mathrm{Out}_R(A)$ (*or equivalently* $\mathrm{Aut}(A)/\mathrm{Aut}_R(A)$).

Proof. (1) Let σ be an automorphism of A. Then $C(A^{\sigma})$ (see Notation 4.3) is isomorphic to the following Morita context $C = (A, A, A^{\sigma}, {}^{\sigma}A, \tau,$

μ_σ), where $\tau(a \otimes b) = ab$ and $\mu_\sigma(a \otimes b) = \sigma(ab)$. The map

$$
\begin{aligned}
{}^{\sigma}A \otimes_A R \otimes_A A^{\sigma} &\xrightarrow{\Psi^{\sigma}} R \\
a \otimes r \otimes b &\mapsto arb
\end{aligned}
$$

is a graded ring isomorphism. This proves that $\mathrm{Orb}_{\mathrm{Out}(A)}([R, f])$ is embedded in the graded isomorphism class of R.

Assume now that (R', f') is an (A, G)-graded ring and $\Phi\colon R \to R'$ is a graded ring isomorphism. Let $\sigma = f'^{-1}\Phi_e f \in \mathrm{Aut}(A)$. Then the following diagram is commutative and consists of ring isomorphisms

$$
\begin{array}{ccccc}
A & \xrightarrow{\mu_\sigma^{-1}} & {}^{\sigma}A \otimes_A A \otimes_A A^{\sigma} & \xrightarrow{1 \otimes f \otimes 1} & {}^{\sigma}A \otimes_A R_e \otimes_A A^{\sigma} \\
{\scriptstyle \sigma^{-1}}\downarrow & & & & \downarrow{\scriptstyle \Psi_e^{\sigma}} \\
A & & \xrightarrow{\qquad f \qquad} & & R_e \\
{\scriptstyle \sigma}\downarrow & & & & \downarrow{\scriptstyle \Phi_e} \\
A & & \xrightarrow{\qquad f' \qquad} & & R'_e
\end{array}
$$

Therefore, $[R', f'] = [R, f]^{\sigma}$ and so (1) holds.

(2) The automorphism σ stabilizes $[R, f]$ if and only if there is a graded isomorphism $\Phi\colon R \to R^{C}$ which makes the diagram

$$
\begin{array}{ccccc}
A & & \xrightarrow{\qquad f \qquad} & & R \\
{\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle \Phi} \\
A & \xrightarrow{\mu_\sigma^{-1}} & {}^{\sigma}A \otimes_A A \otimes_A A^{\sigma} & \xrightarrow{1 \otimes f \otimes 1} & {}^{\sigma}A \otimes_A R \otimes_A A^{\sigma}
\end{array}
$$

commutative. But this will hold if and only if the diagram

$$
\begin{array}{ccccc}
A & & \xrightarrow{\qquad f \qquad} & & R_e \\
{\scriptstyle \sigma^{-1}}\downarrow & & & & \downarrow{\scriptstyle \Phi_e} \\
A & \xrightarrow{M^{-1}} & {}^{\sigma}A \otimes_A A \otimes_A A^{\sigma} & \xrightarrow{1 \otimes f \otimes 1} & {}^{\sigma}A \otimes_A R_e \otimes_A A^{\sigma} \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} \\
& & A & \xrightarrow{\quad f \quad} & R_e
\end{array}
$$

is commutative. Thus σ stabilizes $[R, f]$ if and only if $f\sigma f^{-1}$ extends to a graded automorphism of R, in other words if $\sigma \in \mathrm{Aut}_R(A)$.

(3) is an consequence of (1) and (2). ∎

COROLLARY 4.7. *Given a ring A, there is one-to-one correspondence between the graded isomorphism classes of G-graded rings R, with $R_e = A$, and the orbits of* $\mathrm{Out}(A)$ (*or* $\mathrm{Aut}(A)$) *in* $\mathrm{Gr}(A, G)$.

Our last remark of this section shows that Corollary 4.7 is a generalization of [11, Proposition 2].

Remark 4.8. By [14, Sect. I.1.3], the isomorphism classes of strongly graded rings R such that $R_e = A$ can be given in terms of a group homomorphism $G \to \mathrm{Pic}(A)$ ($g \mapsto [R_g]$) and a set of bimodule homomorphisms $R_g \otimes_A R_h \to R_{gh}$ satisfying certain conditions. Crossed products are determined by parameter sets. In the notation of [11], two crossed products R and R', with $R_e = R'_e = A$, are A-isomorphic if and only if $[R, 1] = [R', 1]$. Further, two parameter sets define A-isomorphic crossed products if and only if they are equivalent in the sense of [11]. Let $\mathrm{CP}(A, G)$ be the subset of $\mathrm{StGr}(A, G)$ formed by the classes that contain a crossed product. The action of $\mathrm{Aut}(A)$ (or $\mathrm{Out}(A)$) on $\mathrm{StGr}(A, G)$ restricts to an action on $\mathrm{CP}(A, G)$ and hence this action can be translated to an action on the set of equivalence classes of parameter sets. Therefore Corollary 4.7 generalizes [11, Proposition 2].

5. STRONGLY GRADED RINGS WITH ISOMORPHIC COMPONENTS

The aim of this section is to show that graded equivalent strongly graded rings with the same homogeneous components are not necessarily graded isomorphic; see Theorem 5.8. We begin with same notation.

Notation 5.1. Fix a unital ring A, a group G, and a strongly (A, G)-graded ring (R, f). Let $\mathrm{Pic}_{(R,f)}(A)$ denote the centralizer of $\{[R_g] : g \in G\}$; i.e.,

$$\mathrm{Pic}_{(R,f)}(A) = \left\{[P] \in \mathrm{Pic}(A) : [P]\left[R_g\right] = \left[R_g\right][P] \text{ for every } g \in G\right\}.$$

Let $\mathrm{StGr}_{(R,f)}(A, G) = \{[R', f'] \in \mathrm{StGr}(A, G) : [R_g] = [R'_g]$ for every $g \in G\}$.

The next lemma is an obvious consequence of Theorem 3.2.

LEMMA 5.2. *There is a map* $\Sigma\colon \mathrm{Pic}_{(R,f)}(A) \to \mathrm{StGr}_{(R,f)}(A, G)$ *given by* $\Sigma([P]) = [R, f]^P$, *whose image is the set of the* $[R', f'] \in \mathrm{StGr}_{(R,f)}(A, G)$ *such that* R' *is graded equivalent to* R.

The significance of Lemma 5.2 is that $\mathrm{Pic}_{(R,f)}(A)/\mathrm{Pic}_R(A)$ parametrizes the class of strongly graded rings that are graded equivalent to R but not graded A-isomorphic to R and that have homogeneous components A-isomorphic to those of R. However, there is another well-documented way to parametrize $\mathrm{StGr}_{(R,f)}(A, G)$ via the cohomology group $H^2(G, U(Z(A)))$.

DEFINITION 5.3. Given $c \in Z^2(G, U(Z(A)))$, R^c denotes the strongly graded ring such that $R^c = R$ as an additive group with the multiplication in R^c defined by

$$r_g \cdot s_h = r_g s_h c(g,h), \qquad g,h \in G, r_g \in R_g, s_h \in R_h.$$

Recall that we have reserved the letter M to denote multiplication maps. To distinguish different multiplication maps induced by an (A,G)-graded ring (R,f), we denote $M_{g,h} = M\colon R_g \otimes_A R_h \to R_{gh}$.

THEOREM 5.4 [14, Theorem A.I.3.16]. *Let (R,f) be a strongly G-graded ring. There is a bijection $\Psi\colon \mathrm{StGr}_{(R,f)}(A,G) \to H^2(G,U(Z(A)))$ given as follows: $\Psi^{-1}(\bar{c}) = [R^c, f]$ and $\Psi([R', f']) = \overline{c_{(R',f')}}$ (the homology induced by $c_{(R',f')}$) where the following map is right multiplication by $c_{(R',f')}(g,h)$, for every $g,h \in G$,*

$$R_{gh} \xrightarrow{M_{g,h}^{-1}} R_g \otimes_A R_h \xrightarrow{\delta_g \otimes \delta_h} R'_g \otimes_A R'_h \xrightarrow{M_{g,h}} R'_{gh} \xrightarrow{\delta_{gh}^{-1}} R_{gh}$$

and $\delta_g\colon R_g \to R'_g$ is an isomorphism of A-bimodules for every $G \in G$.

The key to the main result of this section (Theorem 5.8) requires an example computed in [5] and the map ϕ defined in [6]. Recall that if R is a strongly graded ring, then $\sigma\colon G \to \mathrm{Aut}(Z(R_e))$ denotes the Miyashita action induced by R (see Notation 4.1).

Notation 5.5. Every $[P] \in \mathrm{Pic}(A)$ defines an automorphism α_P of $Z(A)$ defined by $pz = \alpha_P(z)p$ for all $p \in P$ and $z \in Z(A)$. This gives rise to an action $\alpha\colon \mathrm{Pic}(A) \to \mathrm{Aut}(Z(A))$ via $[P] \mapsto \alpha_P$. If $[P] \in \mathrm{Pic}_{(R,f)}(A)$, then α_P commutes with σ_g for every $g \in G$. Therefore, P induces an automorphism β_P of $H^2(G, Z(A))$ and so we have another action β: $\mathrm{Pic}_{(R,f)}(A) \to \mathrm{Aut}(H^2(G,Z(A)))$ via $[P] \mapsto \beta_P$. We denote the image of $x \in \mathrm{Pic}_{(R,f)}(A)$ under β by β_x.

PROPOSITION 5.6 [5]. *There is an exact sequence*

$$1 \to H^1(G,U(Z(A))) \xrightarrow{\alpha_3} \mathrm{Pic}^{\mathrm{gr}}(R) \xrightarrow{\beta_3} \mathrm{Pic}_{(R,f)}(A) \xrightarrow{\phi} H^2(G,U(Z(R_e))),$$

where α_3 and β_3 are the group homomorphisms from Proposition 4.2 and for every $x, y \in \mathrm{Pic}_{(R,f)}(A)$, $\phi(xy) = \phi(x)\beta_x(\phi(y))$.

Now, to use [5], we must show that the maps ϕ and $\Psi \circ \Sigma\colon \mathrm{Pic}_{(R,f)}(A) \to H^2(G,U(Z(A)))$ are strongly related. This is precisely what the next lemma does.

LEMMA 5.7. *$\phi \circ (-)^{-1} = \Psi \circ \Sigma$; i.e., the following diagram*

$$\begin{array}{ccc} \mathrm{Pic}_{(R,f)}(A) & \xrightarrow{(-)^{-1}} & \mathrm{Pic}_{(R,f)}(A) \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathrm{StGr}_{(R,f)}(A,G) & \xrightarrow{\Psi} & H^2(G, U(Z(A))) \end{array}$$

is commutative ($(-)^{-1}$ *denotes the inverse map*).

Proof. Let $[P] \in \mathrm{Pic}_{(R,f)}(A)$ and (A, A, P, Q, τ, μ) be a Morita context. We need to show that $\phi([P]) = (\Psi\Sigma)([Q])$.

Fix $g, h \in G$.

There is an A-bimodule isomorphism $\gamma_g\colon P \to R_{g^{-1}} \otimes_A P \otimes_A R_g$. Following the proof of Proposition 5.6 in [5], one can obtain $\phi([P])$ in terms of the map γ_g. More precisely, $\phi([P]) = \bar{c}$ where $\gamma_{gh}^{-1}(M_{h^{-1},g^{-1}} \otimes 1 \otimes M_{g,h})(1 \otimes \gamma_g \otimes 1)\gamma_h$ is left multiplication by $c(g,h)$.

Let $\delta_g = (\rho_g \otimes 1)(1 \otimes \gamma_g \otimes 1)(1 \otimes \tau^{-1})$ where $\rho_g\colon R_g \otimes_A R_{g^{-1}} \otimes P \to P$ is given by $\rho_g(r_g \otimes s_{g^{-1}} \otimes p) = (r_g s_{g^{-1}})p$.

By Theorem 5.4, $(\Psi \circ \Sigma)([Q]) = \bar{d}$ where, for every $g, h \in G$, $\delta_{gh}^{-1} M_{g,h}(\delta_g \otimes \delta_h) M_{g,h}^{-1}$ is right multiplication by $d(g,h)$. Fix $r \in R_g$ and $s \in R_h$. We show that $rsd(g,h) = rsc(g,h)$ and so it will follow that $c = d$.

We are going to use a Sweedler-like notation. For $p \in P$, let $\gamma_g(p) \in R_{g^{-1}} \otimes_A P \otimes_A R_g$ be denoted by

$$\gamma_g(p) = \sum_{\gamma_g} p_{(g^{-1})} \otimes p_{(g^0)} \otimes p_{(g)}.$$

Set $\tau^{-1}(1) = \sum_i p_i \otimes q_i \in P \otimes_A Q$. Then

$$\begin{aligned} \delta_g(r) &= (\rho_g \otimes 1)(1 \otimes \gamma_g \otimes 1)(1 \otimes \tau^{-1})(r) \\ &= (\rho_g \otimes 1)(1 \otimes \gamma_g \otimes 1)\Big(\sum_i r \otimes p_i \otimes q_i\Big) \\ &= (\rho_g \otimes 1)\Big(\sum_{i,\gamma_g} r \otimes p_{i_{(g^{-1})}} \otimes p_{i_{(g^0)}} \otimes p_{i_{(g)}} \otimes q_i\Big) \\ &= \sum_{i,\gamma_g} r p_{i_{(g^{-1})}} p_{i_{(g^0)}} \otimes p_{i_{(g)}} \otimes q_i. \end{aligned}$$

Similarly,

$$\delta_h(s) = \sum_{j,\gamma_h} s p_{j_{(h^{-1})}} p_{j_{(h^0)}} \otimes p_{j_{(h)}} \otimes q_j.$$

Thus

$$M_{g,h}(\delta_g \otimes \delta_h)M_{g,h}^{-1}(rs)$$

$$= M_{g,h}\left(\left(\sum_{i,\gamma_g} rp_{i_{(g-1)}}p_{i_{(g^0)}} \otimes p_{i_{(g)}} \otimes q_i\right) \otimes \left(\sum_{j,\gamma_h} sp_{j_{(h-1)}}p_{j_{(h^0)}} \otimes p_{j_{(h)}} \otimes q_j\right)\right)$$

$$= \sum_{i,\gamma_g,j,\gamma_h} rp_{i_{(g-1)}}p_{i_{(g^0)}} \otimes p_{i_{(g)}}\mu\left(q_i \otimes sp_{j_{(h-1)}}p_{j_{(h^0)}}\right)p_{j_{(h)}} \otimes q_j$$

$$= \sum_{i,\gamma_g,j,\gamma_h} rp_{i_{(g-1)}}p_{i_{(g^0)}} \otimes p_{i_{(g)}}a_i p_{j_{(h)}} \otimes q_j, \tag{5.1}$$

where $a_i = \mu(q_i \otimes sp_{j_{(h-1)}}p_{j_{(h^0)}})$. Note that

$$\sum_i p_i a_i = \sum_i \tau(p_i \otimes q_i)sp_{j_{(h-1)}}p_{j_{(h^0)}} = sp_{j_{(h-1)}}p_{j_{(h^0)}}. \tag{5.2}$$

Let $1 = \sum_k x_k \otimes y_k$, where $x_k \in R_h$ and $y_k \in R_{h^{-1}}$. Since $\gamma_{gh}^{-1}(M_{h^{-1},g^{-1}} \otimes 1 \otimes M_{g,h})(1 \otimes \gamma_g \otimes 1)\gamma_h$ is left multiplication by $c(g,h)$, one has

$$\gamma_{gh}^{-1}\left(\sum_{\gamma_g} y_k p_{i_{(g-1)}} \otimes p_{i_{(g^0)}} \otimes p_{i_{(g)}}ap_{j_{(h)}}\right)$$

$$= \gamma_{gh}^{-1}(M_{h^{-1},g^{-1}} \otimes 1 \otimes 1)\left(\sum_{\gamma_g} y_k \otimes p_{i_{(g-1)}} \otimes p_{i_{(g^0)}} \otimes p_{i_{(g)}} \otimes ap_{j_{(h)}}\right)$$

$$= \gamma_{gh}^{-1}(M_{h^{-1},g^{-1}} \otimes 1 \otimes 1)(1 \otimes \gamma_g \otimes 1)\left(y_k \otimes p_i \otimes ap_{j_{(h)}}\right)$$

$$= c(g,h)\gamma_h^{-1}\left(y_k \otimes p_i \otimes ap_{j_{(h)}}\right). \tag{5.3}$$

Thus

$$\left(1 \otimes \gamma_{gh}^{-1} \otimes 1\right)\left(\rho_{gh}^{-1} \otimes 1\right)M_{g,h}(\delta_g \otimes \delta_h)M_{g,h}^{-1}(rs)$$

$$= \left(1 \otimes \gamma_{gh}^{-1} \otimes 1\right)\left(\rho_{gh}^{-1} \otimes 1\right)\left(\sum_{i,\gamma_g,j,\gamma_h} rp_{i_{(g-1)}}p_{i_{(g^0)}} \otimes p_{i_{(g)}}ap_{j_{(h)}} \otimes q_j\right) \qquad \text{(by 5.1)}$$

$$= \left(1 \otimes \gamma_{gh}^{-1} \otimes 1\right)\left(\sum_{i,\gamma_g,j,\gamma_h,k} rx_k \otimes y_k p_{i_{(g-1)}} \otimes p_{i_{(g^0)}} \otimes p_{i_{(g)}}ap_{j_{(h)}} \otimes q_j\right)$$

$$= \sum_{i,j,\gamma_h,k} rx_k \otimes c(g,h)\gamma_h^{-1}\left(y_k \otimes p_i \otimes ap_{j_{(h)}}\right) \otimes q_j \qquad \text{(by 5.3)}$$

$$= \sum_{i,j,\gamma_h,k} rx_k \otimes c(g,h)\gamma_h^{-1}\big(y_k \otimes p_i \otimes ap_{j_{(h)}}\big) \otimes q_j$$

$$= \sum_{j,\gamma_h,k} rx_k \otimes c(g,h)\gamma_h^{-1}\big(y_k \otimes sp_{j_{(h^{-1})}} p_{j_{(h^0)}} \otimes p_{j_{(h)}}\big) \otimes q_j \qquad \text{(by 5.2)}$$

$$= \sum_{j,\gamma_h,k} rx_k \otimes c(g,h)\gamma_h^{-1}\big(y_k sp_{j_{(h^{-1})}} \otimes p_{j_{(h^0)}} \otimes p_{j_{(h)}}\big) \otimes q_j$$

$$= \sum_{j,\gamma_h,k} rx_k \otimes y_k sc(g,h)\gamma_h^{-1}\big(p_{j_{(h^{-1})}} \otimes p_{j_{(h^0)}} \otimes p_{j_{(h)}}\big) \otimes q_j$$

$$= \sum_{j,\gamma_h,k} rx_k y_k sc(g,h) \otimes \gamma_h^{-1}\big(p_{j_{(h^{-1})}} \otimes p_{j_{(h^0)}} \otimes p_{j_{(h)}}\big) \otimes q_j$$

$$= \sum_j rsc(g,h) \otimes p_j \otimes q_j$$

$$= rsc(g,h) \otimes \tau^{-1}(1).$$

Finally

$$\begin{aligned} rsd(g,h) &= \delta_{gh}^{-1} M_{g,h}(\delta_g \otimes \delta_h) M_g^{-1}(rs) \\ &= (1 \otimes \tau)\big(1 \otimes \gamma_{gh}^{-1} \otimes 1\big)\big(\rho_{gh}^{-1} \otimes 1\big) M_{g,h}(\delta_g \otimes \delta_h) M_g^{-1}(rs) \\ &= (1 \otimes \tau)\big(c(g,h) rs \otimes \tau^{-1}(1)\big) \\ &= rsc(g,h). \end{aligned}$$

▮

Now we are ready for the main theorem of this section

THEOREM 5.8. *There are strongly G-graded rings R and R′ satisfying*

1. $R_e = R'_e$.
2. *For every* $g \in G$, R_g *and* R'_g *are isomorphic as* R_e*-bimodules*.
3. *R and R′ are graded equivalent*.
4. *R and R′ are not graded isomorphic*.

Proof. By [5], there is a strongly graded ring R (actually a skew group ring) for which the map ϕ is not trivial. By Lemma 5.7, $\Psi\Sigma \neq 1$ which together with Lemma 5.2 proves the theorem. ▮

A specific example of the above theorem now follows:

EXAMPLE 5.9. Let A be a unital ring, G a group, and α: $G \to \mathrm{Aut}(A)$ a group homomorphism. Let $w \in \mathrm{Aut}(A)$ and β: $G \to \mathrm{Aut}(A)$ be given by $\beta(g) = w^{-1}\alpha(g)w$. Let $R = A_\alpha G$ the skew group ring associated to the action α and $R' = A_\beta G$ the skew group ring associated to the action β. Then $[R', 1] = [R, 1]^{A^w}$, so R and R' are graded equivalent. Moreover, if $[A^w] \in \mathrm{Pic}_{(R,f)}(A)$, then $[R'_g] = [R_g]$ for every $g \in G$. However the following example from [5] shows that they may be non-graded isomorphic.

Let $K = \mathbf{Q}(\epsilon, \alpha)$ where ϵ is a primitive third root of unity and $\alpha^3 = d \in \mathbf{Q} - \mathbf{Q}(\epsilon)^3$ (for example, let α be the real cube root of 3). Let $G = \mathrm{Gal}(K, \mathbf{Q})$ be the Galois group of this extension. Then $G = \langle \pi, \tau \rangle \simeq S_3$ where π and τ are given by

$$\epsilon^\pi = \epsilon,\ \alpha^\pi = \epsilon\alpha,\ \epsilon^\tau = \epsilon^2, \text{ and } \alpha^\tau = \alpha.$$

Let $\pi', \tau' \in \mathrm{Aut}(K^2)$ be given by $(a, b)^{\pi'} = (a^\pi, b^\pi)$ and $(a, b)^{\tau'} = (b, a)$. Let A be the skew polynomial ring $K^2[X; \pi']$. Let y be the inner automorphism of A given by (α, α^2) and w the automorphism of A which acts as τ' on K^2 and $X^w = X$.

Let $G = \langle y, w \rangle \subseteq \mathrm{Aut}(A)$, the group of automorphisms of A generated by y and w. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be the inclusion map, $R = A_\alpha G$, $\beta\colon G \to \mathrm{Aut}(A)$ be given by $\beta(g) = w^{-1}\alpha(g)w$ and $R' = A_\beta G$. Clearly $Aw = A^w$ and $[R', 1] = \Sigma([A^w])$. By [5], $[Aw] \in \mathrm{Pic}_{(R,1)}(A)$ and $\phi([Aw]) \neq 1$. Thus, R' and R are graded equivalent and R_g and R'_g are isomorphic as A-bimodules. However, by Lemma 5.7, $\Psi([R', 1]) = (\Psi \circ \Sigma)(A^{w^{-1}}) \neq 1$. Finally, by Theorem 5.4, R' is not graded isomorphic to R.

6. APPLICATION I—STRONGLY GRADED RINGS GRADED EQUIVALENT TO A CROSSED PRODUCT

In this section, we complete our study of Problem A from the Introduction by using the results of the previous sections to show that graded equivalence is a viable tool for reducing the study of strongly graded rings.

By the structure of projective modules over semiperfect rings (see [2, Theorem 27.11]), if A is a basic semiperfect ring, then $\mathrm{Pic}(A) = \mathrm{Out}(A)$. Therefore, strongly graded rings with basic semiperfect coefficient rings are crossed products. Corollary 3.3 now implies the following result which first appeared in [9]:

COROLLARY 6.1. *If R is a strongly graded ring and R_e is semiperfect, then R is graded equivalent to a crossed product R' whose coefficient ring is the basic ring of R_e.*

In order to give our solution to Problem B in the next section, we need to describe the parameter set of the crossed product from the above corollary. Let R be a strongly graded ring, so that $R_e = A$ is semiperfect. Let e be a basic idempotent of A and $C = (A, B = eAe, P = Ae, Q =$

$eA, \tau, \mu)$, the Morita context, where both τ and μ are multiplication maps. Since R^C is a crossed product, for every $g \in G$, $Q \otimes_A R_g \otimes_A P \simeq B$ as right B-modules and so $Q \otimes_A R_g \simeq Q$ as right A-modules. Let $\Psi_g\colon Q \otimes_A R_g \simeq Q$ be an isomorphism of right A-modules for every $g \in G$. Then there is a automorphism α_g of B such that Ψ_g is an isomorphism of B–A-bimodules $Q \otimes_A R_g \simeq {}^{\alpha_g}Q$ [6, Theorem 55.12]. On the other hand, for every $g, h \in G$, the composition of the following isomorphisms

$$\delta_{g,h}\colon Q \xrightarrow{\Psi_h^{-1}} Q \otimes_A R_h \xrightarrow{\Psi_g^{-1}\otimes 1} Q \otimes_A R_g \otimes_A R_h \xrightarrow{1\otimes \rho_{g,h}} Q \otimes_A R_{gh} \xrightarrow{\Psi_{gh}} Q$$

is an isomorphism of right A-modules. Therefore there exists a unit $t(g,h)$ of B so that $\delta_{g,h}(q) = t(g,h)q$, for every $q \in Q$.

LEMMA 6.2. *With the above notation, (α, t) is a parameter set of G over B and R^C is graded isomorphic to $B *_t^\alpha G$.*

Proof. To check that (α, t) is a parameter set, it suffices to check that the multiplication in the crossed product $R *_t^\alpha G$ is associative [15, Lemma 1.1]. So to prove the lemma, we need only to show that there is a bijection $R^C \to R *_t^\alpha G$ that preserves addition and multiplication.

Let $\Gamma\colon R^C \to B *_c^\alpha G$ be defined by $\Gamma(q \otimes r_g \otimes p) = \bar{g}\,\mu(\Psi_g(q \otimes r_g) \otimes p)$ and extended linearly. Γ is an additive group isomorphism and we check that it preserves multiplication.

$$\begin{aligned}
&\Gamma\big((q \otimes r_g \otimes p)(q' \otimes r'_h \otimes p')\big)\\
&\quad= \overline{gh}\mu\big(\Psi_{gh} \otimes 1\big)\big(q \otimes r_g\tau(p \otimes q')r'_h \otimes p'\big)\\
&\quad= \overline{gh}\mu\big(\Psi_{gh}\big(q \otimes r_g\tau(p \otimes q')r'_h\big) \otimes p'\big)\\
&\quad= \overline{gh}\mu\big(\delta_{g,h}\Psi_h\big(\Psi_g \otimes 1\big)\big(1 \otimes \rho_{g,h}^{-1}\big)\big(q \otimes r_g\tau(p \otimes q')r'_h\big) \otimes p'\big)\\
&\quad= \overline{gh}\mu\big(\delta_{g,h}\Psi_h\big(\Psi_g \otimes 1\big)\big(q \otimes r_g \otimes \tau(p \otimes q')r'_h\big) \otimes p'\big)\\
&\quad= \overline{gh}\mu\big(\delta_{g,h}\Psi_h\big(\Psi_g\big(q \otimes r_g\big) \otimes \tau(p \otimes q')r'_h\big) \otimes p'\big)\\
&\quad= \overline{gh}\mu\big(\delta_{g,h}\Psi_h\big(\Psi_g\big(q \otimes r_g\big)\tau(p \otimes q') \otimes r'_h\big) \otimes p'\big)\\
&\quad= \overline{gh}\mu\big(\delta_{g,h}\Psi_h\big(\mu\big(\Psi_g\big(q \otimes r_g\big) \otimes p\big)q' \otimes r'_h\big) \otimes p'\big)\\
&\quad= \overline{gh}\mu\big(\delta_{g,h}\,\mu\big(\Psi_g\big(q \otimes r_g\big) \otimes p\big)^{\alpha_h}\Psi_h(q' \otimes r'_h) \otimes p'\big)\\
&\quad= \overline{gh}t(g,h)\,\mu\big(\Psi_g\big(q \otimes r_g\big) \otimes p\big)^{\alpha_h}\mu\big(\Psi_h(q' \otimes r'_h) \otimes p'\big)\\
&\quad= \bar{g}\,\mu\big(\Psi_g\big(q \otimes r_g\big) \otimes p\big)\bar{h}\,\mu\big(\Psi_h(q' \otimes r'_h) \otimes p'\big)\\
&\quad= \Gamma(q \otimes r_g \otimes p)\Gamma(q' \otimes r'_h \otimes p').
\end{aligned}$$

∎

Our next reduction application comes from a specific family of strongly graded rings and crossed products studied by Saorín [16] and Jespers and Okniński [7].

Remark 6.3. Saorín [16] proved that if $R = A * G$ is a left perfect crossed product such that $R_e/J(R_e)$ is a finite direct product of finite-dimensional simple algebras over an algebraic closed field, then G is finite. Actually his result is stated for strongly graded rings but, as it was pointed out by Jespers and Okniński [7], Saorín's proof is not correct for strongly graded rings because [16, Lemma 7] is based on a false statement in [14]. Jespers and Okniński gave a correct proof in [7] for strongly graded rings. However, relying on Saorín's proof for crossed products, one can easily extend the result for strongly graded rings using Corollary 6.1.

COROLLARY 6.4 [7]. *Let R be a perfect strongly G-graded ring such that $R_e/J(R_e)$ is a finite direct product of finite-dimensional simple algebras over an algebraic closed field. Then G is necessarily finite.*

Proof. Let A be the basic algebra of R_e. By Corollary 6.1, R is graded equivalent to a crossed product $A * G$ and $A/J(A)$ is a finite direct product of copies of an algebraic closed field. Now, by [14], G is finite. ∎

Another reduction application using graded equivalences appears in [10], in which the reduction process is used to simplify the study of finite representation type for orders.

The results above indicate that it is possible, using graded equivalence, to reduce from strongly graded rings to crossed products. If the crossed product obtained is not a twisted group ring we cannot expect to reduce to another twisted group ring (see Corollary 3.4). As our final analysis of Problem A, we wish to investigate when we can make a further reduction to skew group rings. It is well known that, using the Cohen−Montgomery duality theory, every G-graded ring R with G finite is graded equivalent to the skew group ring $(R\#G) * G$ (see [8]). But while this process simplifies the grading, the coefficient ring becomes more complicated (as a subring of $|G| \times |G|$ matrices over R). Consequently, we close this section by considering the question: When is a strongly graded ring R graded equivalent to a skew group ring R' so that $R_e \simeq R'_e$? This question has a very general negative answer.

PROPOSITION 6.5. *Let A be a basic semiperfect ring and let R and R' be graded equivalent strongly graded rings with $R_e \simeq R'_e \simeq A$. If R is a skew group ring, then so is R'.*

Proof. Consider R and R' as (A, G)-graded rings via the isomorphisms $f\colon A \simeq R_e$ and $f'\colon A \simeq R'_e$. By Proposition 4.4, there is $[P] \in \mathrm{Pic}(A)$ so that $[R', f'] = [R, f]^{[P]}$. If A is basic semiperfect, then $P = A^{\beta}$ for some

$\beta \in \mathrm{Aut}(A)$. Since R is a skew group ring, there is a group homomorphism $\alpha: G \to \mathrm{Aut}(A)$ so that $R \simeq A *^{\alpha} G$. For every $g \in G$, let $\bar{g} = 1 \otimes g \otimes 1 \in A^{\beta^{-1}} \otimes_A Ag \otimes A^{\beta}$. Then $\overline{gh} = (1 \otimes g \otimes 1)(1 \otimes h \otimes 1) = 1 \otimes gh \otimes 1$. This shows that $R^{[P]}$ is a skew group ring and hence so is R'. ∎

7. APPLICATION II. STRONGLY GRADED SEMISIMPLE RINGS

In this final section, we analyze Problem B from the Introduction. Our goal is to characterize when a strongly graded ring R is semisimple; see Theorem 7.5. Particular cases appear in Corollaries 7.7 and 7.8.

It is well known that if R is semisimple, then R_e is semisimple. Thus, we assume, for the remainder of this section, that R_e is a direct product of finite matrix rings over division rings.

Our strategy is the following. First we use Lemma 6.2 to compute a crossed product graded equivalent to R. Then we use ideas from [8] to reduce the study to the case of crossed products over division rings. Finally, in the case when R_e is a direct product of matrix rings over fields, we can reduce to crossed products over fields and then use the results of [4] to give specific conditions for the semisimplicity of these crossed products.

Notation 7.1. Given an automorphism $\alpha \in \mathrm{Aut}(D)$ (D a ring) and positive integers n, m, there exists a group automorphism of $M_{n,m}(D)$, given by $(a_{i,j}) \mapsto (a_{i,j}^{\alpha})$. We abuse the notation and denote this map also by α. It is clear that if $a \in M_{n,m}(D)$ and $b \in M_{m,l}(D)$, then $(ab)^{\alpha} = a^{\alpha} b^{\alpha}$.

Given an element x in a direct product $\prod_i X_i$, and $i \in I$, $x(i)$ stands for the ith coordinate of x.

Now we describe, up to graded isomorphisms, all the strongly graded rings R, such that R_e is semisimple. This characterization is essentially based on the discussion in [14, A.I.3] and the use of a factor set, but our description is more explicit.

Fix a semisimple ring $A = \prod_{i=1}^{k} M_{n_i}(D_i)$ where n_i is a positive integer and D_i is a division ring for every i. Assume that if D_i and D_j are isomorphic, then they are equal.

Let P be the subgroup of permutations σ of $\{1, 2, \ldots, n\}$ such that $D_i = D_{\sigma(i)}$, for every i. P acts on $\prod_{i=1}^{k} \mathrm{Aut}(D_i)$ by permuting the coordinates, i.e., $\sigma(\alpha)(i) = \alpha(\sigma^{-1}(i))$, for $\alpha \in \prod_{i=1}^{k} \mathrm{Aut}(D_i)$. Let H be the semidirect product induced by this action, i.e., $H = \prod_{i=1}^{k} \mathrm{Aut}(D_i) \times P$ as a set and the product is given by $(\alpha, \sigma)(\beta, \tau) = (\alpha\sigma(\beta), \sigma\tau)$.

For every $(\alpha, \sigma) \in H$, let $A(\alpha, \sigma) = \prod_{i=1}^{k} M_{n_{\sigma(i)}, n_i}(D_i)$, with the following bimodule structure:

$$(ap)(i) = a(\sigma(i))^{\alpha(\sigma(i))} p(i), \qquad (pa)(i) = p(i)a(i),$$
$$a \in A,\ p \in A(\sigma, \alpha).$$

LEMMA 7.2. *For every* $(\alpha, \sigma) \in H$, $[A(\alpha, \sigma)] \in \mathrm{Pic}(A)$ *and the map* f: $H \to \mathrm{Pic}(A)$, $(\alpha, \sigma) \mapsto [A(\alpha, \sigma)]$, *is a group epimorphism whose kernel is* $\prod_{i=1}^{k} \mathrm{Inn}(D_i) \times 1$.

Proof. Let Φ: $A(\alpha, \sigma) \otimes_A A(\beta, \tau) \to A(\alpha\sigma(\beta), \sigma\tau)$ be the map given by $\Phi(p \otimes q)(i) = p(\tau(i))^{\beta(\tau(i))} q(i)$. To prove that $A(\alpha, \sigma)$ is invertible and f is a group homomorphism it is enough to show that Φ is a bimodule isomorphism. First we check that it is well defined:

$$\begin{aligned}\Phi(pa \otimes q)(i) &= (pa)(\tau(i))^{\beta(\tau(i))} q(i) = (p(\tau(i))a(\tau(i)))^{\beta(\tau(i))} q(i)\\ &= p(\tau(i))^{\beta(\tau(i))} a(\tau(i))^{\beta(\tau(i))} q(i) = p(\tau(i))^{\beta(\tau(i))} (aq)(i)\\ &= \Phi(p \otimes aq)(i).\end{aligned}$$

Next we check that Φ is a bimodule homomorphism:

$$\begin{aligned}\Phi(ap \otimes q)(i) &= (ap)(\tau(i))^{\beta(\tau(i))} q(i)\\ &= \left(a(\sigma\tau(i))^{\alpha(\sigma\tau(i))} p(\tau(i))\right)^{\beta(\tau(i))} q(i)\\ &= a(\sigma\tau(i))^{\alpha(\sigma\tau(i))\beta(\tau(i))} p(\tau(i))^{\beta(\tau(i))} q(i)\\ &= a(\sigma\tau(i))^{(\alpha\sigma(\beta))(\sigma\tau(i))} \Phi(p \otimes q)(i)\\ &= (a\Phi(p \otimes q))(i),\end{aligned}$$

$$\begin{aligned}\Phi(p \otimes qa)(i) &= p(\tau(i))^{\beta(\tau(i))} (qa)(i)\\ &= p(\tau(i))^{\beta(\tau(i))} q(i)a(i)\\ &= \Phi(p \otimes q)(i)a(i)\\ &= (\Phi(p \otimes q)a)(i).\end{aligned}$$

Now we prove that f is surjective. Let $e_1, e_2, \ldots, e_k$ be the primitive central idempotents of A. Let $[P] \in \mathrm{Pic}(A)$. For every $i = 1, 2, \ldots, k$, $(Pe_i)_{Ae_i} \simeq M_{m_i, n_i}(D_i)$ for some $m_i \in N$. Then

$$A = \mathrm{End}(P) \overset{\phi}{\simeq} \prod_{i=1}^{k} \mathrm{End}\left((Pe_i)_{Ae_i}\right) \simeq \prod_{i=1}^{k} M_{m_i}(D_i).$$

Let $f_1, f_2, \dots, f_k$ be the primitive central idempotents of $\prod_{i=1}^k M_{m_i}(D_i)$. Then there is a permutation $\sigma \in P$, such that $m_i = n_{\sigma(i)}$ and $\phi(e_i) = f_{\sigma(i)}$ for every i. Moreover, ϕ restricts to an automorphism $\alpha'(i)$ of $M_{n_i}(D_i)$. By the Skolem–Noether theorem, there is an automorphism $\alpha(i)$ of D_i, such that $\alpha'(i)\alpha(i)^{-1}$ is inner. It follows that $\alpha(i)$ induces an isomorphism of A-bimodules $P \simeq A(\alpha, \sigma)$.

Finally we show that $\operatorname{Ker} f = \prod_{i=1}^k \operatorname{Inn}(D_i) \times 1$. Assume that $(\alpha, \sigma) \in \operatorname{Ker} f$. Then $e_i A(\alpha, \sigma) e_j = A(\alpha, \sigma) e_{\sigma(i)} e_j$ and so $\sigma = 1$. This implies that ${}_A Ae_i \simeq_A A(\alpha, 1) e_i \simeq^{\alpha(i)} (Ae_i)$ for every i, and hence $\alpha(i)$ is inner (in $M_{n_i}(D_i)$) for every i. But this implies that $\alpha(i)$ is inner. ▮

To define a strongly graded ring we need a notion a bit more complicated than a parameter set.

DEFINITION 7.3. Let A be as above and let G be a group. A *factor set* of G in A is a triple of maps

$$\left(\beta : G \to \prod_{i=1}^{k} \operatorname{Aut}(D_i),\ \sigma : G \to P,\ t : G \otimes G \to \prod_{i=1}^{k} D_i^* \right)$$

satisfying the following conditions for every $g, h, k \in G$ and $i = 1, 2, \dots, k$ (the images of g by β, σ, and t are denoted by β_g, σ_g, and $t_{g,'h}$):

1. σ is a group homomorphism.
2. $\beta_{gh} \iota_{t_{g,h}} = \beta_g \sigma_g(\beta_h)$.
3. $t_{gh,k}(i) t_{g,h}(\sigma_k(i))^{\beta_k(\sigma_k(i))} = t_{g,hk}(i) t_{h,k}(i)$.

Given a factor set (β, σ, t), we define the G-graded ring $A(\beta, \sigma, \tau) = \bigoplus_{g \in G} A(\beta_g, \sigma_g)$ where the product is given by

$$(r_g r_h)(i) = t_{g,h}(i) r_g(\sigma_h(i))^{\beta_h(\sigma_h(i))} r_h(i).$$

Two factor sets (β, σ, t) and (β', σ', t') are said to be *equivalent* if $\sigma = \sigma'$ and there exists a map u: $G \to \prod_{i=1}^k D_i^*$ such that

$$\beta_g = \iota_{u_g} \beta'_g \qquad \text{and} \qquad u(gh)(i) t_{g,h}(i) = t'_{g,h}(i) u(g)(\sigma_h(i))^{\beta_h(\sigma_h(i))} u_h(i)$$

for every $g, h \in G$ and $i = 1, 2, \dots, n$.

PROPOSITION 7.4. *Every strongly G-graded ring R, with $R_e = A$, is A-isomorphic to a ring of the form $A(\beta, \sigma, t)$ for some factor set (β, σ, t). Moreover, two factor sets give rise to graded A-isomorphic rings if and only if they are equivalent.*

Proof. By Lemma 7.2, if R is a strongly graded ring so that $R_e = A$, then R_g is isomorphic as an A-bimodule to $A(\beta_g, \sigma_g)$ for some $(\beta_g, \sigma_g) \in H$. We will assume that $R_g = A(\beta_g, \sigma_g)$. If $\rho_{g,h}$: $A(\beta_g, \sigma_g) \otimes_A A(\beta_h, \sigma_h) \to A(\beta_{gh}, \sigma_{gh})$ is the multiplication map and $\Phi_{g,h}$: $A(\beta_g, \sigma_g) \otimes_A A(\beta_h, \sigma_h) \to A(\beta_g \sigma_g(\beta_h), \sigma_g \sigma_h)$ is the isomorphism defined in the proof of Lemma 7.2, then $T_{g,h} = \rho_{g,h}\Phi_{g,h}^{-1}$: $A(\beta_g \sigma_g(\beta_h), \sigma_g \sigma_h) \to A(\beta_{gh}, \sigma_{gh})$ is an isomorphism of A-bimodules. By Lemma 7.2, σ is a group homomorphism and there is a unit $t_{g,h}$ so that $\beta_g \sigma_g(\beta_h) = \beta_{gh} \iota_{t_{g,h}}$. Moreover, one may assume that $T_{g,h}(x) = t_{g,h} x$ for every x. Therefore, the multiplication in R is given by

$$(r_g r_h)(i) = t_{g,h}(i) r_g(\sigma_h(i))^{\beta_h(\sigma_h(i))} r_h(i).$$

Now it is a matter of computation to show that this multiplication is associative if and only if (β, σ, t) is a factor set. This follows from the computations

$$\begin{aligned}((r_g r_h) r_k)(i) &= t_{gh,k}(i)(r_g r_h)(\sigma_k(i))^{\beta_k(\sigma_k(i))} r_k(i) \\ &= t_{gh,k}(i)\big(t_{g,h}(\sigma_k(i)) r_g(\sigma_{hk}(i))\big)^{\beta_h(\sigma_{hk}(i))} \\ &\qquad \times r_h(\sigma_k(i))^{\beta_k(\sigma_k(i))} r_k(i) \\ &= t_{gh,k}(i) t_{g,h}(\sigma_k(i))^{\beta_k(\sigma_k(i))} r_g(\sigma_{hk}(i))^{\beta_h(\sigma_{hk}(i))\beta_k(\sigma_k(i))} \\ &\qquad \times r_h(\sigma_k(i))^{\beta_k(\sigma_k(i))} r_k(i) \\ &= t_{gh,k}(i) t_{g,h}(\sigma_k(i))^{\beta_k(\sigma_k(i))} r_g(\sigma_{hk}(i))^{(\beta_h \sigma)(\beta_k)(\sigma_{hk}(i))} \\ &\qquad \times r_h(\sigma_k(i))^{\beta_k(\sigma_k(i))} r_k(i)\end{aligned}$$

and

$$\begin{aligned}(r_g(r_h r_k))(i) &= t_{g,hk}(i) r_g(\sigma_{hk}(i))^{\beta_{hk}(\sigma_{hi}(i))}(r_h r_k)(i) \\ &= t_{g,hk}(i) r_g(\sigma_{hk}(i))^{\beta_{hk}(\sigma_{hk}(i))} t_{h,k}(i) r_h(\sigma_k(i))^{\beta_k(\sigma_k(i))} r_k(i) \\ &= t_{g,hk}(i) t_{h,k}(i) r_g(\sigma_{hk}(i))^{\beta_h \sigma(h)(\beta(k))(\sigma_{hk}(i))} \\ &\qquad \times r_h(\sigma_k(i))^{\beta_k(\sigma_k(i))} r_k(i).\end{aligned}$$

Assume now that (β, σ, t) and (β', σ', t') are two factor sets and Φ: $A(\beta, \sigma, t) \to A(\beta', \sigma', t')$ is a graded A-isomorphism. Then, for every $g \in G$, the restriction Φ_g of Φ to $A(\beta_g, \sigma_g) \to A(\beta'_g, \sigma'_g)$ is a bimodule isomorphism. By Lemma 7.2, $\sigma = \sigma'$ and hence $A(\beta_g, \sigma_g)$ and $A(\beta'_g, \sigma'_g)$ coincide as left A-modules. Therefore, there exist unit $u(g)$ ($g \in G$) of A, such that $\Phi_g(r_g)(i) = u(g)(i)r_g(i)$, for every $r_g \in A(\beta_g, \sigma_g)$. By straightforward computations, one proves that, if $u(g)$ is a unit for every g, then the map u: $A(\beta, \sigma, t) \to A(\beta', \sigma', t')$, given by $u(r_g)(i) \to u_g(i)r_g(i)$ is an A-isomorphism if and only if $\beta_g = \iota_{u_g}\beta'_g$ and $u(gh)(i)t_{g,h}(i) = t'_{g,h}(i)u(g)(\sigma_h(i))^{\beta_h(\sigma_h(i))}u_h(i)$. Since the matrices having only 0 and 1 as entries are fixed by β_g and β'_g, one concludes that $u(g)(i)$ is a scalar matrix, for every i, and hence we may assume that $u(g)(i) \in D_i^*$. ∎

The significance of Proposition 7.4 is that we need only study the strongly related graded rings of the form $A(\beta, \sigma, t)$ for a factor set (β, σ, t).

Let $R = A(\beta, \sigma, t)$ be such a strongly graded ring. Let

$$C = \left(A, B = \prod_{i=1}^{k} D_i, P = \sum_{i=1}^{k} M_{n_i \otimes 1}(D_i), Q = \prod_{i=1}^{k} M_{1 \otimes n_i}(D_i), \mu, \tau \right)$$

be the obvious Morita context. Then R^C is a crossed product over B. We use Lemma 6.2 to compute the parameter set for this crossed product. For every $g \in G$, let Ψ_g: $Q \otimes_A R_g \to Q$ be the map given by $\Psi_g(q \otimes r)(i) = q(\sigma_g(i))^{\beta_g(\sigma_g(i))}r(i)$. Ψ_g is an isomorphism of right A-modules. Let α_g: $A \to A$ be the map given by $a^{\alpha_g}(i) = a(\sigma_g(i))^{\beta_g(\sigma_g(i))}$. Then α_g is an automorphism of A and Φ_g is an isomorphism of A-bimodules from $Q \otimes_A R_g$ to ${}^{\alpha_g}Q$. So α is the required action.

We show that $t_{g,h}$ is the required cocyle. Indeed, if $q \in Q$, $r_g \in R_g$, and $r_h \in R_h$, then

$$\begin{aligned}
&\big(\Psi_{gh}(1 \otimes \rho_{g,h})(q \otimes r_g \otimes r_h)\big)(i) \\
&\quad = \Psi_{gh}(q \otimes r_g r_h)(i) \\
&\quad = q\big(\sigma_{gh}(i)\big)^{\beta_{gh}(\sigma_{gh}(i))}(r_g r_h)(i) \\
&\quad = q\big(\sigma_{gh}(i)\big)^{\beta_{gh}(\sigma_{gh}(i))} t_{g,h}(i) r_g\big(\sigma_h(i)\big)^{\beta_h(\sigma_h(i))} r_h(i) \\
&\quad = t_{g,h}(i) q\big(\sigma_{gh}(i)\big)^{(\beta_g \sigma_g(\beta_h))(\sigma_{gh}(i))} r_g\big(\sigma_h(i)\big)^{\beta_h(\sigma_h(i))} r_h(i) \\
&\quad = t_{g,h}(i)\Big(q\big(\sigma_{gh}(i)\big)^{\beta_g(\sigma_{gh}(i))} r_g\big(\sigma_h(i)\big)\Big)^{\beta_h(\sigma_h(i))} r_h(i) \\
&\quad = t_{g,h}(i)\big(\Psi_g(q \otimes r_g)(\sigma_h(i))\big)^{\beta_h(\sigma_h(i))} r_h(i) \\
&\quad = t_{g,h}(i)\Psi_h\big(\Psi_g(q \otimes r_g) \otimes r_h\big)(i) \\
&\quad = t_{g,h}(i)\Psi_h\big(\Psi_g \otimes 1\big)\big(q \otimes r_g \otimes r_h\big)(i).
\end{aligned}$$

Therefore, if $\delta_{g,h}$ is the map from Section 5, $\delta_{g,h}(q) = \Psi_{gh}(1 \otimes \rho_{g,h})(\Psi_g^{-1} \otimes 1)\Psi_h^{-1}(q) = t_{g,h}q$.

Let (α, t) be a parameter set in a product of rings $\prod_{i\in I} R_i$. If J is a subset of I and H a subgroup of G, such that $(\prod_{j\in J} R_j)^{\alpha_h} = \prod_{j\in J} R_j$ for every $h \in H$, $(\alpha^{(J)}, t^{(J)})$ denotes the parameter set of H over $\prod_{j\in J} R_i$ given by $x^{\alpha_h^{(J)}}(j) = x^{\alpha_h}(j)$ and $t_{g,h}^{(J)}(j) = t_{g,h}(j)$, for every $x \in \prod_{i\in J} R_i$, $j \in J$, and $h \in H$. If $j \in I$, $\alpha^{(j)}$ (resp. $t^{(j)}$) stands for $\alpha^{(\{j\})}$ (resp. $t^{(\{j\})}$).

Now we are ready to state the main theorem of this section.

THEOREM 7.5. *Let $A = \prod_{i=1}^k M_{n_i}(D_i)$ be a semisimple artinian ring, where every D_i is a division ring and $D_i = D_j$, if they are isomorphic. Let $B = \prod_{i=1}^k D_i$ and let (β, σ, t) be a factor set of a group G over A. For every $g \in G$, let α_g be the automorphism of B given by $a^{\alpha_g}(i) = a(\sigma_g(i))^{\beta_g(\sigma_g(i))}$. Let $j_1, j_2, \ldots, j_n$ be representatives of the orbits of the action σ of G on $\{1, 2, \ldots, k\}$. For every $i = 1, \ldots, k$, let J_i be the orbit of j_i and G_i the stabilizer of j_i. Then (α, t) defines a parameter set of G over B and the following assertions are equivalent*:

1. *$A(\beta, \sigma, t)$ is semisimple.*
2. *$B *_t^\alpha G$ is semisimple.*
3. *$(\prod_{i\in J_i} D_{j_i}) *_{t^{(J_i)}}^{\alpha(j_i)} G$ is semisimple for every $i = 1, 2, \ldots, n$.*
4. *$D_{j_i} *_{t^{(j_i)}}^{\beta^{(j_i)}} G_i$ is semisimple for every $i = 1, 2, \ldots, n$.*

Proof. The fact that (α, t) is a parameter set is a consequence of Lemma 6.2. (1) ⇔ (2) is a consequence of the fact that $A(\beta, \sigma, t)$ and $B *_t^\alpha G$ are graded equivalent.

To prove (2) ⇔ (3) it is enough to realize that

$$B *_t^\alpha G = \prod_{i=1}^n \left(\prod_{i\in J_i} D_{j_i} \right) *_{t^{(J_i)}}^{\alpha(J_i)} G.$$

To simplify the proof of (3) ⇔ (4), we may assume that $n = 1$; i.e., the action of G on $\{1, 2, \ldots, k\}$ induced by σ is transitive. Let $D = D_1$ and let H be the stabilizer of 1 by σ. Then, using ideas from [7], we prove that $R = B *_t^\alpha G$ is isomorphic to $M_k(D *_{t^{(1)}}^{\beta^{(1)}} H)$.

Let $e_1, e_2, \ldots, e_k$ be the primitive idempotents of B. Since the action σ of G on $\{1, 2, \ldots, k\}$ is transitive, for every i, there exists $g \in G$, such that $e_1\bar{g} = \bar{g}e_i$. Thus the map $x \mapsto x\bar{g}$ defines an isomorphism ${}_R Re_1 \simeq_R Re_i$. Therefore ${}_R R \simeq ({}_R Re_1)^k$ and so $R \simeq \mathrm{End}({}_R R) \simeq M_n(\mathrm{End}({}_R(Re_1)) \simeq M_n(e_1 Re_1)$. But $e_1 Re_1 \simeq D *_{t^{(1)}}^{\beta^{(1)}} H$. ∎

PROPOSITION 7.6. *A skew group ring $B *^{\alpha} G$ where B is a direct product of division rings is semisimple if and only if G is finite and there exists $b \in B$, such that $\sum_{g \in G} b^{\alpha_g} = 1$.*

Proof. A very classical argument shows that if $R = B *^{\alpha} G$ is semisimple, then G is finite. Then R is semisimple if and only if it is von Neumann regular. Since B is abelian regular (i.e., every idempotent is central), the result is a consequence of [1]. ∎

As a direct consequence of Theorem 7.5 and Proposition 7.6, we have the following.

COROLLARY 7.7. *Let $A = \prod_{i=1}^{k} M_{n_i}(D_i)$ be a semisimple artinian ring, where every D_i is a division ring and $D_i = D_j$, if they are isomorphic. Let (β, σ, t) be a factor set of a group G over A. Assume that $t_{g,h} = 1$ for every g, h. Then the following are equivalent:*

1. *A is semisimple.*
2. *G is finite and there exists $b \in B$, such that $\sum_{g \in G} b_{\sigma_g(i)}^{\beta_g(\sigma_g(i))} = 1$, for every $1 \le i \le k$.*
3. *If $j_1, j_2, \dots, j_i$ is a set of representatives of the action σ of G on $\{1, 2, \dots, n\}$, then the stabilizer of G_i of j_i by this action is finite and for every $1 \le i \le n$, there exists $x_i \in D_{j_i}$, so that $\sum_{g \in G_i} b^{\beta_g(j_i)} = 1$.*

Finally we consider the case where A is a direct product of matrix rings over fields and the grading group is finite. Using Theorem 7.5, the semisimplicity of a strongly graded ring over A reduces to the semisimplicity of crossed products over fields. This case has been studied recently by Aljadeff and Robinson [4].

Let $R = K *_t^{\alpha} G$ be a crossed product over a field of characteristic $p \neq 0$ and assume that $G = \prod_{i=1}^{r} G_i$ with G_i cyclic of order p^{k_i}. Then, from the discussion in [4], $H^2(G, K^*) \simeq \bigoplus_{i=1}^{r} K^*/(K^*)^{p^{k_i}}$, so every cocycle t of G over K is represented by an r-tuple $(a_1(K^*)^{p^{k_1}}, a_2(K^*)^{p^{k_2}}, \dots, a_r(K^*)^{p^{k_r}})$.

COROLLARY 7.8. *Let $A = \prod_{i=1}^{k} M_{n_i}(K_i)$, where every K_i is a field of characteristic p_i (where p_i could be 0) and $K_i = K_j$, if they are isomorphic. Let (β, σ, t) be a factor set of a group over A and let $j_1, j_2, \dots, j_n$ be representatives of the orbits of the action σ of G on $\{1, 2, \dots, k\}$. For every $i = 1, 2, \dots, n$, let G_i be the stabilizer of j_i under this action and let $H_i = \{g \in G_i\colon \beta_g = 1\}$. If $p_i \neq 0$, let P_i be a Sylow p_i-subgroup of H_i. The following* *assertions are equivalent*:

1. $A(\beta, \sigma, t)$ *is semisimple*.
2. $K_{j_i} *_{t^{(j_i)}}^{\beta^{(j_i)}} G_i$ *is semisimple for every* $i = 1, 2, \ldots, n$.
3. $K_{j_i} *_{t^{(j_i)}} H_i$ *is semisimple for every* $i = 1, 2, \ldots, n$.
4. *For every* $i = 1, 2, \ldots, n$, *either* p_i *does not divide the order of* H_i *or the following conditions hold*:

(a) $|H_i'|$ *is prime to* p_i, *so that* P_i *is abelian, say isomorphic to* $\prod_{i=1}^{r} C_i$ *with* C_i *cyclic of order* p^{k_i}.

(b) *The restriction of* t^{j_i} *to* P_i *is represented an r-tuple*

$$\left(a_1(K_i^*)^{p_i^{k_1}}, a_2(K_i^*)^{p_i^{k_2}}, \ldots, a_r(K_i^*)^{p_i^{k_r}}\right)$$

such that $\{a_1, a_2, \ldots, a_r\}$ *is p-independent over* $K_i^{p_i}$.

Proof. The equivalence of (1) and (2) comes from Theorem 7.5. (2) ⇔ (3) is a consequence of [3, Corollary 4.2] and (3) ⇔ (4) is a consequence of [4, Theorem 2]. ∎

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